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HOPF BIFURCATION FROM VISCOUS SHOCK WAVES*

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Abstract. Using spatial dynamics, we prove a Hopf bifurcation theorem for viscous Lax shocks in viscous conservation laws. The bifurcating viscous shocks are unique (up to time and space translation), exponentially localized in space, periodic in time, and their speed satisfies the Rankine–Hugoniot condition. We also prove an "exchange of spectral stability" result for super- and subcritical bifurcations and outline how our proofs can be extended to cover degenerate, over-, and undercompressive viscous shocks.

Key words. viscous conservation law, Lax shock, Hopf bifurcation

AMS subject classifications. 35L65, 35B32, 35L67

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1. Introduction. The purpose of this paper is to study Hopf bifurcation from viscous shock waves. While Hopf bifurcations from equilibria are well understood in ordinary differential equations (ODEs) and in dissipative partial differential equations (PDEs) on bounded domains, a variety of new phenomena and difficulties arise when studying Hopf bifurcations for PDEs on unbounded domains.

In particular, Hopf bifurcations from travelling waves are complicated by the presence of a neutral mode at the origin which is induced by spatial translation. If the essential spectrum of the linearization around the travelling wave is bounded away from the imaginary axis, appropriate center-manifold reductions and equivariant parametrizations as in [4, 6, 17] show that the bifurcation problem reduces to a standard Hopf bifurcation, and standard results on bifurcation and exchange of stability [2] immediately carry over to this setting [15, section 2]; the only effect of the translation mode is an adjustment of the wave speed. When the Hopf instability is caused by essential spectrum that crosses the imaginary axis, a variety of interesting new phenomena can occur, including failure of bifurcation [15] and bifurcation of multiple solution branches [16, section 2.3]. The situation becomes more involved when the instability caused by the essential spectrum is stationary, as the wave will then typically decay only algebraically at onset which leads to significant complications in the analysis [16, sections 2.1, 2.2, and 3].

From the preceding list, one can easily envision yet another possible scenario where the complex Hopf eigenvalues belong to the point spectrum, whilst the translation mode is embedded in the continuous spectrum. This situation arises, for instance, when the primary wave is not spatially localized, but the Hopf eigenfunctions are localized: Examples are Hopf bifurcations from coherent structures such as sources and sinks in one spatial dimension, and spiral waves in two dimensions. A model problem in higher space dimensions, but with a space-dependent potential, has recently been analyzed in [1]. Viscous shock waves provide another prominent example where the translation mode is embedded into the continuous spectrum. In fact, conservation

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laws can often be derived formally and rigorously in spatially extended systems where the primary pattern breaks the underlying continuous symmetry [3, 7].

In this article, we investigate Hopf bifurcations from viscous shock waves using the spatial-dynamics techniques we developed in [13, 14, 15] for Hopf bifurcations from fronts and pulses in reaction-diffusion systems. Our paper is strongly motivated by recent work of Texier and Zumbrun [19, 20] in which they analyzed oscillatory instabilities of viscous shocks using delicate estimates for the temporal period map of the linearized semigroup. Texier and Zumbrun proved the existence of a continuous branch of oscillatory viscous shocks with a $1/x$ decay estimate at spatial infinity. In a personal communication, Zumbrun asked us whether spatial-dynamics techniques can be used to obtain the same or stronger results than those in [19, 20]. We demonstrate here that the spatial-dynamics approach yields indeed sharper results, while simplifying the analysis and adding geometric insight into the problem: We show that the bifurcating oscillatory shocks are unique, exponentially localized, and depend smoothly on the bifurcation parameter, and we calculate the spectra of the linearization about the bifurcating oscillating shock waves, thereby confirming the expected exchange of stability. Instead of analyzing the temporal semigroup whose linearization has essential spectrum up to the imaginary axis, we consider the spatial evolution of temporally periodic functions for which we gain compactness of the resolvent due to the imposed time periodicity. While this method may appear nonintuitive, it is completely analogous to the usual phase-plane analysis used to prove existence of viscous shocks and to study their stationary bifurcations. After this paper was completed, Texier and Zumbrun were able to extend their approach to prove in [21] exponential localization of the bifurcating solutions for compressible Navier–Stokes and magnetohydrodynamics.

Outline. In section 2, we state our main result on bifurcation and spectral stability of modulated shocks. The bifurcation result is proved in section 3. In section 4, we review the precise characterization of spectra and prove stability and instability in the case of super- and subcritical bifurcations, respectively. We conclude with a discussion of several extensions and generalizations in section 5.

2. Setup and main results.

Consider the viscous conservation law

$$u_t + f(u)_y = u_{yy}, \qquad y \in \mathbb{R},\ u \in \mathbb{R}^n, \tag{2.1}$$

where f is a smooth flux function. We are interested in viscous shocks $q^0(y - c^0 t)$ which connect the constant rest states $u^0_\pm$ at $y = \pm\infty$ so that

$$\lim_{x\to\pm\infty} q^0(x) = u^0_\pm.$$

Viscous shocks are stationary solutions in the moving reference frame $x = y - ct$ in which (2.1) becomes

$$u_t = \partial_x \left[u_x + cu - f(u)\right], \qquad x \in \mathbb{R},\ u \in \mathbb{R}^n, \tag{2.2}$$

and they therefore satisfy the integrated steady state equation

$$u_x = [f(u) - f(u^0_-)] - c[u - u^0_-], \tag{2.3}$$

where the speed c is given necessarily by the Rankine–Hugoniot condition

$$c = \frac{f_j(u^0_+) - f_j(u^0_-)}{u^0_{+,j} - u^0_{-,j}}, \qquad j = 1, \dots, n. \tag{2.4}$$

In particular, $q^0(x)$ lies in the intersection of the unstable manifold $\widetilde{W}^{\mathrm{u}}_{-}$ of u^0_{-} and the stable manifold $\widetilde{W}^{\mathrm{s}}_{+}$ of u^0_{+} for (2.3).

The most common viscous shocks are Lax shocks for which $u^0_{\pm}$ are hyperbolic equilibria of (2.3) with $\dim \widetilde{W}^{\mathrm{u}}_{-} = p+1$ and $\dim \widetilde{W}^{\mathrm{s}}_{+} = n-p$ for some $p \in \{0, \dots, n-1\}$. We assume that the intersection of $\widetilde{W}^{\mathrm{u}}_{-}$ and $\widetilde{W}^{\mathrm{s}}_{+}$ is transverse along q^0 and that the Jacobian $f_u(u^0_{\pm})$ has only real and distinct eigenvalues. If $u^{\varepsilon}_{\pm}$ are smooth curves that depend on a real parameter $\varepsilon \approx 0$, then we find a smooth family of Lax shocks $q^{\varepsilon}(x)$ with a smooth speed relation $c = c^{\varepsilon}$ given by the Rankine–Hugoniot condition. Since the eigenvalues of $f_u(u)$ are the characteristic speeds of propagation at u, the condition on the dimensions of $\widetilde{W}^{\mathrm{s,u}}_{\pm}$ merely states that $p+1$ characteristics enter the shock from the left and $n-p$ characteristics enter from the right.

We are interested in the scenario where the Lax shocks undergo a Hopf instability upon increasing ε through zero. We therefore consider the linearization at the shock which is given by the linear operator

$$\mathcal{L}^{\varepsilon} := \partial_x \left[\partial_x + c^{\varepsilon} - f_u(q^{\varepsilon}(x))\right],$$

which we view as a closed unbounded operator on $L^2(\mathbb{R}, \mathbb{R}^n)$. Its essential spectrum is readily seen to be contained in the closed left half-plane, touching the imaginary axis only at the origin with a quadratic tangency. We assume that the point spectrum lies in the open left half-plane, bounded away from the imaginary axis, except for an isolated pair $\lambda(\varepsilon)$ and $\overline{\lambda(\varepsilon)}$ of simple complex eigenvalues with

$$\lambda(0) = \mathrm{i}\omega_0 \neq 0, \qquad \mathrm{Re}\,\lambda_{\varepsilon}(0) > 0. \tag{2.5}$$

Theorem 2.1 (bifurcation). *Under the above assumptions, there are positive constants K, η, and δ and a smooth function*

$$\begin{aligned} [0, \delta) &\longrightarrow \mathcal{C}^2_{\mathrm{unif}}(\mathbb{R} \times S^1, \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R}, \\ a &\longmapsto (q^*(a), \varepsilon(a), \omega(a)), \end{aligned}$$

so that $u^*(x, t; a) := q^*(x, \omega(a)t; a)$ satisfies (2.2) with $c = c^{\varepsilon(a)}$ for all a,

$$|q^*(x, \tau; a) - u^{\varepsilon(a)}_{\pm}| \leq K\mathrm{e}^{-\eta|x|}, \qquad q^*(x, \tau; 0) = q^0(x), \qquad \omega(0) = \omega_0,$$

and $q^*(x, \cdot; a)$ has minimal period 2π in τ for each $a > 0$. Furthermore, any non-stationary time-periodic solution $u(x, t)$ of (2.2), which is pointwise close to $q^0(x)$ and converges to $u^{\varepsilon}_{\pm}$ as $x \to \pm\infty$, is in fact an appropriate space and time translation of u^*.

Note that $u^*(x, t; a)$ and $q^{\varepsilon(a)}(x)$ have the same asymptotic rest states and travel with the same (average) wave speed. Theorem 2.1 remains true if $f = f(u; \varepsilon)$ depends smoothly on the parameter ε.

Spectral stability of the modulated shocks $u^*(x, t; a)$ is determined by the Floquet spectrum

$$\Sigma = \{\lambda \in \mathbb{C};\ \mathrm{e}^{\lambda T} \in \text{ spectrum of } \Phi_T\},$$

where $T = 2\pi/\omega$ is the temporal period of u^*, and Φ_t is the evolution operator of the linearization

$$v_t = \partial_x[\partial_x + c^{\varepsilon(a)} - f_u(u^*(x, t; a))]v$$

of (2.2) about u^* on L^2 or $\mathcal{C}^0_{\mathrm{unif}}$.

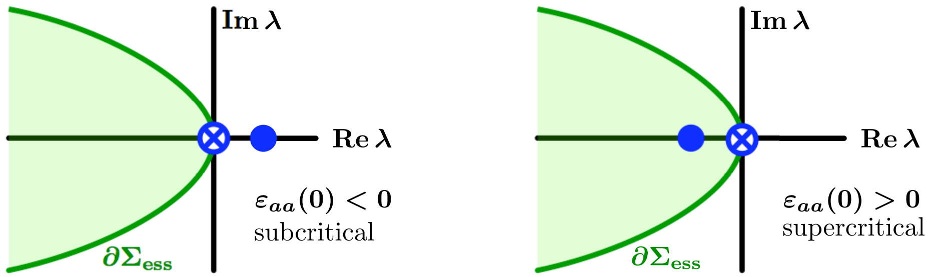

FIG. 1. *The Floquet spectra of the oscillatory shocks u^* from Theorem 2.1 is shown for $a > 0$: $\lambda = 0$ has geometric and algebraic multiplicity two, while the location of the remaining simple Floquet exponent near the origin depends on the sign of $\varepsilon_{aa}(0)$.*

THEOREM 2.2 (stability). *Assume that the hypotheses of Theorem 2.1 are met and that the Evans function associated with q^0 has a simple zero at the origin (see section 4 for details). If $\varepsilon_{aa}(0) \neq 0$, then the Floquet spectrum Σ of the oscillatory shock u^* given in Theorem 2.1 is as indicated in Figure 1.*

We refer the reader to [5, section 3.4] and [23, sections 9, 10] for explicit conditions which imply that the Evans function of q^0 has a simple zero at the origin.

3. Existence of modulated viscous shocks. In this section, we prove Theorem 2.1.

3.1. Preparations. We begin by collecting some properties of the linearization

$$\mathcal{L}^\varepsilon = \partial_x \left[\partial_x + c^\varepsilon - f_u(q^\varepsilon(x))\right]$$

about the viscous shocks that we need later on. Since we assumed in (2.5) that the Hopf eigenvalues $\lambda(0) = \mathrm{i}\omega_0 \neq 0$ and $\overline{\lambda(0)}$ of $\mathcal{L}^0$ are simple, we know that there are nonzero L^2-functions v_j and ψ_j for $j = 1, 2$ that form a basis of the eigenspaces of $\mathcal{L}^0$ and its adjoint $[\mathcal{L}^0]^*$, respectively, associated with these Hopf eigenvalues. We can choose these functions so that

$$\begin{aligned} \langle\psi_i, v_j\rangle_{L^2} = \langle\psi_i, \psi_j\rangle_{L^2} = \delta_{ij}, \\ \mathcal{L}^0 v_1 = -\omega_0 v_2, \qquad \mathcal{L}^0 v_2 = \omega_0 v_1. \end{aligned} \tag{3.1}$$

The result [10, Theorem 5.4 in Chapter II] gives the characterization

$$\mathrm{Re}\,\lambda_\varepsilon(0) = \frac{1}{2}\sum_{j=1}^{2}\langle\psi_j, \partial_\varepsilon\mathcal{L}^\varepsilon|_{\varepsilon=0} v_j\rangle_{L^2} \tag{3.2}$$

of the derivative $\mathrm{Re}\,\lambda_\varepsilon(0)$ which we assumed in (2.5) to be positive.

3.2. Spatial dynamics. To find time-periodic solutions of (2.2), we rescale time $\tau := \omega t$ to get

$$\omega\partial_\tau u + f(u)_x - cu_x = u_{xx},$$

which we then cast as the first-order system

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} v \\ \omega\partial_\tau u + f(u)_x - cv \end{pmatrix} = \begin{pmatrix} v \\ \omega\partial_\tau u + f_u(u)v - cv \end{pmatrix}. \tag{3.3}$$

We view (3.3) as an equation for $U = (u, v)$ in $Y = H^1(S^1) \times H^{1/2}(S^1)$ with $S^1 = [0, 2\pi]/\sim$. The space Y is natural for various reasons: First, we wish to work on spaces of time-periodic functions and shall see later in (3.13), see also [13, section 3], that the spaces $H^{s+1/2}(S^1) \times H^s(S^1)$ for $s \geq 0$ are the only spaces compatible with the linear leading-order part of (3.3). We choose $s = 1/2$ since $H^1(S^1)$ embeds into $C^0(S^1)$ which guarantees that the nonlinearity $f_u(u)$ in (3.3) is well defined and differentiable on Y. Finally, we remark that we shall often use, for convenience, the scalar product

$$\langle U, V\rangle_X := \frac{1}{2\pi}\int_0^{2\pi} \langle U(\tau), V(\tau)\rangle_{\mathbb{R}^{2n}}\, \mathrm{d}\tau$$

of the space $X = L^2(S^1) \times L^2(S^1)$ to define complements and compute adjoints; this scalar product is also an inner product on Y as Y embeds continuously into X.

The system (3.3) is invariant under the S^1-action

$$\Gamma : S^1 \longrightarrow L(Y, Y), \qquad \sigma \longmapsto \Gamma_\sigma, \qquad [\Gamma_\sigma U](\tau) = U(\tau - \sigma). \tag{3.4}$$

We record that the fixed-point space $\operatorname{Fix}\Gamma \cong \mathbb{R}^n \times \mathbb{R}^n$ of this action consists precisely of all time-independent functions, and (3.3) restricted to $\operatorname{Fix}\Gamma$ becomes the usual travelling-wave ODE

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} v \\ f_u(u)v - cv \end{pmatrix} \tag{3.5}$$

which is equivalent to (2.3). Equation (3.5) possesses the equilibria $U_{\mathrm{eq}} = (u, 0)$ for $u \in \mathbb{R}^n$ and the heteroclinic orbits $Q^\varepsilon(x) := (q^\varepsilon, q^\varepsilon_x)(x)$ for $c = c^\varepsilon$: $Q^\varepsilon(x)$ connects $U^-_{\mathrm{eq}}(\varepsilon) = (u^\varepsilon_-, 0)$ to $U^+_{\mathrm{eq}}(\varepsilon) = (u^\varepsilon_+, 0)$ with

$$T_{Q^\varepsilon(x)} W^{\mathrm{u}}_- + T_{Q^\varepsilon(x)} W^{\mathrm{cs}}_+ = \mathbb{R}^{2n}, \tag{3.6}$$

where $W^j_\pm := W^j(U^\pm_{\mathrm{eq}}(\varepsilon))$. The transversality of the intersection in (3.6) is a consequence of the following dimension count for $\varepsilon = 0$. Since $\dim W^{\mathrm{u}}_- = p + 1$ and $\dim W^{\mathrm{cs}}_+ = 2n - p$, it suffices to show that the only nontrivial elements in the intersection of the tangent spaces are multiples of $Q^0_x(x)$. This, in turn, can be seen as follows. Starting with any nontrivial bounded solution $(u, v)(x)$ of the variational equation

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ f_{uu}(q^0)[q^0_x, \cdot] & f_u(q^0) - c^0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.7}$$

of (3.5) about $Q^0 = (q^0, q^0_x)$, we find that its first component $u(x)$ is a nontrivial bounded solution of

$$u_{xx} = [(f_u(q^0) - c^0)u]_x. \tag{3.8}$$

If a nontrivial bounded solution $(u, v)(x)$ of (3.7) lies, in addition, in $T_{Q^\varepsilon(x)} W^{\mathrm{u}}_-$, then $u(x)$ decays exponentially at $x = -\infty$ and is therefore also a nontrivial bounded solution of the variational equation

$$u_x = [f_u(q^0) - c^0]u \tag{3.9}$$

of (2.3) about q^0 since we can integrate (3.8) once, starting at $x = -\infty$. To complete the argument, we recall our assumption that q^0 lies in the transverse intersection of the unstable and stable manifolds of the hyperbolic rest states $u^0_\pm$ of (2.3); this hypothesis implies that each nontrivial bounded solution of (3.9) is necessarily exponentially localized and must, in fact, be a multiple of q^0_x as claimed.

Next, we linearize the full system (3.3) in the solution $Q^0 = (q^0, q^0_x)$ for $\omega = \omega_0$ to get

$$V_x = \begin{pmatrix} 0 & 1 \\ \omega_0\partial_\tau + f_{uu}(q^0)[q^0_x, \cdot] & f_u(q^0) - c^0 \end{pmatrix} V, \qquad V \in Y. \tag{3.10}$$

For $x \to \pm\infty$, we obtain the asymptotic systems

$$V_x = \begin{pmatrix} 0 & 1 \\ \omega_0\partial_\tau & f_u(u^0_\pm) - c^0 \end{pmatrix} V, \qquad V \in Y, \tag{3.11}$$

whose properties we discuss first. Equations (3.10) and (3.11) leave each subspace $Y_k := \{\mathrm{e}^{\mathrm{i}k\tau}\hat{V};\ \hat{V} \in \mathbb{C}^{2n}\}$ invariant for $k \in \mathbb{Z}$. If we restrict (3.11) to Y_k, we obtain the system

$$\hat{V}_x = \begin{pmatrix} 0 & 1 \\ \mathrm{i}k\omega_0 & f_u(u^0_\pm) - c^0 \end{pmatrix} \hat{V}, \qquad \hat{V} \in \mathbb{C}^{2n}, \tag{3.12}$$

where $V = \mathrm{e}^{\mathrm{i}k\tau}\hat{V}$. For $k \neq 0$, the matrices in (3.12) are hyperbolic: $\nu = \mathrm{i}\kappa$ is an eigenvalue if and only if $\det[-\kappa^2 - \mathrm{i}\kappa(f_u(u^0_\pm) - c^0) - \mathrm{i}k\omega_0] = 0$, which is excluded since $f_u(u^0_\pm)$ was assumed to have only real eigenvalues.[1] For $|k| \to \infty$, the eigenvalues of the matrices in (3.12) are

$$\nu_j = \pm\sqrt{\mathrm{i}\omega_0 k}(1 + \mathrm{O}(|k|^{-1/2})) \qquad \text{with eigenfunction } \begin{pmatrix} \nu_j e_j \\ e_j \end{pmatrix}, \tag{3.13}$$

where e_j denotes the canonical basis in $\mathbb{R}^n$. In particular, the stable and unstable eigenspaces have a uniform angle in $H^1(S^1) \times H^{1/2}(S^1)$ as $|k| \to \infty$, and therefore for all $k \neq 0$; see also [13, Lemma 3.3]. Thus, we can apply the results in [11, 13] to conclude that (3.11) restricted to $Y_\mathrm{h} := \overline{\bigoplus_{k\neq 0} Y_k}$ has exponential dichotomies $\Phi^{\mathrm{s,u}}_{\pm,\mathrm{h}}(x, y)$ on $\mathbb{R}^\pm$ since the perturbation

$$\begin{pmatrix} 0 & 0 \\ f_{uu}(q^0(x))[q^0_x(x), \cdot] & f_u(q^0(x)) - f_u(u^0_\pm) \end{pmatrix} : H^1 \times H^{1/2} \longrightarrow H^1 \times H^{1/2}$$

is bounded independently of x and converges to zero as $|x| \to \infty$. We define

$$\nu^\mathrm{s}_\pm := -\frac{1}{2} \sup \left\{ \mathrm{Re}\,\nu_j;\ \mathrm{Re}\,\nu_j < 0,\ \nu_j \text{ is an eigenvalue of } (3.12)_\pm \text{ for some } k \in \mathbb{Z} \right\},$$
$$\nu^\mathrm{u}_\pm := \frac{1}{2} \inf \left\{ \mathrm{Re}\,\nu_j;\ \mathrm{Re}\,\nu_j > 0,\ \nu_j \text{ is an eigenvalue of } (3.12)_\pm \text{ for some } k \in \mathbb{Z} \right\},$$

[1] Purely imaginary spatial eigenvalues $\nu = \mathrm{i}\kappa$ are actually equivalent to essential spectrum at $\lambda = \mathrm{i}\omega_0 k$ so that, for more general viscosity matrices and fluxes, the analysis goes through provided the Hopf eigenvalue $\mathrm{i}\omega_0$ is *not resonant* with essential spectrum on the imaginary axis.

and observe that $\nu_\pm^{\rm s}, \nu_\pm^{\rm u} > 0$ due to (3.13). The spaces

$$E_+^{\rm cs} = \left\{ V_0 \in Y;\ \exists \text{ solution } V(x) \text{ of (3.10) on } \mathbb{R}^+ \text{ with } V(0) = V_0,\ \sup_{x \geq 0} |V(x)| < \infty \right\},$$

$$E_-^{\rm u} = \Big\{ V_0 \in Y;\ \exists \text{ solution } V(x) \text{ of (3.10) on } \mathbb{R}^- \text{ with } V(0) = V_0,$$
$$\sup_{x \leq 0} |V(x)| {\rm e}^{\nu_-^{\rm u}|x|} < \infty \Big\}$$

are closed subspaces of Y.

Claim. *We have*

$$E_+^{\rm cs} \cap E_-^{\rm u} = \mathbb{R} Q_x^0(0) \oplus \mathbb{R} V_1(0) \oplus \mathbb{R} V_2(0), \tag{3.14}$$
$$Y = [E_+^{\rm cs} + E_-^{\rm u}] \oplus \mathbb{R}\Psi_1(0) \oplus \mathbb{R}\Psi_2(0), \tag{3.15}$$

where, using the definitions of v_j and ψ_j from section 3.1*,*

$$\begin{aligned} V_1(x) &:= \cos\tau \begin{pmatrix} v_1 \\ \partial_x v_1 \end{pmatrix}(x) + \sin\tau \begin{pmatrix} v_2 \\ \partial_x v_2 \end{pmatrix}(x), \\ V_2(x) &:= -\sin\tau \begin{pmatrix} v_1 \\ \partial_x v_1 \end{pmatrix}(x) + \cos\tau \begin{pmatrix} v_2 \\ \partial_x v_2 \end{pmatrix}(x) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \Psi_1(x) &:= \cos\tau \begin{pmatrix} \widetilde{\psi}_1 \\ \psi_1 \end{pmatrix}(x) + \sin\tau \begin{pmatrix} \widetilde{\psi}_2 \\ \psi_2 \end{pmatrix}(x), \\ \Psi_2(x) &:= -\sin\tau \begin{pmatrix} \widetilde{\psi}_1 \\ \psi_1 \end{pmatrix}(x) + \cos\tau \begin{pmatrix} \widetilde{\psi}_2 \\ \psi_2 \end{pmatrix}(x) \end{aligned} \tag{3.17}$$

with $\widetilde{\psi}_j := -\partial_x \psi_j - [f_u^T(q^0) - c^0]\psi_j$ for $j = 1, 2$.

Proof. The characterization of $E_+^{\rm cs}$ and $E_-^{\rm u}$ is a consequence of the existence of exponential dichotomies on $Y_{\rm h}$ and the dynamics of the travelling-wave ODE (3.5). First, recall that the dynamics on the Fourier subspaces Y_k decouple, so that we can write

$$E_+^{\rm cs} = \bigoplus_{k\in\mathbb{Z}} (E_+^{\rm cs} \cap Y_k), \qquad E_-^{\rm u} = \bigoplus_{k\in\mathbb{Z}} (E_-^{\rm u} \cap Y_k).$$

We know that the strong unstable manifold $W^{\rm u}(U_{\rm eq}^-(0))$ and the center-stable manifold $W^{\rm cs}(U_{\rm eq}^+(0))$ of (3.5) intersect transversely along $Q^0(x)$; see (3.6). Thus,

$$\operatorname{span} Q_x^0(0) = E_+^{\rm cs} \cap E_-^{\rm u} \cap Y_0.$$

Next, $V_0 \in Y_{\rm h}$, the subspace of nonzero Fourier modes $k \neq 0$, lies in $E_+^{\rm cs} \cap E_-^{\rm u}$ if and only if $V(x)$ satisfies (3.11) on $\mathbb{R}$ with $V(x) \to 0$ exponentially as $|x| \to \infty$. Since (3.11) on Y decouples, we find that such a solution can be taken in the form $V(x) = {\rm e}^{{\rm i}k\tau}(v, v_x)(x)$ for some integer $k \neq 0$. In particular, $v(x)$ satisfies

$$\mathcal{L}^0 v = {\rm i}k\omega_0 v,$$

and is therefore an L^2-eigenfunction of $\mathcal{L}^0$ to the eigenvalue $\lambda = \mathrm{i}k\omega_0$. Inspecting our hypotheses on $\mathcal{L}^0$, (3.14) follows. To prove (3.15), we consider the adjoint equation

$$\Psi_x = -\begin{pmatrix} 0 & -\omega_0\partial_\tau + f_{uu}^T(q^0)[q_x^0,\cdot] \\ 1 & f_u^T(q^0) - c^0 \end{pmatrix}\Psi, \qquad \Psi \in Y, \tag{3.18}$$

of (3.11), taken with respect to the inner product in the space $X = L^2(S^1) \times L^2(S^1)$. We note that the functions $\Psi_j(x)$ from (3.17) satisfy (3.18). A calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle V(x), \Psi(x)\rangle_X = 0 \qquad \text{for all } x \in \mathbb{R}$$

for solutions $V(x)$ of (3.11) and $\Psi(x)$ of (3.18); see [14] for similar arguments. Using the relation between (3.18) and $[\mathcal{L}^0]^*$, we conclude that (3.15) is met. ◻

Note that the direction $Q_x^0(0) \in E_+^{\mathrm{cs}} \cap E_-^{\mathrm{u}}$ corresponds to the flow direction. To remove it, we shall later use the hyperplane

$$\mathcal{S} := [\mathbb{R}Q_x^0(0)]^\perp \subset Y. \tag{3.19}$$

We are now ready to discuss the nonlinear equation (3.3) near the orbit $Q^0(x)$ for ω close to ω_0 and ε close to zero. It is convenient to set

$$\omega = \omega_0 + \Omega$$

and to consider

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} v \\ (\omega_0 + \Omega)\partial_\tau u + f_u(u)v - c^\varepsilon v \end{pmatrix} \tag{3.20}$$

near the orbit $Q^0 = (q^0, q_x^0)$ for (ε, Ω) close to zero. We employ the smooth coordinate change

$$z = x\sqrt{\omega_0 + \Omega}, \qquad (\widetilde{u}, \widetilde{v}) = \left(u, v/\sqrt{\omega_0 + \Omega}\right)$$

which transforms (3.20) into the equation

$$\begin{pmatrix} \widetilde{u}_z \\ \widetilde{v}_z \end{pmatrix} = \begin{pmatrix} \widetilde{v} \\ \partial_\tau \widetilde{u} + (\omega_0 + \Omega)^{-1/2}[f_u(\widetilde{u}) - c^\varepsilon]\widetilde{v} \end{pmatrix}. \tag{3.21}$$

The advantage of (3.21) over (3.20) is that the Ω-dependent part of the right-hand side of (3.21) is a smooth mapping from Y into itself which depends smoothly on (ε, Ω) for (ε, Ω) near zero; in contrast, the dependence on Ω of the right-hand side of (3.20) is through the term $u \mapsto \Omega\partial_\tau u$ which is not even bounded from H^1 into $H^{1/2}$. Using the fact that the linearized equation (3.10) can be solved using exponential dichotomies (whose existence we established above), we can proceed as in [13, section 3.5] and [22] to prove the existence of unstable and center-stable manifolds for (3.21), and therefore for (3.20), near the viscous shock. More precisely, there exist constants $\delta > 0$ and $K > 0$ such that

$$\begin{aligned} \mathcal{W}_{\varepsilon,\Omega}^{\mathrm{u}} := \{&U_0 \in Y;\ \exists \text{ solution } U(x) \text{ of (3.20) on } \mathbb{R}^- :\ U(0) = U_0,\ |U_0 - Q^0(0)| < \delta, \\ &|U(x) - U_{\mathrm{eq}}^-(\varepsilon)| \le K\mathrm{e}^{-\nu_-^{\mathrm{u}}|x|} \text{ for } x \le 0\}, \\ \mathcal{W}_{\varepsilon,\Omega}^{\mathrm{cs}} := \{&U_0 \in Y;\ \exists \text{ solution } U(x) \text{ of (3.20) on } \mathbb{R}^+ :\ U(0) = U_0,\ |U_0 - Q^0(0)| < \delta, \\ &\exists U_{\mathrm{eq}}^+ \in Y_0 \text{ with } |U_{\mathrm{eq}}^+ - U_{\mathrm{eq}}^+(0)| < \delta \text{ so that } |U(x) - U_{\mathrm{eq}}^+| \le K\mathrm{e}^{-\nu_+^{\mathrm{s}}|x|} \\ &\text{for } x \ge 0\} \end{aligned}$$

are smooth manifolds that are invariant under the action of the group Γ defined in (3.4) and that depend smoothly on (ε, Ω) near zero (smoothness with respect to the parameters follows from [22] since the right-hand side of the rescaled equation (3.21) is smooth in the parameters). Moreover, $Q^\varepsilon(0) \in \mathcal{W}^{\rm u}_{\varepsilon,\Omega} \cap \mathcal{W}^{\rm cs}_{\varepsilon,\Omega}$, and the tangent spaces of the invariant manifolds at this point of intersection are given by

$$T_{Q^0(0)}\mathcal{W}^{\rm u}_{0,0} = E^{\rm u}_-, \qquad T_{Q^0(0)}\mathcal{W}^{\rm cs}_{0,0} = E^{\rm cs}_+.$$

Note that the center-stable manifold $\mathcal{W}^{\rm cs}_{\varepsilon,\Omega}$ is in effect given as the union of stable manifolds to the manifold $\{U^+_{\rm eq} = (u, 0);\ u \in \mathbb{R}^n\}$ of asymptotic states, and therefore unique.

Finding solutions of (2.2), with temporal frequency ω near ω_0, that converge asymptotically to constants as $x \to \pm\infty$ is therefore equivalent to finding elements U_0 in the intersection

$$\mathcal{W}^{\rm u}_{\varepsilon,\Omega} \cap \mathcal{W}^{\rm cs}_{\varepsilon,\Omega} \cap \left[Q^0(0) + \mathcal{S}\right] \tag{3.22}$$

for Ω close to zero, with $\mathcal{S}$ as in (3.19). Note that U_0 will have nontrivial time-τ dependence if and only if U_0 has a nonzero $Y_{\rm h}$-component. The minimal period will be $2\pi/\omega$ if the component of U_0 in Y_1 does not vanish. We use Lyapunov–Schmidt reduction to determine the intersection (3.22). To this end, we write

$$E^{\rm cs}_+ \cap \mathcal{S} = \widetilde{E}^{\rm cs}_+ \oplus \operatorname{span}\{V_1(0), V_2(0)\}, \qquad E^{\rm u}_- \cap \mathcal{S} = \widetilde{E}^{\rm u}_- \oplus \operatorname{span}\{V_1(0), V_2(0)\}.$$

There are then unique smooth maps

$$\begin{aligned} G^{\rm cs}(\cdot;\varepsilon,\Omega): &\quad \widetilde{E}^{\rm cs}_+ \oplus \operatorname{span}\{V_1(0), V_2(0)\} \longrightarrow \widetilde{E}^{\rm u}_- \oplus \operatorname{span}\{\Psi_1(0), \Psi_2(0)\}, \\ G^{\rm u}(\cdot;\varepsilon,\Omega): &\quad \widetilde{E}^{\rm u}_- \oplus \operatorname{span}\{V_1(0), V_2(0)\} \longrightarrow \widetilde{E}^{\rm cs}_+ \oplus \operatorname{span}\{\Psi_1(0), \Psi_2(0)\} \end{aligned}$$

with

$$Q^\varepsilon(0) + \operatorname{graph} G^j(\cdot;\varepsilon,\Omega) = \mathcal{W}^j_{\varepsilon,\Omega} \cap [Q^0(0) + \mathcal{S}], \qquad j = \text{cs}, \text{u},$$

and $D_U G^j(0;0,0) = 0$ for $j = \text{cs}, \text{u}$. In particular, both maps are equivariant under the S^1-action Γ. Thus, intersections of $\mathcal{W}^{\rm u}_{\varepsilon,\Omega}$ and $\mathcal{W}^{\rm cs}_{\varepsilon,\Omega}$ in $Q^0(0) + \mathcal{S}$ are in one-to-one correspondence with the zeroes of the mapping

$$\begin{aligned} G(\cdot;\varepsilon,\Omega): &\quad \mathbb{R} \times \widetilde{E}^{\rm u}_- \times \widetilde{E}^{\rm cs}_+ \longrightarrow \widetilde{E}^{\rm u}_- \oplus \widetilde{E}^{\rm cs}_+ \oplus \operatorname{span}\{\Psi_1(0), \Psi_2(0)\}, \\ &(a, w^{\rm u}, w^{\rm cs}) \longmapsto w^{\rm u} + G^{\rm u}(w^{\rm u} + aV_1(0);\varepsilon,\Omega) - [w^{\rm cs} + G^{\rm cs}(w^{\rm cs} + aV_1(0);\varepsilon,\Omega)], \end{aligned}$$

where we factored out the nontrivial S^1-action on $\operatorname{span}\{V_1(0), V_2(0)\}$. Lyapunov–Schmidt reduction shows that there is a unique map

$$W: \quad U_\delta(0) \subset \mathbb{R}^3 \longrightarrow \widetilde{E}^{\rm u}_- \times \widetilde{E}^{\rm cs}_+, \quad (a,\varepsilon,\Omega) \longmapsto (W^{\rm u}(a,\varepsilon,\Omega), W^{\rm cs}(a,\varepsilon,\Omega)),$$

so that $G(a, w^{\rm u}, w^{\rm cs};\varepsilon,\Omega) = 0$ if and only if

$$\langle \Psi_j(0), G(a, W(a,\varepsilon,\Omega);\varepsilon,\Omega)\rangle_X = 0 \qquad \text{for } j = 1, 2.$$

Furthermore, W is smooth in (a,ε,Ω) and we have $D_{(a,\varepsilon,\Omega)}W(0,0,0) = 0$. In fact, since $G(0,0,0;\varepsilon,\Omega) \equiv 0$ due to $Q^\varepsilon(0) \in \mathcal{W}^{\rm u}_{\varepsilon,\Omega} \cap \mathcal{W}^{\rm cs}_{\varepsilon,\Omega}$ for all (ε,Ω), we have in addition that $W(0,\varepsilon,\Omega) = 0$ for all small (ε,Ω), so that

$$W(a,\varepsilon,\Omega) = a\mathrm{O}(|a| + |\varepsilon| + |\Omega|). \tag{3.23}$$

It suffices therefore to solve the reduced equations

$$\langle \Psi_j(0), G(a, W(a,\varepsilon,\Omega); \varepsilon, \Omega)\rangle_X = 0 \qquad \text{for } j = 1, 2. \tag{3.24}$$

To derive an expression for (3.24), we write (3.20) as

$$U_x = F(U, \varepsilon, \Omega). \tag{3.25}$$

Using the coordinates $U = Q^\varepsilon + \widetilde{U}$, we find that $\widetilde{U}$ satisfies

$$\widetilde{U}_x = F_U(Q^0, 0, 0)\widetilde{U} + \mathcal{N}(\widetilde{U}, \varepsilon, \Omega, x), \tag{3.26}$$

where

$$\begin{aligned} \mathcal{N}(\widetilde{U}, \varepsilon, \Omega, x) &:= F(Q^\varepsilon + \widetilde{U}, \varepsilon, \Omega) - F(Q^\varepsilon, \varepsilon, \Omega) - F_U(Q^0, 0, 0)\widetilde{U} \\ &= \mathrm{O}(|\widetilde{U}|(|\widetilde{U}| + |\varepsilon| + |\Omega|)). \end{aligned} \tag{3.27}$$

Using the variation-of-constant formula that captures unstable and center-stable manifolds (see, e.g., [22] or [13, Proposition 3.13]) and the fact that $\Psi_j(x)$ satisfies (3.18) together with [14, Lemma 5.1], we find that (3.24) is given by

$$\int_{-\infty}^{\infty} \Big\langle \Psi_j(x), \mathcal{N}(\widetilde{U}^\pm(x), \varepsilon, \Omega, x)\Big\rangle_X \, \mathrm{d}x = 0, \qquad j = 1, 2, \tag{3.28}$$

where $\widetilde{U}^\pm(x)$ satisfies (3.26) on $\mathbb{R}^\pm$ with $\widetilde{U}^-(0) = aV_1(0) + W^{\mathrm{u}}(a, \varepsilon, \Omega)$ and $\widetilde{U}^+(0) = aV_1(0) + W^{\mathrm{cs}}(a, \varepsilon, \Omega)$. If we write (3.28) as $\Pi(a, \varepsilon, \Omega) = 0$, then we know from the preceding discussion that $\Pi(0, \varepsilon, \Omega) = 0$ for all (ε, Ω): this solution corresponds to the persisting Lax shocks in $\mathrm{Fix}\,\Gamma$. To obtain genuinely time-periodic solutions corresponding to $a \neq 0$, we write

$$\Pi(a, \varepsilon, \Omega) = a\widetilde{\Pi}(a, \varepsilon, \Omega) \tag{3.29}$$

and consider $\widetilde{\Pi}(a, \varepsilon, \Omega) = 0$, which can be solved by the implicit function theorem provided the 2×2 matrix $D_{(\varepsilon,\Omega)}\widetilde{\Pi}(0, 0, 0)$ is invertible. Equations (3.29) and (3.28) show that

$$\begin{aligned} D_{(\varepsilon,\Omega)}\widetilde{\Pi}(0, 0, 0) &= D_a D_{(\varepsilon,\Omega)}\Pi(0, 0, 0) \\ &= \left[D_a D_{(\varepsilon,\Omega)} \int_{-\infty}^{\infty} \langle \Psi_j(x), \mathcal{N}(\widetilde{U}^\pm(x), \varepsilon, \Omega, x)\rangle_X \, \mathrm{d}x \Big|_{(a,\varepsilon,\Omega)=0} \right]_{j=1,2} \end{aligned} \tag{3.30}$$

which we now compute. We know that

$$\widetilde{U}^+(x) = aV_1(x) + W^{\mathrm{cs}}(a, \varepsilon, \Omega)(x), \qquad \widetilde{U}^-(x) = aV_1(x) + W^{\mathrm{u}}(a, \varepsilon, \Omega)(x)$$

which we rewrite as

$$\widetilde{U}^\pm(x) = a[V_1(x) + \widetilde{W}^\pm(x; a, \varepsilon, \Omega)] \tag{3.31}$$

with

$$\begin{aligned} \widetilde{W}^-(x; a, \varepsilon, \Omega) &:= \frac{1}{a} W^{\mathrm{u}}(a, \varepsilon, \Omega)(x) = \mathrm{O}(|a| + |\varepsilon| + |\Omega|), \qquad x \in \mathbb{R}^-, \\ \widetilde{W}^+(x; a, \varepsilon, \Omega) &:= \frac{1}{a} W^{\mathrm{cs}}(a, \varepsilon, \Omega)(x) = \mathrm{O}(|a| + |\varepsilon| + |\Omega|), \qquad x \in \mathbb{R}^+, \end{aligned} \tag{3.32}$$

due to the estimate (3.23). Thus,

$$\frac{\mathrm{d}^2}{\mathrm{d}(\varepsilon,\Omega)\mathrm{d}a}\mathcal{N}(\widetilde{U}^{\pm}(x),\varepsilon,\Omega,x)|_{(a,\varepsilon,\Omega)=0} = D_{(\varepsilon,\Omega)}F_U(Q^{\varepsilon}(x),\varepsilon,\Omega)|_{(a,\varepsilon,\Omega)=0}V_1(x).$$

Upon comparing (3.25) with (3.20), we see that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\Omega\mathrm{d}a}\mathcal{N}|_{(a,\varepsilon,\Omega)=0} &= \begin{pmatrix} 0 & 0 \\ \partial_\tau & 0 \end{pmatrix} V_1,\\ \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon\mathrm{d}a}\mathcal{N}|_{(a,\varepsilon,\Omega)=0} &= \begin{pmatrix} 0 & 0 \\ \partial_\varepsilon(f_{uu}(q^\varepsilon)[q_x^\varepsilon,\cdot])|_{\varepsilon=0} & \partial_\varepsilon[f_u(q^\varepsilon)-c^\varepsilon]|_{\varepsilon=0} \end{pmatrix} V_1.\end{aligned}$$

Substituting the expressions (3.16) and (3.17) for V_1 and Ψ_j and using the normalization (3.1), we obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\Omega\mathrm{d}a}\int_{-\infty}^{\infty}\left\langle \Psi_j(x),\mathcal{N}(\widetilde{U}^{\pm}(x),\varepsilon,\Omega,x)\right\rangle_X \mathrm{d}x\Big|_{(\varepsilon,\Omega)=0}\right]_{j=1,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2. \tag{3.33}$$

An analogous computation for the derivative with respect to ε gives

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon\mathrm{d}a}\int_{-\infty}^{\infty}\left\langle \Psi_1(x),\mathcal{N}(\widetilde{U}^{\pm}(x),\varepsilon,\Omega,x)\right\rangle_X \mathrm{d}x\Big|_{(\varepsilon,\Omega)=0}\right]\\ &= \frac{1}{2}\sum_{j=1}^{2}\left\langle \psi_j,\partial_\varepsilon\partial_x\left[f_u(q^\varepsilon)v_j - c^\varepsilon v_j\right]_{\varepsilon=0}\right\rangle_{L^2}\\ &= -\frac{1}{2}\sum_{j=1}^{2}\langle \psi_j,\partial_\varepsilon\mathcal{L}^\varepsilon|_{\varepsilon=0}v_j\rangle_{L^2}\\ &= -\operatorname{Re}\lambda_\varepsilon(0),\end{aligned}$$

where we used (3.2) to obtain the last step. Hence, we find that the Jacobian in (3.30) is given by

$$D_{(\varepsilon,\Omega)}\widetilde{\Pi}(0,0,0) = \begin{pmatrix} -\operatorname{Re}\lambda_\varepsilon(0) & 0 \\ \star & 1 \end{pmatrix} \tag{3.34}$$

which is invertible due to our hypothesis on the transverse crossing of the Hopf eigenvalues.

Upon applying the implicit function theorem to solve $\widetilde{\Pi}(a,\varepsilon,\Omega)=0$, we conclude that there exist unique functions $(\varepsilon_*,\Omega_*)(a)\in\mathbb{R}^2$ and $Q^*(0;a)\in Y$, defined for $|a|<\delta$, so that

$$Q^*(0;a)\in\mathcal{W}^{\mathrm{u}}_{(\varepsilon_*,\Omega_*)(a)}\cap\mathcal{W}^{\mathrm{cs}}_{(\varepsilon_*,\Omega_*)(a)}\cap\left[Q^0(0)+\mathcal{S}\right].$$

These functions are smooth and satisfy $\partial_a(\varepsilon_*,\Omega_*)|_{a=0}=0$, $Q^*(0;0)=Q^0(0)$, and

$$\begin{aligned}Q^*(x;a) &= Q^{\varepsilon_*(a)}(x) + a[V_1(x)+\widetilde{W}^{\pm}(x;a,\epsilon_*(a),\Omega_*(a))]\\ &=: Q^{\varepsilon_*(a)}(x) + a\widetilde{Q}^*(x;a).\end{aligned}\tag{3.35}$$

By construction, we have $Q^*(x;a)\to U^-_{\mathrm{eq}}(\varepsilon_*(a))$ as $x\to-\infty$. Furthermore, we have $Q^*(x;a)\in\mathcal{W}^{\mathrm{cs}}_{(\varepsilon_*,\Omega_*)(a)}$ from which we infer that there exists a $U^+_*(a)\in\mathbb{R}^n$ with

$|U_*^+(a) - U_{\rm eq}^+(\varepsilon_*(a))| < \delta$, so that $Q^*(x;a) \to U_*^+(a)$ exponentially as $x \to \infty$ with rate $\nu_+^{\rm s}$. We claim that $U_*^+(a) = U_{\rm eq}^+(\varepsilon_*(a))$. To prove this claim, consider the smooth functional

$$\mathcal{E}: \quad Y \longrightarrow \mathbb{R}^n, \quad (u,v) \longmapsto \int_0^{2\pi} [v - f(u) + cu] \, \mathrm{d}\tau. \tag{3.36}$$

This functional is conserved under the evolution of (3.3). If $U(x) = (u,v)(x) \in Y$ is a solution of (3.3), then $v = u_x$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{E}(U(x)) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2\pi} [v - f(u) + cu]\,\mathrm{d}\tau \\ &= \int_0^{2\pi} [v_x - f_u(u)v + cv]\,\mathrm{d}\tau \\ &\overset{(3.3)}{=} \int_0^{2\pi} \omega u_\tau \,\mathrm{d}\tau = 0 \end{aligned} \tag{3.37}$$

since u is 2π-periodic in τ. Furthermore, for $U = (u_0, v_0) \in \operatorname{Fix}\Gamma \subset Y$, we have

$$D_U\mathcal{E}(u_0,v_0)\begin{pmatrix}\widetilde{u}\\ \widetilde{v}\end{pmatrix} = \int_0^{2\pi} [\widetilde{v} - f_u(u_0)\widetilde{u} + c\widetilde{u}]\,\mathrm{d}\tau = \widetilde{v}_0 - f_u(u_0)\widetilde{u}_0 + c\widetilde{u}_0 \in \mathbb{R}^n.$$

In particular, $D_U\mathcal{E}(u_\pm^0, 0)$ restricted to $\mathbb{R}^n \times \{0\} \subset Y_0$ is invertible, since we assumed that none of the characteristic speeds vanishes in the frame that moves with speed c^0. Since $Q^{\varepsilon_*(a)}(x)$ connects $U_{\rm eq}^-(\varepsilon_*(a))$ to $U_{\rm eq}^+(\varepsilon_*(a))$, they have the same $\mathcal{E}$-values, and the preceding argument shows that there is no other equilibrium $U_{\rm eq}^+$ near $U_{\rm eq}^+(\varepsilon_*(a))$ with the $\mathcal{E}$-value of $U_{\rm eq}^+(\varepsilon_*(a))$. Therefore, $Q^*(x;a) \to U_{\rm eq}^\pm(\varepsilon_*(a))$ for $x \to \pm\infty$.

This completes the existence proof of the bifurcating oscillatory viscous shock waves. The uniqueness statement in Theorem 2.1 is a consequence of our construction which captures all solutions that lie in the intersection of $\mathcal{W}_{\varepsilon,\Omega}^{\rm u}$ and $\mathcal{W}_{\varepsilon,\Omega}^{\rm cs}$. We remark that (3.37) also shows that any time-periodic localized travelling viscous shock wave satisfies the Rankine–Hugoniot condition (2.4).

To prepare the ground for the following spectral stability proof, we derive an expression for $\varepsilon_{aa}(0)$. First, we set $(\varepsilon,\Omega) = 0$ and compute the derivatives

$$\frac{\mathrm{d}^j}{\mathrm{d}a^j}\Pi_i(a,0,0) = \frac{\mathrm{d}^j}{\mathrm{d}a^j}\int_{-\infty}^{\infty} \Big\langle \Psi_i(x), \mathcal{N}(\widetilde{U}^\pm(x),0,0,x)\Big\rangle_X \,\mathrm{d}x$$

at $a = 0$ for $i = 1,2$. Using the expressions (3.27) for $\mathcal{N}$ and (3.31) for $\widetilde{U}^\pm$ together with the estimate (3.32) for $\widetilde{W}(x;a,0,0)$, we easily find that the first and second derivatives vanish at $a = 0$ for $i = 1,2$, while

$$\begin{aligned} \kappa_3 := {} & \frac{\mathrm{d}^3}{\mathrm{d}a^3}\Pi_1(a,0,0) \\ = {} & \int_{-\infty}^{\infty} \Big\langle \Psi_1(x), F_{UUU}(Q^0(x),0,0)[V_1(x)]^3 + \\ & \qquad + 3F_{UU}(Q^0(x),0,0)[V_1(x), \widetilde{W}_a(x;0,0,0)]\Big\rangle_X \,\mathrm{d}x. \end{aligned} \tag{3.38}$$

A straightforward calculation using (3.34) then shows that

$$\varepsilon_{aa}(0) = \frac{\kappa_3}{3\operatorname{Re}\lambda_\varepsilon(0)}. \tag{3.39}$$

4. Stability of the bifurcating modulated viscous shocks. This section is devoted to the proof of Theorem 2.2. Our goal is to determine the Floquet spectrum

$$\Sigma = \{\lambda \in \mathbb{C};\ \mathrm{e}^{2\pi\lambda} \in \text{ spectrum of } \Phi_{2\pi}\}$$

associated with the evolution Φ_t of the linearization

$$\omega v_\tau = \partial_x[\partial_x + c - f_u(q^*(x,\tau;a))]v$$

of (2.2) about q^* on $\mathcal{C}^0_{\mathrm{unif}}$. Note that (ε,ω) and q^* depend smoothly on the parameter a introduced in section 3, and so do the wave speed $c = c^\varepsilon$ and the asymptotic rest states $u^\varepsilon_\pm$ through $\varepsilon = \varepsilon_*(a)$; we will suppress this dependence for most of the proof.

The Floquet spectrum Σ is the disjoint union of the essential spectrum Σ_{ess} and the point spectrum Σ_{pt}, which consists, by definition, of all isolated eigenvalues with finite multiplicity. Since the modulated shock $q^*(x,\tau;a)$ converges exponentially to the constants $u^\varepsilon_\pm$ as $x \to \pm\infty$, uniformly in τ, the set Σ_{ess} is bounded to the right by the essential spectra

$$\Sigma^\pm_{\mathrm{ess}} = \{\lambda \in \mathbb{C};\ \det(k^2 + \mathrm{i}k[f_u(u^\varepsilon_\pm) - c^\varepsilon] + \lambda) = 0 \text{ for some } k \in \mathbb{R}\}$$

of $u^\varepsilon_\pm$ (see, for instance, [14, Proposition 2.10]), which touch the imaginary axis at $\lambda = 0$ and lie otherwise in the open left half-plane due to our hypothesis that the eigenvalues of $f_u(u^\varepsilon_\pm)$ are real and simple.

It therefore suffices to locate point spectrum, that is, isolated Floquet exponents λ which are captured, via the ansatz $v(x,\tau) = \mathrm{e}^{\lambda\tau}u(x,\tau)$ with $u(x,\tau+2\pi) = u(x,\tau)$ for all τ, by the equation

$$\omega_*(a)u_\tau + \lambda u = \partial_x[\partial_x + c^{\varepsilon_*(a)} - f_u(q^*(x,\tau;a))]u$$

which we rewrite as

$$\begin{aligned} V_x &= \begin{pmatrix} 0 & 1 \\ \omega\partial_\tau + \lambda + f_{uu}(q^*)[q^*_x,\cdot] & f_u(q^*) - c \end{pmatrix} V \\ &= [F_U(Q^*(x;a),\epsilon_*(a),\Omega_*(a)) + \lambda\mathcal{B}]V, \qquad V \in Y, \end{aligned} \tag{4.1}$$

with $Q^*(x;a) = (q^*, q^*_x)(x,\cdot;a)$ from (3.35).

Since we assumed spectral stability for $\epsilon = 0$ except for the Hopf eigenvalues and the translational eigenvalue at the origin (which all contribute to the Floquet exponent $\lambda = 0$), it suffices to find all isolated Floquet exponents of (4.1) in a fixed small neighborhood of the origin. We choose an open set $\Omega \subset \mathbb{C}$ as indicated in Figure 2. Standard theory implies that $\lambda \in \Omega$ is a Floquet exponent if and only if (4.1) has a nontrivial exponentially decaying solution on $\mathbb{R}$. Taking the limit $x \to \pm\infty$ in (4.1), we obtain the asymptotic operators

$$\begin{pmatrix} 0 & 1 \\ \omega\partial_\tau + \lambda & f_u(u_\pm) - c \end{pmatrix}. \tag{4.2}$$

We denote the eigenvectors and eigenvalues of $[f_u(u_\pm) - c]$ by $r^\pm_j$ and $\nu^\pm_j$, respectively. As discussed in section 3, the operators in (4.2) are hyperbolic for $\lambda = 0$ except for the n-fold eigenvalue $\mathcal{V} = 0$ with eigenvectors $(r^\pm_j, 0) \in Y_0$. This eigenvalue and the associated eigenvectors become

$$\mathcal{V}^\pm_j = -\frac{\lambda}{\nu^\pm_j} + \mathrm{O}(\lambda^2), \qquad R^\pm_j = \begin{pmatrix} r^\pm_j \\ \mathcal{V}^\pm_j r^\pm_j \end{pmatrix}, \qquad j = 1,\dots,n, \tag{4.3}$$

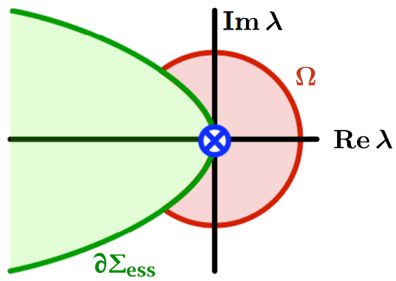

FIG. 2. *The definition of the open set $\Omega \subset \mathbb{C}$ in the complex Floquet plane is shown. The embedded Floquet exponent at the origin has multiplicity at least equal to two with eigenfunctions q_x^* and q_τ^*.*

for λ near zero. For $\lambda \in \Omega$, the unstable eigenspace $E_-^\infty(\lambda, a)$ at $x = -\infty$ and the stable eigenspace $E_+^\infty(\lambda, a)$ at $x = \infty$ are therefore given by

$$E_+^\infty := E_+^{\mathrm{s}} \oplus \mathcal{R}_+, \qquad E_-^\infty := E_-^{\mathrm{u}} \oplus \mathcal{R}_-, \qquad \mathcal{R}_\pm = \operatorname{span}\{R_j^\pm;\ \nu_j^\pm \gtrless 0\},$$

and these spaces depend smoothly on a and are analytic in λ for λ near zero. Note that $\dim \mathcal{R}^+ = p$ and $\dim \mathcal{R}^- = n - p - 1$ with p as in section 2.

LEMMA 4.1. *There are unique closed subspaces $E_\pm(\lambda, a)$ of Y, defined and analytic in λ near zero and smooth in $a \geq 0$, such that $V(x)$ is a bounded solution of (4.1) on $\mathbb{R}^\pm$ for some $\lambda \in \Omega$ if and only if $V(0) \in E_\pm(\lambda, a)$.*

Proof. We begin by considering (4.1) with Q^* replaced by Q^ε. In this case, (4.1) decouples on each Fourier space Y_k, and the claimed statement holds for Q^ε due to the Gap lemma [5, 9] applied in Y_0 and exponential dichotomy theory together with estimates as in [13, Lemma 3.3] in the other Fourier spaces. Since the difference of Q^ε and Q^* is small for all x and decays to zero exponentially as $|x| \to \infty$, these results carry over to (4.1) using, for instance, the integral formulation in [12, (4.12) in section 4.3]; see also [18, section 7.6] for a slightly different proof. ∎

Lemma 4.1 shows that Floquet exponents in Ω can be found by seeking nontrivial intersections of $E_-(\lambda, a)$ and $E_+(\lambda, a)$. To determine their intersections, we first set $(\lambda, a) = 0$ to see what these spaces look like at onset and then use perturbation theory to analyze the case when $(\lambda, a) \neq 0$.

Hence, let $\lambda = 0$, then (4.1) is simply the variational equation

$$V_x = F_U(Q^*(x; a), \epsilon_*(a), \Omega_*(a))V, \qquad V \in Y, \tag{4.4}$$

of the modulated wave Q^*. When $a = 0$, we have $Q^* = Q^0$, and (4.4) describes Floquet exponents at $\lambda = 0$ of the unperturbed viscous shock q^0. In particular, (4.4) decouples on each Fourier space Y_k, and our hypotheses on the Evans function and the spectral properties of the viscous shock imply that

$$\begin{aligned} E_+(0,0) \cap E_-(0,0) &= \operatorname{span}\{Q_x^0(0), V_1(0), V_2(0)\}, \\ [E_+(0,0) + E_-(0,0)]^\perp &= \operatorname{span}\{\Psi_0, \Psi_1(0), \Psi_2(0)\} \end{aligned} \tag{4.5}$$

for an appropriate nonzero vector $\Psi_0 \in Y_0$. Next, consider (4.4) for an arbitrary a near zero. First, note that the gradient of the jth component $\mathcal{E}_j$ of the conserved quantity $\mathcal{E}$ from (3.36), computed in the $X = L^2(S^1) \times L^2(S^1)$ scalar product, is given by

$$\nabla \mathcal{E}_j(u, v) = \nabla \langle \mathcal{E}(u, v), e_j \rangle_{\mathbb{R}^n} = \begin{pmatrix} -[f_u^T(u) - c]e_j \\ e_j \end{pmatrix}, \qquad j = 1, \ldots, n, \tag{4.6}$$

where e_j denotes the jth canonical basis vector in $\mathbb{R}^n$. Our analysis of $\mathcal{E}$ in section 3 implies that these n gradients are linearly independent for $a = 0$, and we therefore have $\dim E^*(a) = n$ for all small a where

$$E^*(a) := \text{span}\,\{\nabla\langle\mathcal{E}(Q^*(0;a)), e_j\rangle_{\mathbb{R}^n};\ j = 1,\ldots,n\}\,.$$

The gradients in (4.6) also satisfy the adjoint equation of (4.4), again computed in X, which shows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\langle\nabla\langle\mathcal{E}(Q^*(x;a)), e_j\rangle_{\mathbb{R}^n}, V(x)\rangle_X \equiv 0, \qquad j = 1,\ldots,n, \tag{4.7}$$

for each solution $V(x)$ of (4.4); this can also be verified directly by evaluating (4.7). We denote by $\ell_j^\pm(a)$ the smooth eigenvectors of $[f_u^T(u_\pm^\varepsilon) - c^\varepsilon]$ at $\varepsilon_*(a)$ associated with the eigenvalues $\nu_j^\pm$ and define

$$E_\pm^*(a) = \text{span}\left\{\nabla\langle\mathcal{E}(Q^*(0;a)), \ell_j^\pm(a)\rangle_{\mathbb{R}^n};\ \nu_j^\pm \lessgtr 0\right\} \subset E^*(a). \tag{4.8}$$

Set $a = 0$, then $\dim E_+^*(0) = n - p$ and $\dim E_-^*(0) = p + 1$. Equations (4.3), (4.6), and (4.7) imply that $E_\pm^*(0) \perp_X E_\pm(0,0)$ and, in fact, that $E_\pm^*(0)$ are perpendicular to each solution of (4.4) at $a = 0$ that decays exponentially at $x = -\infty$ or $x = \infty$. Equation (4.5) implies then that $E_+^*(0) \cap E_-^*(0) = \text{span}\{\Psi_0\}$, and therefore $\dim[E_+^*(0) + E_-^*(0)] = n$. Since $E_\pm^*(a) \subset E^*(a)$ for all a, and the latter space is n-dimensional for all a, we conclude that $E_+^*(a) + E_-^*(a) = E^*(a)$, and the dimensions of the sum and intersection of $E_\pm^*(a)$ cannot change for a close to zero. Hence, we can choose a nonzero basis vector $\Psi_0^*(0;a)$ in the one-dimensional intersection $E_+^*(a) \cap E_-^*(a)$ that depends smoothly on a as well as linearly independent smooth elements $\Psi_j^\pm(0;a) \in E_\pm^*(a)$, with $j = 1,\ldots,n-p-1$ for the $+$ sign and $j = 1,\ldots,p$ for the $-$ sign, so that $\Psi_j^\pm(0;a) \perp_X \Psi_0^*(0;a)$ for all j. Using (4.3), (4.6), and (4.8), we see that

$$\Psi_j^\pm(0;a) \perp_X E_\pm(0,a), \qquad \Psi_0^*(0;a) \perp_X [E_+(0,a) + E_-(0,a)] \tag{4.9}$$

for all a. Lastly, we define

$$\Psi_{1,2}^*(0;a) := [1 - P(a)]\Psi_{1,2}(0;a), \tag{4.10}$$

where $P(a)$ is the orthogonal projection in X onto $E_+^*(a) + E_-^*(a)$.

Having prepared the ground for the forthcoming analysis, we now return to the full eigenvalue problem (4.1)

$$V_x = [F_U(Q^*(x;a), \epsilon_*(a), \Omega_*(a)) + \lambda\mathcal{B}]V.$$

We seek solutions $V^\pm(x)$ on $\mathbb{R}^\pm$ of the form

$$V^\pm(x) = b_0 Q_x^*(x;a) + b_1 V_1(x) + b_2 \widetilde{Q}_\tau^*(x;a) + \widetilde{V}^\pm(x;\lambda,a)b \tag{4.11}$$

with $b = (b_0, b_1, b_2)$, $Q^* = Q^\varepsilon + a\widetilde{Q}^*$ as in (3.35), and

$$\widetilde{V}^\pm(0;\lambda,a)b \perp \text{span}\{Q_x^*(0;a), V_1(0), \widetilde{Q}_\tau^*(0;a)\} \tag{4.12}$$

for all b, so that

$$V^+(0) - V^-(0) \in \text{span}\{\Psi_j^*(0;a);\ j = 0,1,2\}, \tag{4.13}$$

$$\text{dist}\left(\frac{1}{|V^\pm(x)|_X}V^\pm(x), E_\pm^\infty(\lambda,a)\right) \to 0 \text{ as } x \to \pm\infty. \tag{4.14}$$

Using exponential dichotomies as in section 3, we can then easily show that the system (4.1), (4.11)–(4.14) has unique solutions for each $b \in \mathbb{R}^3$ and (λ, a) near zero and that these solutions depend analytically on λ and smoothly on a. In particular, $E_+(\lambda, a) \cap E_-(\lambda, a) \neq \{0\}$ if and only if $\det \mathcal{D}(\lambda, a) = 0$ where

$$\mathcal{D}(\lambda, a): \quad \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad b \longmapsto \mathcal{D}(\lambda, a)b = \big(\langle \Psi_j^*(0; a), V^+(0) - V^-(0)\rangle_X\big)_{j=0,1,2}.$$

We will now compute the Taylor expansion of $\mathcal{D}$ to solve $\det \mathcal{D}(\lambda, a) = 0$.

First, we set $a = 0$ so that $Q^* = Q^0$, $Q_x^0 =: V_0$, and $\widetilde{Q}_\tau^* = V_2$. A calculation similar to the derivation of (3.28) gives

$$\begin{aligned} \partial_\lambda \mathcal{D}(0,0) &= \left(\int_{\mathbb{R}} \langle \Psi_i^*(x; 0), \mathcal{B}V_j(x)\rangle_X \,\mathrm{d}x\right)_{i,j=0,1,2} \\ &= \operatorname{diag}\left(\int_{\mathbb{R}} \langle \Psi_j(x), \mathcal{B}V_j(x)\rangle_X \,\mathrm{d}x\right) \\ &= \operatorname{diag}(M_0, 1, 1), \end{aligned}$$

where we used the normalization (3.1). Our hypothesis that $\lambda = 0$ is a simple zero of the Evans function of the viscous shock at $\varepsilon = 0$ implies that $M_0 \neq 0$.

Next, we set $\lambda = 0$ and compute derivatives with respect to a. Since $\lambda = 0$, the eigenvalue problem reduces to the variational equation (4.4). In particular, both $\partial_x Q^*(x; a)$ and $\partial_\tau \widetilde{Q}^*(x; a)$ are solutions of (4.4) that satisfy (4.14), and we can set $b_0 = b_2 = 0$ as they make no contribution to $\mathcal{D}(0, a)$. We focus therefore on $V^\pm(x) = V_1(x) + \widetilde{V}^\pm(x; 0, a)$ for which (4.9) and (4.14) together imply

$$\langle \Psi_0^*(0; a), V^+(0) - V^-(0)\rangle_X = 0 \tag{4.15}$$

for all a. The equation for $\widetilde{V}$ is

$$\begin{aligned} \widetilde{V}_x &= F_U(Q^0(x), 0, 0)\widetilde{V} \\ &\quad + [F_U(Q^*(x; a), \varepsilon_*(a), \Omega_*(a)) - F_U(Q^0(x), 0, 0)](V_1(x) + \widetilde{V}) \end{aligned} \tag{4.16}$$

and, proceeding as before and using (3.35), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}a} \left\langle \Psi_j^*(0; a), \widetilde{V}^+(0; 0, a) - \widetilde{V}^-(0; 0, a)\right\rangle_X \Big|_{a=0} \\ &= \int_{\mathbb{R}} \left\langle \Psi_j(x), F_{UU}(Q^0(x), 0, 0)[V_1(x), V_1(x)]\right\rangle_X \,\mathrm{d}x \end{aligned}$$

for $j = 1, 2$. Inspecting (3.16) and (3.17), we see that the integrands vanish pointwise for each x. Summarizing the findings obtained so far, we have

$$\mathcal{D}(\lambda, a) = \begin{pmatrix} M_0\lambda & 0 & 0 \\ 0 & \lambda + \mathrm{O}(a^2) & 0 \\ 0 & \mathrm{O}(a^2) & \lambda \end{pmatrix} + \mathrm{O}(|\lambda|(|\lambda + |a|)).$$

Thus, it remains to compute the diagonal $\mathrm{O}(a^2)$ term. Expanding (4.16), we see that the second derivative with respect to a of this diagonal term is given by

$$\begin{aligned}\partial_a^2\mathcal{D}_{22}(0,0) &= \int_{-\infty}^{\infty}\langle\Psi_1(x), F_{UUU}(Q^0(x),0,0)[V_1(x)]^3\\ &\quad + 3F_{UU}(Q^0(x),0,0)[V_1(x),\partial_a\widetilde{V}(x;0,0)]\\ &\quad + \varepsilon_{aa}(0)D_\varepsilon(F_U(Q^\varepsilon(x),\varepsilon,0))|_{\varepsilon=0}V_1(x)\\ &\quad + \Omega_{aa}(0)\partial_\Omega F_U(Q^0(x),0,0)V_1\rangle_X\,\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\langle\Psi_1(x), F_{UUU}(Q^0(x),0,0)[V_1(x)]^3\\ &\quad + 3F_{UU}(Q^0(x),0,0)[V_1(x),\partial_a\widetilde{V}(x;0,0)]\rangle_X\,\mathrm{d}x - \varepsilon_{aa}(0)\operatorname{Re}\lambda_\varepsilon(0),\end{aligned}$$

where the term involving Ω vanishes for the same reason that shows that the first component in (3.33) is zero. Comparing the integral term in the above expression with (3.38), we see that they coincide provided $\partial_a\widetilde{V}(x;0,0) = \partial_a\widetilde{W}(x;0,0,0)$. The following lemma, whose proof we postpone until after we finished the discussion of $\mathcal{D}(\lambda,a)$, states that this identity indeed holds.

LEMMA 4.2. *We have* $\partial_a\widetilde{V}(x;0,0) = \partial_a\widetilde{W}(x;0,0,0)$.

Thus, we can conclude that

$$\partial_a^2\mathcal{D}_{22}(0,0) = \kappa_3 - \varepsilon_{aa}(0)\operatorname{Re}\lambda_\varepsilon(0) \overset{(3.39)}{=} 2\varepsilon_{aa}(0)\operatorname{Re}\lambda_\varepsilon(0)$$

and consequently

$$\mathcal{D}(\lambda,a) = \begin{pmatrix} M_0\lambda & 0 & 0\\ 0 & \lambda+\varepsilon_{aa}(0)\operatorname{Re}\lambda_\varepsilon(0)a^2+\mathrm{O}(a^3) & 0\\ 0 & \mathrm{O}(a^2) & \lambda\end{pmatrix} + \mathrm{O}(|\lambda|(|\lambda+|a|)).$$

The equation $\det\mathcal{D}(\lambda,a) = 0$ has therefore precisely three solutions, counted with multiplicity, near zero which are given by $\lambda = 0$ with multiplicity two and a simple zero at

$$\lambda_*(a) = -\varepsilon_{aa}(0)\operatorname{Re}\lambda_\varepsilon(0)a^2 + \mathrm{O}(a^3),$$

so that $\lambda_*(a)$ and $\varepsilon_*(a)$ have opposite signs since we assumed that $\operatorname{Re}\lambda_\varepsilon(0) > 0$. Subject to establishing Lemma 4.2, this completes the proof of Theorem 2.2.

Proof of Lemma 4.2. Expanding the relevant equations for $\widetilde{V}$ and $\widetilde{W}$, we find that both $\partial_a\widetilde{V}^\pm(x;0,0)$ and $\partial_a\widetilde{W}^\pm(x;0,0,0)$ satisfy the linear inhomogeneous differential equation

$$V_x = F_U(Q^0(x),0,0)V + F_{UU}(Q^0(x),0,0)[V_1(x),V_1(x)].$$

The asymptotic boundary conditions in the hyperbolic directions coincide for both functions, but differ for the center directions. We shall show that the center components of $\partial_a\widetilde{V}^\pm(0;0,0)$ and $\partial_a\widetilde{W}^\pm(0;0,0,0)$ are equal to each other from which we can infer that the two solutions coincide as claimed.

We begin by discussing $\widetilde{V}^\pm(x;0,0)$. Equation (4.15) implies that the $\Psi_0^*(0;a)$ components of $\widetilde{V}^\pm(0;0,a)$ coincide for all a. Since $\Psi_j^\pm(0;a)$ is perpendicular to the space on the right-hand side of (4.12), we also have

$$\langle\Psi_j^\pm(0;a),\widetilde{V}^+(0;0,a) - \widetilde{V}^-(0;0,a)\rangle_X = 0$$

for all j and all a, and we conclude that the center components of $\widetilde{V}^{\pm}(0;0,a)$ coincide for all a. Since $V^{\pm}(0) \in E_{\pm}(0,a)$ for all a, (4.9) gives

$$\langle \Psi_j^{\pm}(0;a), V^{\pm}(0)\rangle_X = 0, \qquad \langle \Psi_0^*(0;a), V^{\pm}(0)\rangle_X = 0$$

for all a, and a Taylor expansion gives

$$\begin{aligned} &\langle \partial_a \Psi_j^{\pm}(0;0), V_1(0)\rangle_X + \langle \Psi_j^{\pm}(0;0), \partial_a \widetilde{V}^{\pm}(0;0,0)\rangle_X = 0 \qquad \text{for all } j,\\ &\langle \partial_a \Psi_0^*(0;0), V_1(0)\rangle_X + \langle \Psi_0^*(0;0), \partial_a \widetilde{V}^{\pm}(0;0,0)\rangle_X = 0. \end{aligned} \tag{4.17}$$

We now turn to $\partial_a \widetilde{W}^{\pm}(0;0,0,0)$. We set $(\varepsilon,\Omega)=0$ and consider the solution pieces

$$U^{\pm}(x) = Q^0(x) + a[V_1(x) + \widetilde{W}^{\pm}(x;a,0,0)]$$

from section 3. By construction, we have $U^{\pm}(x) \in \mathcal{W}^{\mathrm{u}}_{0,0}$, and the conserved quantity $\mathcal{E}(U^-(x))$ does, therefore, not depend on a. Its derivative with respect to a is given by

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}a}\mathcal{E}(Q^0(0) + a[V_1(0) + \widetilde{W}^-(0;a,0,0)])\\ &= \langle \nabla\mathcal{E}(Q^0(0) + a[V_1(0) + \widetilde{W}^-(0;a,0,0)]), V_1(0) + \widetilde{W}(0;a,0,0)\rangle_X\\ &= \langle \nabla\mathcal{E}(Q^0(0) + a[V_1(0) + \mathrm{O}(a)]), V_1(0) + a\widetilde{W}_a^-(0;0,0,0) + \mathrm{O}(a^2)\rangle_X\\ &= a\left[\left\langle \frac{\mathrm{d}}{\mathrm{d}a}\nabla\mathcal{E}(Q^0(0) + aV_1(0))|_{a=0}, V_1(0)\right\rangle_X + \langle \nabla\mathcal{E}(Q^0(0)), \widetilde{W}_a^-(0;0,0,0)\rangle_X\right]\\ &\quad + \mathrm{O}(a^2). \end{aligned}$$

Thus, we get

$$\begin{aligned} &\langle \partial_a \Psi_j^{\pm}(0;0), V_1(0)\rangle_X + \langle \Psi_j^{\pm}(0;0), \widetilde{W}_a^-(0;0,0,0)\rangle_X = 0,\\ &\langle \partial_a \Psi_0^*(0;0), V_1(0)\rangle_X + \langle \Psi_0^*(0;0), \widetilde{W}_a^-(0;0,0,0)\rangle_X = 0. \end{aligned} \tag{4.18}$$

We can proceed as before to show continuity of $\widetilde{W}_a^{\pm}(0;0,0,0)$ in the center components. This fact, together with the continuity of $\widetilde{V}_a(0;0,0)$ in the center directions and (4.17) and (4.18), shows that the center components of $\partial_a \widetilde{V}^{\pm}(0;0,0)$ and $\partial_a \widetilde{W}^{\pm}(0;0,0,0)$ are indeed equal to each other as claimed. ∎

5. Discussion.

There are numerous possible generalizations and extensions of our results. The crucial ingredient is the existence of the evolution operators $\Phi_{\pm}^{\mathrm{s,u}}$ for the spatial dynamical system, which requires some hyperbolicity in the spatial dynamics. The results are clearly not dependent on the particular form of the viscosity matrix: *nonlinear viscosity* $B(u)u_{xx}$ is allowed as long as the essential spectrum is nonresonant with the Hopf eigenvalue (uniform positivity is typically sufficient). We can also allow *parameter-dependent fluxes*: the parameter ε may appear explicitly in the viscosity matrix and the flux $f = f(u;\varepsilon)$.

Under- and overcompressive shocks can be treated similarly. All viscous shocks can be viewed as heteroclinic orbits in the travelling-wave ODE (3.5)

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} v \\ f_u(u)v - cv \end{pmatrix} \tag{5.1}$$

which connect families of equilibria at $x = \pm\infty$. To set up the problem, we can, for instance, prescribe the values of u on ingoing characteristics. Choose manifolds $\mathcal{S}_\pm$ of $\mathbb{R}^n$ so that $T_{u^0_\pm}\mathcal{S}_\pm \oplus \mathcal{I}_\pm = \mathbb{R}^n$, where $\mathcal{I}_\pm$ is the eigenspace belonging to eigenvalues $\nu^\pm \lessgtr 0$ of $f_u(u^0_\pm) - c^0$. We then seek viscous shock waves in the intersection of $\mathcal{W}^{\mathrm{u}}(\mathcal{S}_-)$ and $\mathcal{W}^{\mathrm{s}}(\mathcal{S}_+)$, where we regard $\mathcal{S}_\pm$ as subsets of the manifold $\mathbb{R}^n \times \{0\} \subset Y_0$ of equilibria of (5.1). Both manifolds are n-dimensional, and we will assume that their intersection along the viscous shock is transverse in the parameter c; this is equivalent to the assumption that $\lambda = 0$ is a simple root of the Evans function associated with the PDE linearization at the shock [5, 8, 23]. One can now vary $\mathcal{S}_\pm = \mathcal{S}^\varepsilon_\pm$ and continue the transverse intersection provided the speed $c = c^\varepsilon$ is adjusted appropriately. If a pair of complex eigenvalues crosses the imaginary axis at $\varepsilon = 0$, the analysis in this paper can be adapted easily to show that there is a unique family of oscillatory under- or overcompressive shocks bifurcating from the primary viscous shock. As for Lax shocks, the bifurcating oscillatory shocks converge exponentially to time-independent rest states as $|x| \to \infty$ due to the presence of the n conservation laws (3.36). We remark that undercompressive shocks occur as weak denotations in combustion, which makes them interesting from an applied viewpoint.

The analysis extends also to the case of *degenerate shock waves*, where we allow for an additional center direction within the travelling-wave ODE in $\mathrm{Fix}\,\Gamma$ at either u^0_- or u^0_+. Again, suitable transversality conditions on the intersections of $\mathcal{W}^{\mathrm{u}}_-$ and $\mathcal{W}^{\mathrm{s}}_+$ together with appropriate assumptions on the nonlinear behavior of the zero characteristic speed near the shock are needed.

Problems posed in *infinite cylinders*,

$$u_t = \Delta u + \sum_j \partial_{x_j} f_j(u), \qquad x \in \mathbb{R} \times \Omega,$$

for bounded cross sections $\Omega \subset \mathbb{R}^N$ and with Neumann boundary conditions on $\mathbb{R}\times\partial\Omega$, say, can also be treated. The existence of exponential dichotomies for this problem follows from [11, 14].

The major open problem that we did not address in this paper is nonlinear stability of the bifurcating oscillatory viscous shocks. It should be possible to establish nonlinear stability using a combination of the approach via pointwise estimates developed by Howard and Zumbrun in [8, 23] and our spatial-dynamics technique which can be used to obtain the necessary estimates for the Green's function; this will be pursued elsewhere.

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