Hamiltonian Systems Near Relative Periodic Orbits

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Abstract. We give explicit differential equations for a symmetric Hamiltonian vector field near a relative periodic orbit. These decompose the dynamics into periodically forced motion in a Poincaré section transversal to the relative periodic orbit, which in turn forces motion along the group orbit. The structure of the differential equations inherited from the symplectic structure and symmetry properties of the Hamiltonian system is described, and the effects of time reversing symmetries are included. Our analysis yields new results on the stability and persistence of Hamiltonian relative periodic orbits and provides the foundations for a bifurcation theory. The results are applied to a finite dimensional model for the dynamics of a deformable body in an ideal irrotational fluid.

Key words. relative periodic orbits, equivariant Hamiltonian systems, noncompact groups

AMS subject classifications. 37J15, 37J20, 53D20, 70H33

PII. S1111111101387760

1. Introduction. Relative periodic orbits are periodic solutions of a flow induced by an equivariant vector field on a space of group orbits. In applications they typically appear as oscillations of a system which are periodic when viewed in some rotating or translating frame. They therefore generalize relative equilibria, for which the “shape” of the system remains constant in an appropriate frame. Relative periodic orbits are ubiquitous in Hamiltonian systems with symmetry. For example, generalizations of the Weinstein–Moser theorem show that they are typically present near stable relative equilibria [25, 39, 43] and can therefore be found in virtually any physical application with a continuous symmetry group. Specific examples for which relative periodic orbits have been discussed or could be found by applying the Weinstein–Moser theorem to stable relative equilibria include rigid bodies [1, 31, 28, 24], deformable bodies [8, 27, 13], gravitational N-body problems [32, 47], molecules [17, 19, 20, 34, 48], and point vortices [26, 46, 38].

Existing theoretical work on Hamiltonian relative periodic orbits includes results on their stability [41, 42] and on their persistence to nearby energy-momentum levels in the case of compact symmetry groups [33]. However, stability, persistence, and bifurcations are still a long way from being well understood, especially in the presence of actions of noncompact symmetry groups with nontrivial isotropy subgroups. Our main aim with this paper is to

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*Received by the editors April 12, 2001; accepted for publication (in revised form) by J. Marsden October 3, 2001; published electronically April 8, 2002. This work was partially supported by UK EPSRC Research grant GR/L60029 and by European Commission funding for the Research Training Network “Mechanics and Symmetry in Europe (MASIE).”

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provide a local description of Hamiltonian vector fields near relative periodic orbits that can be used to develop stability and bifurcation theories.

In [55], the “bundle” structure of a general vector field near a relative periodic solution is analyzed for proper actions of arbitrary Lie groups. The dynamics near a relative periodic orbit is decomposed as periodically forced motion in a Poincaré section, which in turn forces motion along the group orbit. In this way, the study of bifurcations from relative periodic solutions is reduced to the study of bifurcations from discrete rotating waves in systems with compact symmetry. These are treated in [23]. The aim of this paper is to extend the results of [55] from general systems to Hamiltonian systems by taking into account the symplectic structure and conserved quantities of the problem.

In addition to [55], we draw on several other sources for inspiration. In the absence of symmetry, it is well known that the dynamics near a Hamiltonian periodic orbit can be described as periodically forced motion in a Poincaré section within an energy level set [2]. Finite symmetries are treated in [7]. We combine these ideas with the local description of Hamiltonian vector fields near a relative equilibrium given in [50], and, since these are present in most Hamiltonian systems, we include the effects of time reversing symmetries by extending the paper [22] to Hamiltonian vector fields.

In section 3.1, we show that a Poincaré section transverse to a relative periodic orbit of a Hamiltonian system decomposes into a part tangent to the energy level set describing rigid body motion, another part tangent to the energy level set describing vibrational motion, and a part parametrizing energy. Then, in section 3.2, we present our central result, the differential equations in these bundle coordinates. In section 4, we use them to deduce a number of new results on stability and persistence. These include Proposition 4.3, describing the block structure of the linearization of a Hamiltonian vector field at a relative periodic orbit, the stability result Corollary 4.5, Corollary 4.8 on the persistence of relative periodic orbits with generic momenta to nearby energy-momentum level sets, and Theorem 4.9 on persistence in the case of nongeneric momenta and finite isotropy subgroups, in the spirit of [33, 40]. Whereas the results of [33, 40] build on topological methods which require compact symmetry groups, our persistence results apply to noncompact symmetry groups as well. (For a more detailed comparison, see section 4.2.2.) Moreover, we will see that in the case of generic momenta, bifurcations from relative periodic orbits reduce to fixed point bifurcations of symplectic maps which are twisted semiequivariant with respect to compact symmetry groups. The latter bifurcations are studied, for example, in [6, 9] for equivariant symplectic maps and in [10] for reversible symplectic maps. All of these results are simple corollaries of the bundle equations. We will present a more general and detailed study of bifurcations and persistence in future work.

In this paper, we restrict our attention to algebraic symmetry groups. These are groups defined by polynomial equations and include compact, Euclidean, and the classical Lie groups, so this assumption is usually satisfied in applications. If this assumption is not satisfied, then there might be no comoving frame in which the relative periodic orbit becomes periodic (i.e., Lemma 2.1 would not apply). Since in this case the bundle structure near relative periodic orbits already becomes more complicated for general systems [55], we deal only with algebraic symmetry groups in this paper.

The paper is organized as follows. In section 2, we recall the bundle structure theorem of
on relative periodic orbits of general systems. In section 3, we study this structure for Hamiltonian systems and present the differential equations in bundle coordinates. In section 4, we discuss linear stability, persistence, and bifurcation from relative periodic solutions. In section 5, we illustrate these results with an application to the dynamics of a finite dimensional model of a deformable body in an ideal irrotational fluid. Section 6 is devoted to the proofs of the bundle structure theorems of section 3.

2. Relative periodic orbits in general systems. In this section, we recall the results of [52], [55], and [22], giving a parametrization of a manifold in the neighborhood of a relative periodic orbit of a reversible equivariant vector field and the form that the vector field takes in these coordinates.

2.1. Reversible equivariant vector fields. We consider an ordinary differential equation on a manifold $M$,

$$\frac{dx}{dt} = f(x), \ x \in M,$$

that is equivariant with respect to a smooth, proper action of a finite dimensional algebraic Lie group $\Gamma$ on $M$:

$$\gamma f(x) = f(\gamma x) \quad \text{for all} \quad \gamma \in \Gamma.$$  

If $x(t)$ is a solution of the equation and $\gamma \in \Gamma$, then $\gamma x(t)$ is also a solution. We call a diffeomorphism $\gamma$ satisfying (2.2) a symmetry of the vector field $f(x)$.

We also include the possibility that the vector field is reversible, i.e., there exists a reversing symmetry $\rho$ such that

$$\rho f(x) = -f(\rho x).$$

This implies that if $x(t)$ is a solution of (2.1), then so is $\rho x(-t)$. Note that if $\rho$ is a reversing symmetry, then $\rho \gamma$ is also a reversing symmetry for every $\gamma \in \Gamma$.

In the reversible case, the symmetries and reversing symmetries together form the reversing symmetry group $G$ of the vector field. The group of symmetries, $\Gamma$, is a normal subgroup of $G$ of index two, i.e., the quotient $G/\Gamma$ is isomorphic to $\mathbb{Z}_2$. It is useful to describe this structure by introducing a character (group homomorphism) $\chi : G \mapsto \{\pm 1\}$, such that $\chi(\gamma) = 1$ for all $\gamma \in \Gamma$, and $\chi(\rho) = -1$ for all $\rho \in G \setminus \Gamma$. This map is called a reversible sign or temporal character [50, 35]. Using this notation, (2.2) and (2.3) are equivalent to the single equation

$$gf(x) = \chi(g)f(gx) \quad \text{for all} \quad g \in G.$$  

We say that a vector field $f$ satisfying (2.4) is (infinitesimally) $(G, \chi)$-reversible-equivariant or $(G, \chi)$-semiequivariant. The corresponding flow $\Phi_t(\cdot)$ is $(G, \chi)$-semiequivariant in the sense of diffeomorphisms:

$$g\Phi_t = (\Phi_t)^{\chi(g)}g \quad \text{for all} \ g \in G \text{ and } t \in \mathbb{R}.$$  

We usually omit the $(G, \chi)$ prefix when it is obvious from the context.

Throughout this paper, we assume that the symmetry group $\Gamma$ is algebraic. Algebraic groups include all compact and Euclidean groups, and so the assumption is usually satisfied in applications.
2.2. Symmetry groups of relative periodic orbits. In this subsection, we recall the notion of a relative periodic orbit and its symmetry groups. Denote the isotropy subgroup of a point \( p \) in \( M \) by \( G_p \):

\[
G_p = \{ g \in G \mid gp = p \},
\]

and let \( \Gamma_p = G_p \cap \Gamma \). Either \( \Gamma_p = G_p \) or \( \Gamma_p \) is a normal subgroup of \( G_p \) of index 2. The groups \( G_p \) and \( \Gamma_p \) are compact because the action of \( G \) on \( M \) is assumed to be proper.

A solution \( x(t) \) of (2.1) is said to lie on a relative periodic orbit if there exists \( T > 0 \) such that \( x(T) \) lies in the group orbit \( \Gamma x(0) \) of \( x(0) = p \), i.e., there exists \( \sigma \in \Gamma \) with

\[
\Phi_T(p) = \sigma p.
\]

The infimum of the numbers \( T \) with this property is called the relative period of the relative periodic orbit, and the corresponding \( \sigma \) is called a spatio-temporal symmetry, phase-shift symmetry, or reconstruction phase \([5, 30, 31]\) of the relative periodic orbit with respect to \( p \). Note that \( \sigma \) determines the drift direction of the relative periodic orbit. A simple calculation shows that \( \sigma \) must lie in \( N_{\Gamma}(\Gamma_p) \), the normalizer of \( \Gamma_p \) in \( \Gamma \). We will always assume that time has been parametrized so that the relative period is 1 and so \( \Phi_1(p) = \sigma p \).

The relative periodic orbit itself is defined to be the submanifold of \( M \) given by

\[
P = \{ \gamma \Phi_t(p) \mid \gamma \in \Gamma, \ t \in \mathbb{R} \}.
\]

Thus relative periodic orbits are periodic orbits for the induced flow on the space of orbits of the action of \( \Gamma \) on \( M \), just as relative equilibria are equilibria in the space of group orbits.

Note that the \((G, \chi)\)-semiequivariance of the flow on \( M \) does not imply that the flow descends to a flow on the space of orbits for the full action of \( G \), and so it does not make sense to replace \( \Gamma \) by \( G \) in the definition of a relative periodic orbit. However, we can consider the action of \( G/\Gamma \cong \mathbb{Z}_2 \) on the space of \( \Gamma \) orbits and define a relative periodic orbit to be reversible if it is invariant under this action and to be nonreversible otherwise. If \( P \) is nonreversible, then \( G_p = \Gamma_p \) for all \( p \in P \). It is shown in \([22]\) that \( P \) is a reversible relative periodic orbit if and only if there exists a point \( p \in P \) such that \( G_p \) contains a reversing symmetry, and so \( \Gamma_p \) is a normal subgroup of \( G_p \) of index two. We call such a point a brake point of the relative periodic orbit and will always choose \( p \) in such a way. Moreover, it is easily shown that the spatio-temporal symmetry \( \sigma \) of a reversible relative periodic orbit satisfies \( \rho \sigma \rho^{-1} \in \sigma^{-1} \Gamma_p \) for each \( \rho \in G_p \setminus \Gamma_p \).

Examples of both reversible and nonreversible relative periodic orbits are provided by the (relative) nonlinear normal modes of relative equilibria. If the relative equilibrium is not reversible, then none of its normal modes will be reversible. If the relative equilibrium is elliptic, nonresonant, and reversible for some involutory reversing symmetry, then its normal modes are also reversible. Consider, for example, the relative equilibria of an ellipsoidal rigid body in an irrotational, ideal fluid modelled by Kirchhoff’s equations \([24]\). Assume that the body is neutrally buoyant but has noncoincident centers of gravity and buoyancy so that it "feels" gravity. Then relative equilibria for which the body is translating vertically are not reversible since the time reversed motion cannot be obtained by a symmetry transformation which preserves the direction of gravity \([56]\). However, horizontally translating relative equilibria and
their normal modes are reversible. Other examples of reversible relative periodic orbits of neutrally buoyant ellipsoidal deformable bodies in irrotational, ideal fluids can be found in section 5.1.

Let $\mathfrak{g}$ denote the Lie algebra of $\Gamma$ and hence also of $G$. The adjoint action of $G$ on $\mathfrak{g}$ is defined by

$$\text{Ad}_g(\xi) = g\xi g^{-1}. $$

We define the $\chi$-dual of any representation of $G$ to be the new representation obtained by composing the map representing $g \in G$ with $\chi(g)$. Let $Z(g)$ denote the centralizer of $g \in G$, and let $z(g)$ denote its Lie algebra. For a subgroup $K$ of $G$, let $z^K(G)$ denote the centralizer, or fixed point subspace, of $K$ in $\mathfrak{g}$ with respect to the $\chi$-dual action of $K$ on $\mathfrak{g}$:

$$z^K(G) = \{ \xi \in \mathfrak{g} : \chi(g)\text{Ad}_g\xi = \xi \text{ for all } g \in K \}. $$

The following lemma states that every relative periodic orbit in a system with an algebraic symmetry group becomes periodic in a comoving frame which respects the isotropy of the relative periodic orbit.

**Lemma 2.1 (see [22]).** Assume that $\Gamma$ is an algebraic Lie group, and let $\tilde{\sigma} \in \Gamma$ be a spatio-temporal symmetry of a relative periodic orbit $P$ with respect to $p \in P$. Then there exists a choice of $\sigma$ in $\tilde{\sigma}\Gamma_p$ and $\alpha \in \Gamma, \xi \in z(\sigma)$, and $n \in \mathbb{N}$ such that

$$\sigma = \alpha \exp(\xi), \quad \alpha^n = 1, \quad \text{and} \quad \xi \in z^K(G_p). $$

If $\Gamma$ is not algebraic, then the conclusions of Lemma 2.1 are in general not satisfied, the bundle structure near relative periodic orbits becomes more complicated [55, 22], and Theorem 2.1 on the bundle structure near relative periodic orbits does not apply. Since most groups in applications are algebraic, we restrict our attention to such symmetry groups.

Following [22], we define the twist diffeomorphism $\phi : G_p \to G_p$ determined by $\sigma \in \Gamma$ to be

$$(2.6) \quad \phi(g_p) = \sigma^{-1}g_p\sigma^{\chi(g_p)}. $$

Lemma 2.1 implies that there is a choice of $\sigma$ in $\tilde{\sigma}\Gamma_p$ such that the order of $\phi$ is finite and that we may replace $\sigma$ by $\alpha$ in the definition of $\phi$. If we denote the order of $\phi$ by $k$, then $k$ divides $n$. In general, $\phi$ is not a group automorphism. However, its restriction $\phi |_{\Gamma_p}$ is the automorphism of $\Gamma_p$ given by

$$\phi(\gamma_p) = \sigma^{-1}\gamma_p\sigma \quad \text{for all } \gamma_p \in \Gamma_p. $$

For any multiple $r$ of $k$, we define the group $L_r$ to be the index $r$ extension of $G_p$ by an abstract element $Q$ of order $r$ such that

$$(2.7) \quad Q^{-1}g_pQ^{\chi(g_p)} = \phi(g_p) \quad \text{for all } g_p \in G_p. $$

If an operator $Q$ satisfies this equation, we say that its inverse $Q^{-1}$ is twisted semiequivariant or twisted reversible equivariant [22]. Replacing $Q$ by $\alpha$ identifies $L_n$ with the subgroup of $G$ generated by $G_p$ and $\alpha$. For orientability reasons the index two extension $L_{2n} = L_n \times \mathbb{Z}_2$
of this group is needed in the results below. We call the groups $L_r$ \textit{reduced spatio-temporal symmetry groups} of the relative periodic orbit because the group $L_{2n}$ or $L_n$ (if the bundle is orientable) is the spatio-temporal symmetry group of the periodic orbit for the symmetry reduced dynamics; cf. section 2.3. We will label elements of $L_r$ by pairs $(g_p, i)$, where $g_p \in G_p$ and $i \in \mathbb{Z}_r$.

If the relative periodic orbit $\mathcal{P}$ is nonreversible and so $\Gamma_p = G_p$ for all $p \in \mathcal{P}$, then $L_r = \Lambda_r := \Gamma_p \times \mathbb{Z}_r$. If the relative periodic orbit is reversible, we have $L_r = (\Lambda_r)_p$, where $p \in G_p \setminus \Gamma_p$ and $(\Lambda_r)_p$ is the index two extension of $\Lambda_r$ generated by $p \in G_p \setminus \Gamma_p$ using (2.7). For a reversible relative periodic orbit, the group $L_r / \Gamma_p$ is isomorphic to the dihedral group of order $2r$, $\mathbb{D}_{2r}$, while for a nonreversible relative periodic orbit $L_r / \Gamma_p \cong \mathbb{Z}_r$.

\subsection*{2.3. Differential equations near a relative periodic orbit.} The following theorems describe the bundle structure near a relative periodic orbit and the form that the differential equations (2.1) take in coordinates adapted to this structure. As mentioned in the introduction, these coordinates decompose the dynamics into a periodically forced motion inside a Poincaré section $N$ which drives drift dynamics on the group.

\textbf{Theorem 2.1 (see \cite{55, 22}).} Let $p$ lie on a relative periodic orbit $\mathcal{P}$ with relative period 1 so that $\Phi_1(p) = \sigma p$ for some $\sigma \in \Gamma$. If $\mathcal{P}$ is reversible, assume $p$ is a brake point. Let $\sigma = \alpha \exp(\xi)$ as in Lemma 2.1. Then in a frame moving uniformly with velocity $\xi$, a $G$-invariant neighborhood $\mathcal{U}$ of $\mathcal{P}$ in $\mathcal{M}$ can be parametrized by

\begin{equation}
\mathcal{U} \equiv (G \times \mathbb{R}/2n\mathbb{Z} \times N)/L_{2n},
\end{equation}

where $N$ is a $G_p$-invariant complement to $T_p\mathcal{P}$ in $T_p\mathcal{M}$ at $p = (\text{id}, 0, 0)$ and the quotient by $L_{2n}$ is with respect to the following action of $L_{2n}$ on $G \times \mathbb{R}/2n\mathbb{Z} \times N$:

\begin{equation}
(g_p, i)(g, \theta, v) = (g \alpha^{-i} g_p^{-1}, \chi(g_p)(\theta + i), g_p Q_N^1 v) \quad \text{for all } g_p \in G_p, \ i \in \mathbb{Z}_{2n}.
\end{equation}

Here $Q_N$ is a linear transformation of $N$ of order $2n$ which is orthogonal with respect to a $G_p$-invariant inner product on $N$ and such that $Q_N^{-1}$ is $G_p$ twisted semiequivariant.

Note that $N$ is a Poincaré section transverse to the relative periodic orbit $\mathcal{P}$ at $p$. The transformation $Q_N$ is determined by the linear map $\sigma^{-1} D\Phi_1(p)$ at the relative periodic orbit; for details see \cite{55, 22} and section 6.1 of this paper. In some cases, the action of $L_{2n}$ can be replaced by an action of $L_n$, and the transformation $Q$ can be chosen to have order $n$; see \cite{55}. Whether or not this is possible depends on orientability properties of the bundle. For Hamiltonian systems it is always possible, as we will see in section 3.1.

The following theorem describes how the differential equation (2.1) lifts to a differential equation on $G \times \mathbb{R}/2n\mathbb{Z} \times N$ under the isomorphism given by Theorem 2.1.

\textbf{Theorem 2.2 (see \cite{55, 22}).} The differential equations in coordinates adapted to the bundle structure given by (2.8) have the form

\begin{equation}
\begin{aligned}
\dot{g} &= \chi(g) g f_G(\theta, v), \\
\dot{\theta} &= \chi(g) f_\theta(\theta, v), \\
\dot{v} &= \chi(g) f_N(\theta, v),
\end{aligned}
\end{equation}
where $f_G$, $f_\Theta$, and $f_N$ are functions on $\mathbb{R}/2n\mathbb{Z} \times N$ taking values in $g$, $\mathbb{R}$, and $N$, respectively, and are $L_{2n}$-semiequivariant:

\begin{align}
  f_G(\chi(g_p)\theta, g_p v) &= \chi(g_p)\text{Ad}_{g_p} f_G(\theta, v), & f_G(\theta + 1, Q_N v) &= \text{Ad}_{q_p} f_G(\theta, v), \\
  f_\Theta(\chi(g_p)\theta, g_p v) &= f_\Theta(\theta, v), & f_\Theta(\theta + 1, Q_N v) &= f_\Theta(\theta, v), \\
  f_N(\chi(g_p)\theta, g_p v) &= \chi(g_p)g_p f_N(\theta, v), & f_N(\theta + 1, Q_N v) &= Q_N f_N(\theta, v)
\end{align}

for all $g_p \in G_p$.

Note that the vector field is in fact determined by its restriction to a $\Gamma$-invariant neighborhood of $P$ in $M$, and so the equations in the theorem can be restricted to $g \in \Gamma$. The coefficients $\chi(g)$ then “disappear” from the equations.

The $(\theta, v)$ equations form a closed subsystem that is semiequivariant with respect to the action of $L_{2n}$ on $\mathbb{R}/2n\mathbb{Z} \times N$. In particular, $f_N$ and $f_\Theta$ are $2n$-periodic in $\theta$, and by a time reparametrization we can assume that $f_\Theta \equiv 1$ so that we obtain a periodically forced equation $\dot{v} = f_N(t, v)$ on the Poincaré section $N$. Furthermore, the relative periodic orbit $P$ of (2.1) reduces to a periodic orbit of the $(\theta, v)$ subsystem with a finite order phase shift symmetry, a discrete rotating wave [23]. Thus the study of bifurcations from (reversible) relative periodic orbits reduces to that of bifurcations from (reversible) discrete rotating waves. For general nonreversible non-Hamiltonian systems, these are studied in [23].

3. Relative periodic orbits of Hamiltonian systems. In this section, we combine the local bundle structure near relative periodic orbits of general systems described in section 2 with the methods used in [50] to obtain equations near Hamiltonian relative equilibria and thereby obtain local descriptions of Hamiltonian systems of equations near relative periodic orbits.

We consider a Hamiltonian ordinary differential equation on a smooth finite dimensional symplectic manifold $M$ with symplectic two-form $\omega$. For each $x \in M$, the restriction of $\omega$ to the tangent space $T_x M$ is denoted by $\omega_x$. Let $G$ be a finite dimensional Lie group, let $\chi : G \to \mathbb{Z}_2$ be a group homomorphism, and let $\Gamma = \ker \chi$. We say that $G$ acts $\chi$-semisymplectically on $M$ if [35, 50]

$$
\omega_{gx}(gu, gv) = \chi(g) \omega_x(u, v) \quad \text{for all } x \in M, g \in G, u, v \in T_x M.
$$

A Hamiltonian vector field

$$
\dot{x} = f_H(x)
$$

is generated by a smooth function, the Hamiltonian $H : M \to \mathbb{R}$, via the relationship

$$
\omega_x(f_H(x), v) = DH(x)v, \quad x \in M, v \in T_x M.
$$

If $H$ is invariant under the action of $G$, then the vector field $f_H$ is $(G, \chi)$-semiequivariant. As before, we denote the flow of (3.1) by $\Phi_t(\cdot)$.

By Noether’s theorem, locally there is a conserved quantity $J_x$ for each continuous symmetry $\xi \in g$ of the system; see, e.g., [1]. The map $J_x(x) = J(x)(\xi)$ is linear in $\xi$ so that $J$ is a map from a neighborhood of $x \in M$ to $g^*$, called a momentum map. Here $g^*$ is the dual of the Lie algebra $g$ of $G$. We assume that the momentum map $J : M \to g^*$ exists globally and
is $G$-equivariant with respect to the action of $G$ on $\mathcal{M}$ and the $\chi$-dual of the coadjoint action, or $\chi$-coadjoint action, of $G$ on $g^*$ [35, 50]:

$$J(gx) = \chi(g)(\text{Ad}_g^*)^{-1}J(x), \quad x \in \mathcal{M}, \; g \in G.$$ 

Here $\text{Ad}_g^*$ is the dual operator to $\text{Ad}_g$, i.e., $\text{Ad}_g^*\mu(\xi) = \mu(\text{Ad}_g\xi)$ for all $\mu \in g^*, \xi \in g$. Note that, since we are interested only in the dynamics inside a $G$-invariant neighborhood $U$ of the relative periodic orbit $P$, it suffices to make the above assumptions about the momentum map $\mathcal{M}$ or, alternatively, to set $\mathcal{M} := U$.

Let $P$ be a relative periodic orbit of the Hamiltonian system (3.1) of relative period 1, and assume that $p \in P$ satisfies $\Phi_1(p) = \sigma p$ for $\sigma \in \Gamma$. If the relative periodic orbit is reversible, assume that $p$ is a brake point. We will assume, without loss of generality, that $H(p) = 0$. As before, let $G_p$ denote the isotropy subgroup of $p$. Let $\mu = J(p)$ be the momentum of the point $p$, and let

$$G_\mu = \{ g \in G : \chi(g)\text{Ad}_g^*\mu = \mu \}$$

be the momentum isotropy subgroup for the $\chi$-dual of the coadjoint action of $G$ on $g^*$.

### 3.1. Bundle structure near Hamiltonian relative periodic orbits.

As before, we assume that the symmetry group $\Gamma$ is algebraic. Theorem 2.1 describes the bundle structure near relative periodic orbits of general $(G, \chi)$-semiequivariant vector fields. In this subsection, we will describe the additional structure that is present for Hamiltonian systems.

Let $P$ be a relative periodic orbit of (3.1), and let $p = \sigma^{-1}\Phi_1(p) \in P$, $\sigma \in \Gamma$. Note that $G_p \subset G_\mu$, $\Gamma_p \subset \Gamma_\mu$, and that $\sigma \in \Gamma_\mu$ since

$$\sigma\mu = \sigma J(p) = J(\sigma p) = J(\Phi_1(p)) = J(p) = \mu.$$ 

As $\sigma \in N(\Gamma_p)$, we conclude that $\sigma \in N_G(\Gamma_p) = N(\Gamma_p) \cap \Gamma_\mu$, which gives a restriction on possible drift directions of Hamiltonian relative periodic orbits, as we will see in the examples in section 5.4. If $\Gamma$ is algebraic, then so is $\Gamma_\mu = \{ \gamma \in \Gamma, \gamma \mu = \mu \}$, and it follows immediately from Lemma 2.1 that we can choose $\sigma$ such that it decomposes as $\sigma = \alpha \exp(\xi)$ with

$$\alpha \in \Gamma_\mu, \quad \alpha^n = 1, \quad \xi \in g_\mu \cap z(\sigma) \cap z^\chi(G_p).$$

As before, identify $L_n \subset G_\mu$ with the compact group generated by $\alpha$ and $G_p$. Choose $L_n$-invariant complements $m_\mu$ to $g_p$ in $g_\mu$ and $n_\mu$ to $g_\mu$ in $g$. Then $g = g_p \oplus m_\mu \oplus n_\mu$, and $g^* = \text{ann}(m_\mu) \oplus \text{ann}(g_p) \oplus \text{ann}(m_\mu)$. These choices of complements define $L_n$-equivariant linear isomorphisms [50]

$$\begin{align*}
\text{ann}(n_\mu) &\cong g^*_\mu, \\
\text{ann}(m_\mu) &\cong \text{ann}_{g^*_\mu}(m_\mu) \cong g^*_p, \\
\text{ann}(g_p) &\cong \text{ann}_{g^*_\mu}(g_p) \cong (g_p/g^*_p)^*,
\end{align*}$$

where $\text{ann}(\cdot)$ denotes an annihilator in $g^*$ and $\text{ann}_{g^*_\mu}(\cdot)$ an annihilator in $g^*_\mu$.

Theorem 2.1 states that in a frame moving with velocity $\xi \in g_\mu$ the bundle near the relative periodic orbit $P$ is periodic with period $2n$. The following result shows that in the
Hamiltonian case the period can be reduced to $n$ and the Poincaré section $N$ can be further decomposed into three subspaces.

Theorem 3.1. Let $P$ be a relative periodic orbit, and let $p = \sigma^{-1}\Phi_1(p) \in P$. Then the $G_p$-invariant Poincaré section $N$ at $p$ of Theorem 2.1 can be chosen to decompose as

\begin{equation}
N = N_0 \oplus N_1 \oplus N_2,
\end{equation}

where

\begin{align}
N_0 &= \text{ker} DH(p) \cap (\text{ker} DJ(p))^\perp \cap N \simeq (\mathfrak{g}_\mu/\mathfrak{g}_p)^*, \\
N_1 &= \text{ker} DH(p) \cap \text{ker} DJ(p) \cap N, \\
N_2 &= (\text{ker} DH(p))^\perp \cap \text{ker} DJ(p) \cap N \simeq \mathbb{R}.
\end{align}

Here $\perp$ denotes orthogonal complements with respect to an appropriate $G_p$-invariant inner product on $T_pM$. The spaces $N_0$, $N_1$, and $N_2$ are all $G_p$-invariant, and $N_1$ is a symplectic subspace of $T_pM$.

The operator $Q_N$ in Theorem 2.1 can be chosen to have order $n$, and so the action of the group $L_{2n}$ on $N$ factors through an action of $L_n$. The actions of $G_p$ and $Q_N$ on $N$ now have the forms

\begin{equation}
g_p(\nu, w, E) = (\chi(g_p)(\text{Ad}_{g_p}^* - 1)\nu, g_p w, E) \quad \text{for all } g_p \in G_p
\end{equation}

and

\begin{equation}
Q_N(\nu, w, E) = (Q_0\nu, Q_1 w, E) \quad \text{with } Q_0 = (\text{Ad}_0^*)^{-1}.
\end{equation}

The linear map $Q_1 : N_1 \to N_1$ is orthogonal with respect to the restricted $G_p$-invariant inner product on $N_1$ and symplectic with respect to the restricted $G_p$-semi-invariant symplectic form $\omega_{N_1} := \omega|_{N_1}$. Its inverse $Q_1^{-1}$ is twisted semiequivariant with respect to the action of $G_p$ on $N_1$.

Moreover, the identification (2.8) of a $G$-invariant neighborhood of $U$ with $(G \times \mathbb{R}/n\mathbb{Z} \times N)/L_n$ is a symplectomorphism, and the $\Gamma$-reduced phase space $U/\Gamma \equiv (\mathbb{R}/n\mathbb{Z} \times N)/(\Gamma_p \times \mathbb{Z}_n)$ is a Poisson space.

This theorem will be proved in sections 6.1–6.7 below. The tangent space decomposition is derived in section 6.2, the Poisson-structure on the $\Gamma$-reduced bundle is described in section 6.6, and the symplectic structure of the bundle is described in section 6.7. That $Q_N$ can always be chosen to have order $n$ is proved in sections 6.4 and 6.5 and is related to the connectedness of groups of symplectic transformations.

We call $N_1$ the symplectic normal space and denote its complex structure by $J_{N_1}$. In [16] it is shown that every semi-invariant symplectic form on a vector space has a semiequivariant complex structure $J$ satisfying $J^2 = -\text{id}$. We will always choose $J_{N_1}$ in such a way. If $G_p$ is finite and so $J$ is nonsingular at $p$, then $N_1$ can be identified with the intersection of the Poincaré section $N$ with the tangent space to the energy-momentum level set through $p$. It can be interpreted as the space of all small shape oscillations near the relative periodic orbit. In a similar way, $\nu \in N_0 \simeq (\mathfrak{g}_\mu/\mathfrak{g}_p)^*$ parametrizes the momenta of the rigid motion, expressed in body coordinates, while $E$ parametrizes the difference in energy from $H(p)$ (see Remark 3.4(e)).
3.2. Equations near Hamiltonian relative periodic orbits. In this subsection, we formulate the central results of this paper. These describe the form taken by a Hamiltonian vector field near a relative periodic orbit in the bundle coordinates given by Theorems 2.1 and 3.1. In the absence of any symmetries it is well known that the dynamics near a Hamiltonian periodic orbit can be described as periodically forced motion in a Poincaré section inside an energy level set [2]. Here we show how this can be generalized to Hamiltonian relative periodic orbits by combining Theorems 2.1 and 2.2 of section 2 and Theorem 3.1 with techniques used for Hamiltonian relative equilibria in [50].

First we need to recall some preliminaries.

Proposition 3.2 (see [50]).
(a) Let $G$ be a Lie group, let $g$ be its Lie algebra, and let $\mu$ be any point in $g^*$. Let, as above, $n_\mu$ be a complement to $g_\mu$ in $g$, and let $P_{\text{ann}(g_\mu)}$ be the projection from $g^*$ to $\text{ann}(g_\mu)$ with kernel $\text{ann}(n_\mu)$. Then for each $\zeta$ sufficiently close to 0 in $\text{ann}(n_\mu)$ and each $\xi \in g_\mu$, the equation

$$P_{\text{ann}(g_\mu)} \left( \text{ad}_{\xi + \eta}(\mu + \zeta) \right) = 0$$

has a unique solution $\eta = \eta_\mu(\xi, \zeta) \in n_\mu$. The map $\eta_\mu : g_\mu \oplus \text{ann}(n_\mu) \to n_\mu$, defined on the whole of $g_\mu$ and a neighborhood of 0 $\in$ ann$(n_\mu)$, is smooth and linear in $\xi$ and satisfies $\eta_\mu(\xi, 0) = 0$ for all $\xi \in g_\mu$ and $\eta_\mu(\xi, \lambda \zeta) = \eta_\mu(\xi, \zeta)$ for all $\lambda \in \mathbb{R}$.

(b) If $\mu = J(p)$ and $n_\mu$ is $G_p$-invariant, then $\eta_\mu(\xi, \zeta)$ is $G_p$-equivariant with respect to the adjoint action of $G_p$ on $g$ and $G_\mu$-equivariant with respect to the $\chi$-coadjoint action of $G_\mu$ on $g^*$.

(c) Let $G_\mu^0$ denote the identity component of $G_\mu$. If $n_\mu$ is a $G_\mu^0$-invariant complement to $g_\mu$ in $g$, then $\eta_\mu \equiv 0$.

For each sufficiently small $\zeta \in g^*$, we define the linear map $j_\mu : g_\mu \to g$ by

$$j_\mu(\zeta)\xi = \xi + \eta_\mu(\zeta, \xi).$$

Now let $p \in \mathcal{P}$ lie on a relative periodic orbit $\mathcal{P}$. If $\mu$ satisfies the condition in (c), i.e., the $L_\mu$-complement $n_\mu$ to $g_\mu$ in $g$ can be chosen to be $G_\mu^0$-invariant, then we say that $\mu$ is split; see [14, 50].

Since the linear action of the compact group $G_p$ on $N_1$ is semisymplectic, there exists a momentum map $L_{N_1} : N_1 \to g_p^*$ which is equivariant with respect to the $\chi$-coadjoint action of $G_p$ on $g_p^*$. Using the complement $m_\mu$ to $g_\mu$ in $g_\mu$, we can identify $g_p^* \simeq \text{ann}_{g_p^*}(m_\mu) \subset g_\mu^*$ (see (3.4)) and so embed the Poincaré section $N = N_0 \oplus N_1 \oplus N_2 \cong (g_\mu/g_p)^* \oplus N_1 \oplus N_2$ into the extended Poincaré section

$$\tilde{N} = g_\mu^* \oplus N_1 \oplus N_2$$

by the map from $N_0 \oplus N_1$ to $g_\mu^* \oplus N_1$ given by

$$(\nu, w) \mapsto (\nu + L_{N_1}(w), w), \quad \nu \in (g_\mu/g_p)^*, \ w \in N_1.$$  

The action of $L_\mu$ on $N_0 \oplus N_1$ defined by Theorem 3.1 extends to an action on $g_\mu^* \oplus N_1$ by extending the action of $Q_0 = (\text{Ad}_{\mu}^{-1})$ on $(g_\mu/g_p)^*$ to the whole of $g_\mu^*$. The choice of $m_\mu$ to be $\text{Ad}_{\mu}$-invariant implies that this action preserves the subspace $\text{ann}_{g_\mu^*}(m_\mu) \simeq g_p^*$. Since $L_\mu$
is compact, the momentum map $L_{N_1}$ can be assumed to be $L_n$-equivariant by averaging. It follows that the embedding (3.12) will also be $L_n$-equivariant.

Let $\hat{h} = \hat{h}(\theta, \nu, w, E)$ denote the lift of the $G$-invariant Hamiltonian $H$ back to the space $G \times \hat{\mathbb{R}}/n\hat{\mathbb{Z}} \times (N_0 \oplus N_1 \oplus N_2)$ under the map given by Theorems 2.1 and 3.1. The function $\hat{h}$ is $L_n$-invariant:

$$\hat{h}(\chi(g_p)\theta, \chi(g_p)(\text{Ad}_{g_p}^*)^{-1}\nu, g_p w, E) = \hat{h}(\theta, \nu, w, E)$$ for all $g_p \in G_p$, and

$$\hat{h}(\theta + 1, (\text{Ad}_{\theta}^*)^{-1}\nu, Q_1 w, E) = \hat{h}(\theta, \nu, w, E).$$

In particular, $\hat{h}$ is periodic in $\theta$ with period $n$. We can extend $\hat{h}$ to a $L_n$-invariant function $\hat{h}(\theta, \zeta, w, E)$ on $\hat{\mathbb{R}}/n\hat{\mathbb{Z}} \times \hat{N}$ by setting $\hat{h}(\theta, \zeta, w, E) = \hat{h}(\theta, \nu, w, E)$, where $\zeta = \nu + \zeta_p \in g^*_\mu$, $\nu \in (g^*/g_p)^*$, $\zeta_p \in g^*_p$.

**Theorem 3.3.** Let $P$ be a relative periodic orbit, and let $p = \sigma^{-1}\Phi_1(p) \in P$. Assume time is parametrized so that the phase dynamics near the relative periodic orbit is given by $\dot{\theta} \equiv 1$ in the equations of Theorem 2.2. Then the Hamiltonian $\hat{h}$ in bundle coordinates is of the form

$$(3.13) \quad \hat{h}(\theta, \nu, w, E) = h(\theta, \nu, w) + E$$

for some $L_n$-invariant function $h$ on $\hat{\mathbb{R}}/n\hat{\mathbb{Z}} \times (N_0 \oplus N_1)$. As above, $h$ extends to an $L_n$-invariant function $h(\theta, \zeta, w) \in \hat{\mathbb{R}}/n\hat{\mathbb{Z}} \times (g^*_\mu \oplus N_1)$. We have $D_{(\zeta, w)}h(\theta, 0, 0) = (\xi, 0)$, and the differential equations for the motion in bundle coordinates

$$(g, \theta, \zeta = \nu + L_{N_1}(w), w, E) \in \Gamma \times \hat{\mathbb{R}}/n\hat{\mathbb{Z}} \times \hat{N}$$

take the form

$$(3.14) \quad \begin{align*}
\dot{g} &= g j_\mu(\zeta)D_{\zeta}h(\theta, \zeta, w), \\
\dot{\theta} &= 1, \\
\dot{\zeta} &= \text{ad}_{j_\mu(\zeta)D_{\zeta}h(\theta, \zeta, w)}(\mu + \zeta), \\
\dot{w} &= J_{N_1}D_w h(\theta, \zeta, w), \\
\dot{E} &= -D_{\theta}h(\theta, \zeta, w).
\end{align*}$$

A proof of Theorem 3.3 is given in section 6.8 below.

**Remarks 3.4.**

(a) Note that the $(\theta, \zeta, w)$ subsystem of (3.14) decouples from and forces the $(g, E)$ equations. Hence (3.14) has a skew-product structure.

(b) The $(\zeta, w)$ subsystem on $g_\mu^* \oplus N_1$ forms a $G_p$-semiequivariant Poisson system that is periodically forced with period $n$. Since the action of $G_p$ on $g_\mu^* \oplus N_1$ is semi-Poisson, the dynamics of this subsystem preserves a $G_p$ momentum map $L_{g_\mu^* \oplus N_1}$; see section 6.6.

(c) If $\mu$ is split, for example, if $G_\mu$ is compact, then $\eta_\mu(\zeta) \equiv 0$ and $j_\mu(\zeta)D_{\zeta}h = D_{\zeta}h$. 

(d) As in the case of relative equilibria [50], the momentum map $J$ is given in bundle coordinates by $J(g, \theta, \nu, w, E) = \chi(g)(\text{Ad}_p^*)^{-1}(\mu + \nu + L_{N_1}(w))$. This can easily be verified using the symplectic form in bundle coordinates, described in section 6.7.

(e) Because of (3.13) the energy level sets $H \equiv c$ with $c \approx 0$ are given in bundle coordinates by $E = E(\theta, \nu, w) = c - h(\theta, \nu, w)$. Since $H(p) = h(0) = 0$, the parameter $E$ therefore parametrizes the difference in energy from $H(p)$.

The next theorem gives the equations that are obtained by projecting the $\dot{\zeta}$ equation on $g^*_\mu$ back to $N_0 \approx (g^*_\mu/g^*_p)$ and hence provides explicit differential equations in the bundle coordinates $(g, \theta, \nu, w, E)$. First we recall some notation for the operator obtained by projecting the coadjoint action of $g_\mu$ to a neighborhood of $0$.

\[ \text{ad}^* : g^*_\mu \to g^*_p \]

\[ \text{ad}^*(\eta) = \{\eta, \cdot\}, \quad \text{ad}^*(\eta) = \nu (\{\xi, \eta\}_{g^*_\mu}). \]

Note that, in general, the bracket $[\cdot, \cdot]_{g^*_\mu}$ and the operators $\text{ad}$ and $\text{ad}^*$ depend on the choice of $g^*_\mu$. Moreover, $[\cdot, \cdot]_{g^*_\mu}$ does not satisfy the Jacobi identity and so is not a Lie bracket. However, in the (very special) case when $g_\mu$ is a normal subalgebra of $g_\mu$ the quotient $g^*_\mu/g^*_p$ is again a Lie algebra, and $[\cdot, \cdot]_{g^*_\mu}$ is equal to its natural Lie bracket for any choice of complement $g^*_\mu$.

Similarly, $\text{ad}^*$ is the usual coadjoint action of $g^*_\mu/g^*_p$ on its dual in this case [50].

**Theorem 3.5.** Coordinates $(g, \theta, \nu, w, E)$ can be chosen on $\Gamma \times \mathbb{R}/n\mathbb{Z} \times N = \Gamma \times \mathbb{R}/n\mathbb{Z} \times ((g^*_\mu/g^*_p)^* \oplus N_1 \oplus N_2)$ so that the restriction of the $(G, \chi)$-semiequivariant Hamiltonian system (3.1) to a neighborhood of $P$ can be lifted to the following system on $\Gamma \times \mathbb{R} \times N$:

\[
\begin{align*}
\dot{g} &= g (D_\theta h(\theta, \nu, w) + \hat{\eta}(\theta, \nu, w)), \\
\dot{\theta} &= 1, \\
\dot{\nu} &= \text{ad}^*_{D_\nu h(\theta, \nu, w)}(\nu), \\
\dot{w} &= J_{N_1} D_w h(\theta, \nu, w), \\
\dot{E} &= -D_\theta h(\theta, \nu, w),
\end{align*}
\]

where the map $\hat{\eta} : \mathbb{R} \times N \to n^*_\mu$ is given by

\[ \hat{\eta}(\theta, \nu, w) = \eta_{\mu}(D_\nu h(\theta, \nu, w), \nu + L_{N_1}(w)) \]

and $P$ is the projection from $g^*$ to $\text{ann}(g^*_p + n^*_\mu) \cong (g^*_\mu/g^*_p)^*$ with kernel $\text{ann}(m^*_\mu)$.

This theorem is obtained from Theorem 3.3 in the same way as the analogous result for relative equilibria in [50].

**4. Stability and bifurcations.** In this section, we outline some straightforward applications of Theorems 3.3 and 3.5. The first subsection describes the linearization of a Hamiltonian vector field at a relative periodic orbit, while the second gives two persistence theorems. The main aim of the section is to indicate potential applications of the theorems. These will be explored in greater depth in future work.
4.1. Stability of relative periodic orbits. In this subsection, we present some simple implications of Theorem 3.5 for the stability of relative periodic orbits. A relative periodic orbit $P$ of the $\Gamma$-equivariant, but not necessarily Hamiltonian, differential equation (2.1) is said to be (orbitally Liapounov) stable or $\Gamma$-stable if $P$ is a (Liapounov stable) periodic orbit for the flow on $\mathcal{M}/\Gamma$. It is said to be exponentially unstable if there exist solutions which start close to $P$ but leave a neighborhood of $P$ in $\mathcal{M}/\Gamma$ exponentially fast. Theorem 2.1 implies that stability or exponential instability of $P$ is equivalent to the stability or exponential instability of the periodic solution $\{\theta \in \mathbb{R}, v = 0\}$ of the $(\theta, v)$ subsystem of (2.10).

Proposition 4.1. Let $p = \sigma^{-1}\Phi_1(p) \in \mathcal{P}$, $\sigma \in \Gamma$, and let $M = \sigma^{-1}D\Phi_1(p)$. Then the following hold.

(a) The map $M$ has the following structure with respect to the decomposition $T_p\mathcal{M} = g\mathfrak{p} \oplus \mathbb{R} \oplus \mathfrak{N}$:

$$M = \begin{pmatrix} \pi_m\text{Ad}_\sigma^{-1}|_m & 0 & D \\ 0 & 1 & \Theta \\ 0 & 0 & M_N \end{pmatrix},$$

where $m \cong g\mathfrak{p} \cong g/g_{\mathfrak{p}}$ is an $L_m$-invariant complement to $g_{\mathfrak{p}}$ in $g$ and $\pi_m$ is the projection from $g$ to $m$ with kernel $g_{\mathfrak{p}}$. 

(b) If time is reparametrized so that $f_\theta(\theta, v) \equiv 1$ and $\Phi_{1,0}^{N}$ is the time 1 map of the periodically forced system on $N$, then $Q_N^{-1}\Phi_{1,0}^{N}$ is a (symmetry reduced) Poincaré map for the periodic solution of the $(\theta, v)$ system with $v = 0$ as fixed point. The block $M_N$ in (4.1) is the linearization of this map: $M_N = Q_N^{-1}D\Phi_{1,0}^{N}(0)$.

Proof. It is easily checked that $f_H(p)$ is a right eigenvector of $M$ with eigenvalue 1. Moreover, for $\xi \in g$ we have

$$\sigma^{-1}D\Phi_1(p)\xi p = \sigma^{-1}\xi\Phi_1(p) = \sigma^{-1}\xi p = (\text{Ad}_\sigma^{-1}\xi)p,$$

which shows that $M\xi p = \text{Ad}_\sigma^{-1}\xi p$ for $\xi \in g$. Therefore, $M$ has the structure shown in (4.1).

As a consequence, $\mathcal{P}$ is exponentially unstable if and only if $M_N$ has eigenvalues outside the unit circle.

Definition 4.2. We call a relative periodic orbit $\mathcal{P}$ spectrally stable if all the eigenvalues of $M_N$ lie within or on the unit circle.

In Hamiltonian systems, (relative) periodic orbits are typically not orbitally Liapounov stable. However, the above spectral stability theory for general systems remains applicable. Criteria for Liapounov stability of Hamiltonian relative periodic orbits that apply in special cases can be found in [41, 42]. In this section, we will describe the extra structure that $M$ and $M_N$ have for Hamiltonian systems.

As usual, let $\mu = J(p)$ and $\mathfrak{m}_\mu$ and $\mathfrak{n}_\mu$ be as in section 3, and so $\mathfrak{m} = \mathfrak{n}_\mu + \mathfrak{m}_\mu$. Let $\pi_{\mathfrak{m}_\mu}$ be the projection from $g$ to $\mathfrak{m}_\mu$ with kernel $\mathfrak{n}_\mu \oplus \mathfrak{g}_p$, and let $\pi_{\mathfrak{n}_\mu}$ be the projection from $g$ to $\mathfrak{n}_\mu$ with kernel $\mathfrak{m}_\mu \oplus \mathfrak{g}_p$. We will now define an analogue of the operators $\text{ad}$, $\text{ad}^*$ introduced in section 3.2 for actions of $g \in G_\mu$ on $\mathfrak{m}_\mu \simeq g_{\mathfrak{u}}/g_p$ and $\text{ann}_{g_{\mathfrak{n}}}(g_p) = (g_{\mathfrak{n}}/g_p)^*$. For $g \in G_\mu$, $\eta \in \mathfrak{m}_\mu$, and $\nu \in \text{ann}_{g_{\mathfrak{n}}}(g_p)$, let

$$\text{Ad}_g\eta = \pi_{\mathfrak{m}_\mu}\text{Ad}_g\eta, \quad (\text{Ad}_g^*(\nu))(\eta) = \nu(\text{Ad}_g\eta).$$
Note that $\overline{\text{Ad}}_g$ and $\overline{\text{Ad}}^*_g$ depend on the choice of the complement $m_\mu$ of $g_\mu$ in $g_\mu$ and vary if $g$ is varied in $g\Gamma_p$.

**Proposition 4.3.** With respect to the tangent space decomposition $T_p\mathcal{M} = T \oplus N$, where

\[ T = T_p\mathcal{P} = T_0 \oplus T_1 \oplus T_2, \quad \text{with} \quad T_0 = g_\mu p, \quad T_1 = n_\mu p, \quad T_2 = \text{span}(f_H(p)), \]

and $N = N_0 \oplus N_1 \oplus N_2$, the linearization $M$ at $p \in \mathcal{P}$ has the following block structure:

\[
M = \sigma^{-1}D\Phi_1(p) = \begin{pmatrix}
\overline{\text{Ad}}^{-1}_\sigma & \pi_{m_\mu}\overline{\text{Ad}}^{-1}_\sigma|_{n_\mu} & 0 & D_0 & D_1 & D_2 \\
0 & \pi_{n_\mu}\overline{\text{Ad}}^{-1}_\sigma|_{n_\mu} & 0 & D_3 & 0 & 0 \\
0 & 0 & 1 & \Theta_0 & \Theta_1 & \Theta_2 \\
0 & \overline{\text{Ad}}^*_\sigma & 0 & 0 \\
0 & M_{10} & M_1 & M_{12} \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

All the subblocks of $M$ are twisted semiequivariant with respect to the appropriate actions of $G_\mu$ on the subspaces $T_i$ and $N_i$, $i = 0, 1, 2$. Moreover, $M$ is symplectic and so the subblocks are related to each other by the equations given in Lemma 6.4.

This proposition is proved in section 6.3, where Lemma 6.4 is stated. Results for relative equilibria of compact group actions analogous to this and the following proposition can be found in [44, 45].

**Proposition 4.4.** Consider the decomposition of $M$ given by Proposition 4.3.

(a) If $1 \notin \text{spec}(M_1)$, then the tangent space decomposition can be chosen so that $\Theta_1 = M_{12} = 0$.

(b) If $\text{spec}(\overline{\text{Ad}}_\sigma) \cap \text{spec}(M_1) = \emptyset$, then the tangent space decomposition can be chosen so that $D_1 = M_{10} = 0$.

(c) If $\text{spec}(\overline{\text{Ad}}_\sigma) \cap \text{spec}(\pi_{n_\mu}\overline{\text{Ad}}_\sigma|_{n_\mu}) = \emptyset$ or $\mu$ is split, then the tangent space decomposition can be chosen so that $\pi_{m_\mu}\overline{\text{Ad}}_\sigma|_{n_\mu} = 0$ and $D_3 = 0$.

(d) If time is parametrized so that $\theta \equiv 1$, then $D_2 = \Theta_0 = \Theta_1 = \Theta_2 = M_{12} = 0$.

**Proof.** Parts (a) and (b) and the first statement of (c) are linear algebra. For the second statement of part (c), observe that if $\mu$ is split, then $n_\mu$ can be chosen to be both $\text{Ad}_\sigma$ and $G_\mu^0$-invariant and hence also $\text{Ad}_\sigma$-invariant since $\sigma = \alpha \exp(\xi)$, $\xi \in g_\mu$. That $D_3 = 0$ then follows from (6.10) below. For Part (d) note that Theorem 3.3 implies that in this case the Hamiltonian $h(\nu, w, \theta)$ in bundle coordinates does not depend on $E$.

The following corollary is a direct consequence of Proposition 4.3.

**Corollary 4.5.** The relative periodic orbit $\mathcal{P}$ is spectrally stable if and only if all the eigenvalues of $M_1 : N_1 \rightarrow N_1$ lie on the unit circle, and those of $\overline{\text{Ad}}_\sigma^* : N_0 \rightarrow N_0$ lie within or on the unit circle.

Note that spectral stability does not depend on the choice of $\sigma$ within the coset $\sigma\Gamma_p$. As for general systems (Proposition 4.1), we see that $\mathcal{P}$ is spectrally stable if and only if 0 lies on a spectrally stable periodic orbit of the periodically forced $(\nu, w)$ subsystem.

For many groups relevant in applications, the spectrum of $\overline{\text{Ad}}_\sigma^*$ automatically lies on the unit circle. For example, this is satisfied if there is a $G_\mu^0$-invariant inner product on $g^*$ and so for all compact groups $G$. It is also satisfied by Euclidean groups and therefore in most applications.
The twisted semiequivariance of the diagonal subblocks of $M$ may imply additional block structure for the subblocks. For example, if the twist diffeomorphism $\phi$ is trivial, so that $M$ is $G_p$-semiequivariant, then the block $M_1$ maps isotypic components of $N_1$ with respect to the $\Gamma_p$-action into themselves [42]. The structure of equivariant symplectic linear maps is described in general in [36]. Extensions to reversible equivariant symplectic linear maps can be deduced from the results on infinitesimally symplectic linear maps in [16].

4.2. Bifurcation of relative periodic orbits. In this section, we describe some simple implications of Theorem 3.5 for bifurcations of relative periodic orbits. Relative periodic orbits which lie near $\mathcal{P}$ correspond bijectively to relative periodic orbits of the $L_n$-semiequivariant $(\theta, \nu, w)$ subsystem of (3.16) on $\mathbb{R}/n\mathbb{Z} \times (g_\mu / g_\nu)^* \oplus N_1$ [55, 22]. The original relative periodic orbit $\mathcal{P}$ itself corresponds to the set $\{ \theta \in \mathbb{R}/n\mathbb{Z}, (\nu, w) = 0 \}$. It is therefore a periodic orbit with finite phase-shift symmetry $\mathbb{Z}_n$, i.e., a discrete rotating wave of the $(\theta, \nu, w)$ subsystem of (3.16). As $L_n$ is compact, the problem of describing bifurcations from relative periodic orbits in systems with noncompact (reversing) symmetry groups is therefore reduced to that of describing bifurcations from discrete rotating waves in systems with compact (reversing) symmetry groups.

The description of all generic bifurcations in the $(\theta, \nu, w)$ subsystem of (3.16) is a difficult problem which we will tackle in future work. In this paper, we content ourselves with describing briefly some easily obtained results for special cases. The case of “minimal” $\mu$ is considered in section 4.2.1. The case of split $\mu$ and the finite isotropy subgroup $\Gamma_p$ is discussed in section 4.2.2.

4.2.1. Minimal momenta. A momentum $\mu \in g^*$ is said to be minimal if $\dim(g_\mu)$ is minimal [50, section 4.2]. It is shown in [12] that the set of minimal $\mu$ is open and dense in $g^*$ and that the isotropy subgroup $\Gamma_\mu$ of a minimal $\mu$ is Abelian. The following result is proved in exactly the same way as the analogous result for relative equilibria [50, Proposition 4.2].

Proposition 4.6. If $\mu$ is minimal, then $\dot{\nu} \equiv 0$ in (3.16), and bifurcation from relative periodic orbits reduces to bifurcation from discrete rotating waves of the $\nu$-dependent periodically forced $w$ equation of (3.16).

As a corollary, we obtain a persistence result for nondegenerate relative periodic orbits.

Definition 4.7. The relative periodic orbit $\mathcal{P}$ is nondegenerate if 1 is not an eigenvalue of the block $M_1$ in $M = \sigma^{-1}D\Phi_1(p)$ defined in Proposition 4.3.

The following generalizes a persistence result for relative equilibria [44, 50] to relative periodic orbits of noncompact groups.

Corollary 4.8. Let $p = \sigma^{-1}\Phi_1(p)$ lie on a nondegenerate relative periodic orbit $\mathcal{P}$ with minimal momentum $\mu$ and energy $e$, and assume that $\Gamma_p$ is trivial. Let $\sigma = \alpha \exp(\xi), \alpha \in \Gamma_\mu, \alpha^n = \text{id}$, and $\xi \in z(\sigma) \cap z^c(G_p)$ as in (3.3). Then the following hold.

(a) For each momentum $\hat{\mu}$ near $\mu$ with $\text{Ad}_{\hat{\mu}}^\gamma \hat{\mu} = \hat{\mu}$ and each energy $\hat{e}$ near $e$, there exists a unique relative periodic orbit near $\mathcal{P}$ with momentum $\hat{\mu}$, energy $\hat{e}$, and relative period close to that of $\mathcal{P}$.

(b) Assume, in addition, that $\mathcal{P}$ is nondegenerate when considered as a relative periodic orbit of relative period $n$; i.e., the block $M_1$ in Proposition 4.3 does not have nth roots of unity as eigenvalues. Then, for each momentum $\hat{\mu}$ near $\mu$ and each energy $\hat{e}$ near $e$, there exists a unique relative periodic orbit near $\mathcal{P}$ with momentum $\hat{\mu}$, energy $\hat{e}$,
and relative period close to some $\ell \in \mathbb{N}$ with $\ell | n$. The union of these relative periodic orbits is a symplectic submanifold of $\mathcal{M}$ of dimension $\dim \Gamma + \dim \Gamma_{\mu} + 2$.

**Proof.** For simplicity, assume that time has been reparametrized such that $f_\theta \equiv 1$. From Propositions 4.3 and 4.6 we see that $M_1 = Q_1^{-1}D\Phi_{1,0}^N(\cdot, \nu)$, where $\Phi_{1,0}^N(\cdot, \nu)$ is the $\nu$-dependent time-evolution of the $w$ equation of (3.16). Since the relative periodic orbit is nondegenerate, we conclude that 1 is not an eigenvalue of $Q_1^{-1}D\Phi_{1,0}^N(0)$. So we can apply the implicit function theorem to the $\nu$-dependent fixed point equation for $Q_1^{-1}D\Phi_{1,0}^N(\cdot, \nu)$ to conclude that there is a family of periodic orbits of the $w$ equation of (3.16) parametrized by $\nu \in \mathfrak{g}_{\mu}^*$, with initial value $w(0) = 0$. Since the $\dot{E}$ equation does not depend on $E$, the $E$-initial value provides an additional parameter. For $\nu \in \mathfrak{g}_{\mu}^*$ with $\Ad_{\alpha}^* \nu = \nu$, we obtain a family $\mathcal{P}_{\nu, E}$ of relative periodic orbits of (3.1) with relative period one. Because of Remarks 3.4 (d), (e) on the momentum map in bundle coordinates and energy parametrization, this family of relative periodic orbits provides exactly one relative periodic orbit for each energy-momentum pair $(\dot{e}, \dot{\mu})$ with $\dot{\mu} = \Ad_{\alpha}^* \dot{\mu}$ close to $(e, \mu)$. This proves part (a).

To prove (b) let $\mathcal{P}$ be nondegenerate as a relative periodic orbit of relative period $n$. Then the fixed point equation for $\Phi_{n,0}^N(\cdot, \nu)$ can be solved uniquely for any small $\nu \in \mathfrak{g}_{\mu}^*$ giving a family $\mathcal{P}_{\nu, E}$ of relative periodic orbits of (3.1) which have relative periods $\ell$ for some $\ell | n$. The dimension formula then follows from the observation that each relative periodic orbit $\mathcal{P}_{\nu, E}$ has dimension $\dim(\Gamma) + 1$. Symplecticity of the submanifold formed by the union of the family $\mathcal{P}_{\nu, E}$ of relative periodic orbits is a consequence of the fact that by Theorem 3.1 a $G$-invariant neighborhood $U$ of the relative periodic orbit $\mathcal{P}$ is symplectomorphic to the symplectic manifold $(G \times \mathbb{R}/n\mathbb{Z} \times N)/L_n$ and that the union of the family of relative periodic orbits $\mathcal{P}_{\nu, E}$ is a manifold of the form $(G \times \mathbb{R}/n\mathbb{Z} \times N_0 \oplus \{0\} \oplus N_2)/L_n$. That this is a symplectic submanifold of $(G \times \mathbb{R}/n\mathbb{Z} \times N)/L_n$ can be seen from the symplectic form in bundle coordinates given in section 6.7. ■

An extension of this result which describes nearby relative periodic orbits with the same isotropy subgroup $\Gamma_\mu$ as $\mathcal{P}$ in the case of general nonfree actions can easily be obtained by applying the method of [40]: just replace $\mathcal{M}$ by the corresponding fixed point space $\text{Fix}_{\Gamma_\mu}(\mathcal{M})$ and $\Gamma$ by $N(\Gamma_\mu)/\Gamma_\mu$.

To study bifurcations of relative periodic orbits with less spatio-temporal symmetry (including subharmonic branching) and bifurcations from degenerate relative periodic orbits, the results in [6, 9, 10] on bifurcations from fixed points of equivariant and reversible symplectic maps can be applied to the $\nu$-dependent symplectic $G_\mu$-semiequivariant map $\Phi_{1,h}^N(\cdot, \nu)$ on $N_1$ provided that $Q_1 = \text{id}$.

**4.2.2. Split momenta and finite isotropy subgroups.** If $\mu$ is split, i.e., if the $L_\mu$-invariant complement $n_\mu$ to $\mathfrak{g}_\mu$ in $\mathfrak{g}$ can be chosen to be $G_\mu^0$-invariant (cf. section 3.2), then the term $P(\text{ad}_{\theta}^*(\nu + L_{N_1}(w)))$ in the $\dot{\nu}$ equation in (3.16) vanishes and the equation becomes

$$
\dot{\nu} = \text{ad}_{D_v h(\theta, \nu, w)}^* (\nu) + \text{ad}_{D_v h(\theta, \nu, w)}^* (L_{N_1}(w)).
$$

If $\Gamma_\mu$ is finite, then $\text{ad}_{D_v h}^* = \text{ad}_{D_v h}$ and $L_{N_1} \equiv 0$, and so the $(\theta, \nu, w)$ equations simplify to

$$
\dot{\theta} = 1, \quad \dot{\nu} = \text{ad}_{D_v h(\theta, \nu, w)}^* (\nu), \quad \dot{w} = J_{N_1} D_w h(\theta, \nu, w).
$$

(4.4)
These equations define an $L_p$-semiequivariant Poisson system on $\mathbb{R}/n\mathbb{Z} \times N$ and a $G_p$-semiequivariant periodically forced Poisson system on $g_p^* \oplus N_1$. The following persistence result for relative periodic orbits of noncompact group actions is inspired by a similar theorem of Montaldi [33] for free actions of compact groups and a result on the persistence of relative equilibria of compact groups with finite isotropy [34]. Montaldi uses the compactness of the coadjoint orbits to infer the existence of relative periodic orbits on each nearby energy-momentum level set. Without this compactness assumption, the topological techniques that he employs no longer apply; however, using the isotropy of the relative periodic orbit, we can get similar results. In contrast to the generalization of the persistence result of [33] given in [40], compactness of the normalizer of the isotropy of the relative periodic orbit is not required, and the spatio-temporal and reversing symmetries of the persisting relative periodic orbits are described.

Theorem 4.9. Let $p = \sigma^{-1}\Phi_1(p)$ lie on a relative periodic orbit $P$ with split momentum $\mu$ and finite isotropy subgroup $\Gamma_p$. Let $L_p = \alpha \exp(\xi)$, where $\alpha^n = \text{id}$ as in (3.3), and let $\Lambda_n$ (resp., $L_n$) be the group generated by $\Gamma_p$ (resp., $G_p$) and $\alpha$. Assume that $P$ is nondegenerate when considered as a relative periodic orbit of relative period $n$; i.e., the block $M_1$ in Proposition 4.3 does not have $n$th roots of unity as eigenvalues.

Let $\tilde{\nu} \in g_p^*$, $\tilde{\nu} \approx 0$, be such that $\text{Fix}_{g_p}(\tilde{\nu}) \cap g\dot{\nu} = \{0\}$, where $\dot{\Gamma}_p = \Gamma_p \cap \dot{\mu}$ and $\dot{\mu} = \mu + \tilde{\nu}$. Then the following hold.

(a) The group $\dot{\Lambda} := \Lambda_n \cap \dot{\mu}$ is a cyclic extension of $\dot{\Gamma}_p$: there exists $\ell \in \mathbb{N}$ with $\ell | n$ such that $\dot{\Lambda}/\dot{\Gamma}_p \simeq \mathbb{Z}_n/\ell$. Moreover, either $\dot{\Lambda} := L_n \cap \dot{\mu}$ equals $\dot{\Lambda}$, or $\dot{\Lambda}$ is a normal subgroup of $\dot{\Lambda}$ of index two. In the latter case, there exists $\rho \in \dot{\Lambda} \setminus \dot{\Lambda}$ such that $\dot{\Lambda} = \dot{\Lambda}_\rho$.

(b) There is a family $P_E(\tilde{\nu})$ of relative periodic orbits close to $P$ which is parametrized by $E \approx 0$ with points $\dot{p}_E \in P_E(\tilde{\nu})$ such that

$$J(\dot{p}_E) = \dot{\mu}, \quad \dot{\Gamma}_p = \dot{\Gamma}_p, \quad G_{\dot{p}_E} = \dot{G}_p := \begin{cases} \Gamma_{\dot{p}_E} & \text{if } \dot{\Lambda} = \dot{\Lambda}, \\ (\Gamma_{\dot{p}_E})_\rho & \text{if } \dot{\Lambda} = \dot{\Lambda}_\rho. \end{cases}$$

The relative period of the relative periodic orbit $P_E(\tilde{\nu})$ is $\dot{\ell} \in \mathbb{N}$, where $\dot{\ell} | n$ is such that $\dot{\Lambda}/\dot{\Gamma}_p \simeq \mathbb{Z}_{n/\ell}$, and we have $\dot{\sigma}^{-1} \Phi_1(\dot{p}_E) = \dot{p}_E$, where $\dot{\sigma} = \sigma^\ell \gamma \exp(\xi)$, with $\alpha^\ell \gamma \in \dot{\Lambda}$ for some $\gamma_p \in \Gamma_p$, and $\xi \in \mathfrak{z}(\dot{\mathcal{G}}_p) \cap g\dot{\mu}$ is small.

Proof. (a) Let $\gamma \in \dot{\Lambda} = \Lambda_n \cap \dot{\mu}$. Since $\Gamma_p$ is normal in $\Lambda_n$, we have $\gamma \gamma_p \gamma^{-1} \in \Gamma_p \cap \dot{\mu} = \dot{\Gamma}_p$ for $\gamma_p \in \dot{\Gamma}_p$. Therefore, $\dot{\Lambda} \subseteq N(\dot{\Gamma}_p)$. We now show that $\dot{\Lambda}/\dot{\Gamma}_p$ is cyclic. Let $\gamma \in \dot{\Lambda}$. Then $\gamma = \gamma_p \alpha^i$ for some $\gamma_p \in \Gamma_p$, $i \in \{0, \ldots, n-1\}$. Moreover, $i \neq 0$ if $\gamma_p \in \Gamma_p \setminus \dot{\Gamma}_p$ by definition of $\dot{\Gamma}_p$. As a consequence, $\gamma^n \in \dot{\Gamma}_p$, and, if $\gamma \in \dot{\Lambda}$, $\tilde{\gamma} = \gamma_p \alpha^i$, for some $\tilde{\gamma}_p \in \Gamma_p$, then $\gamma \gamma^{-1} = \gamma_p \tilde{\gamma}^{-1} \in \dot{\Gamma}_p$. Hence $\dot{\Lambda}/\dot{\Gamma}_p$ is a subgroup of $\mathbb{Z}_n$ and therefore cyclic, i.e., there is some $\dot{\ell} | n$ with $\dot{\Lambda}/\dot{\Gamma}_p \simeq \mathbb{Z}_{n/\dot{\ell}}$.

Now assume that $\dot{\Gamma}_p \neq \dot{\Lambda}$, and let $\rho \in \dot{\Lambda} \setminus \dot{\Lambda}$. Since $\Lambda_n$ is normal in $L_n$, we have $\rho \gamma \rho^{-1} \in \Lambda_n \cap \dot{\mu} = \Lambda$ for $\gamma \in \dot{\Lambda}$ and $\dot{\Lambda} = \Lambda_p$.

(b) 1. We first prove that the periodically forced Poisson system on $N_0 \oplus N_1$ (the $(\nu, w)$ subsystem of (3.16)) has an $n$-periodic solution with isotropy $\Gamma_p$. Let $\Psi_{t,t_0} = (\Psi^{\nu}_{t,t_0}; \Psi^{w}_{t,t_0})$ denote the time-evolution of this subsystem. The original relative periodic orbit corresponds to the origin: $\Psi_{t,0}(0) = 0$. Because it is Poisson, $\Psi_{t,t_0}$ restricts to a symplectic map on
the symplectic leaves $\mathcal{O} \times N_1$, where $\mathcal{O}$ is a coadjoint orbit of $\Gamma_\mu$ in $\mathfrak{g}_\mu^*$. Moreover, $\Psi_{t,0}$ is $\Gamma_\mu$-equivariant. As a consequence, $\Psi_{t,0}$ maps $(\text{Fix}_{\mathfrak{g}^*_\mu}(\hat{\Gamma}_p) \cap \Gamma_\mu \hat{\nu}) \times \text{Fix}_{N_1}(\hat{\Gamma}_p)$ into itself. Since by assumption $\text{Fix}_{\mathfrak{g}^*_\mu}(\hat{\Gamma}_p) \cap \mathfrak{g}_\mu \hat{\nu} = \{0\}$, the path-connected component of the point $\hat{\nu}$ in $\text{Fix}_{\mathfrak{g}^*_\mu}(\hat{\Gamma}_p) \cap \Gamma_\mu \hat{\nu}$ is just $\{\hat{\nu}\}$, and so we can conclude that

\begin{equation}
(4.5) \quad \Psi_{t,0}^\nu(\hat{\nu}, \hat{w}) = \hat{\nu} \quad \text{for all } t \in \mathbb{R}, \hat{w} \in \text{Fix}_{N_1}(\hat{\Gamma}_p).
\end{equation}

The nondegeneracy condition implies that $D_{\hat{w}}(\Psi_{t,0}^w(0)) - \text{id}$ is invertible. So we can solve the equation $\Psi_{t,0}^w(\nu, w) = w$ uniquely for $w = w(\nu)$ if $\nu \in N_0 \simeq \mathfrak{g}_\mu^*$ is small. Therefore, we have proved that $(\hat{\nu}, \hat{w}) = \Psi_{n,0}(\hat{\nu}, \hat{w})$, with $\hat{w} = w(\hat{\nu})$, lies on an $n$-periodic solution of the $(\nu, w)$ system. Moreover, since $\Psi_{t,0}$ is $\Gamma_\mu$-equivariant, $w(\cdot)$ is a $\Gamma_\mu$-equivariant map from $\mathfrak{g}_\mu^*$ to $N_1$, and therefore $(\hat{\nu}, \hat{w}) \in \text{Fix}_{N_0 \oplus N_1}(\hat{\Gamma}_p)$.

2. Now we investigate the spatio-temporal symmetry of this periodic solution of the $(\theta, \nu, w)$ system. Let $\ell \neq 0$ be minimal with $\alpha^\ell \gamma_p \in \hat{\Lambda}$ for some $\gamma_p \in \Gamma_\mu$. Then $\hat{\Lambda}/\hat{\Gamma}_p \simeq \mathbb{Z}_{n/\ell}$ by (a). Define $\Pi(\nu, w) = \gamma_p^{-1} Q_N^{-\ell} \Psi_{t,0}(\nu, w)$. We want to show that $\Pi(\hat{\nu}, \hat{w}) = (\hat{\nu}, \hat{w})$. Because of (4.5) and because $\alpha^\ell \gamma_p \in \hat{\Lambda}$, we conclude that $\Pi^\ell(\hat{\nu}, \hat{w}) = \hat{\nu}$. Since

$$
\Pi^{n/\ell} = \gamma_p Q_N^{-\ell} \Psi_{n,0} = \gamma_p \Psi_{n,0},
$$

where $\gamma_p \in \hat{\Gamma}_p$ by part (a), we also have $\Pi^{n/\ell}|_{\text{Fix}(\hat{\Gamma}_p)} = \Psi_{n,0}|_{\text{Fix}(\hat{\Gamma}_p)}$ and, therefore, $\Pi^{n/\ell}(\hat{\nu}, \hat{w}) = \Pi(\hat{\nu}, \hat{w})$. Since $\gamma_p^{-1} \alpha^{-\ell} \in N(\hat{\Gamma}_p)$ and $Q_1$ and $\alpha$ generate the same twist diffeomorphism on $G_\mu$, $\Pi^{\nu}(\hat{\nu}, \hat{w}) \in \text{Fix}_{N_1}(\hat{\Gamma}_p)$ also. Hence $\Psi_{n,0}(\Pi(\hat{\nu}, \hat{w})) = \Pi(\hat{\nu}, \hat{w})$. Since $\Pi^\nu(\hat{\nu}, \hat{w}) = \hat{\nu}$ and $\hat{w} = w(\hat{\nu}) = \Pi^\nu(\hat{\nu}, \hat{w}) = \Pi(\hat{\nu}, \hat{w})$ is locally unique, we conclude that $\Pi(\hat{\nu}, \hat{w}) = (\hat{\nu}, \hat{w})$. So $(\hat{\nu}, \hat{w})$ lies on an $n$-periodic solution of the periodically forced $(\nu, w)$ system of (3.16) with isotropy $\hat{\Gamma}_p$ and spatio-temporal symmetry $Q_N^h \gamma_p$.

3. Next we study the reversing symmetries of the periodic solution of the $(\theta, \nu, w)$ system. So let $\hat{L} \neq \hat{\Lambda}$ and $\rho \in \hat{L} \setminus \hat{\Lambda}$. We are looking for a brake point on the periodic solution of the $(\theta, \nu, w)$ system with reversing symmetry $\rho$. We have $\rho = g_p \alpha^i$ for some $g_p \in G_\mu \setminus \Gamma_\mu$ and some $i \in \{0, \ldots, n-1\}$. By (2.9) $\rho = (g_p, i)$ acts on $(\theta, \nu, w)$ as

$$(g_p, i)(\theta, \nu, w) = (\chi(g_p)(\theta + i), g_p Q_0^i \nu, g_p Q_1^i w).$$

By definition of $\hat{L}$ we have $g_p Q_0^i \hat{\nu} = (\text{Ad}_{g_p \alpha^i}^*)^{-1} \hat{\nu} = \hat{\nu}$. Moreover,

$$
\chi(g_p)(\theta + i) = \hat{\theta} \quad \text{for} \quad \hat{\theta} := -i/2.
$$

The periodic solution of the $(\theta, \nu, w)$ system is given by $\{(\theta, \hat{\nu}, \Psi_{\theta,0}(\hat{\nu}, \hat{w})), \theta \in \mathbb{R}/n\mathbb{Z}\}$. Let $\Phi_{\theta,0}^{\text{red}}$ denote the $L_n$-semiequivariant flow of the $(\theta, \nu, w)$ system. Since $\Phi_{\theta,0}^{\text{red}}(\hat{\theta}, \hat{\nu}, \Psi_{\theta,0}(\hat{\nu}, \hat{w})) = (\hat{\theta}, \hat{\nu}, \Psi_{\theta,0}^{\text{red}}(\hat{\nu}, \hat{w}))$, we have

$$
(\hat{\theta}, \hat{\nu}, g_p Q_1^i \Psi_{\theta,0}^{\text{red}}(\hat{\nu}, \hat{w})) = (g_p, i) \Phi_{\theta,0}^{\text{red}}(\hat{\theta}, \hat{\nu}, \Psi_{\theta,0}^{\text{red}}(\hat{\nu}, \hat{w})) = \Phi_{\theta,0}^{\text{red}}(\hat{\theta}, \hat{\nu}, g_p Q_1^i \Psi_{\theta,0}^{\text{red}}(\hat{\nu}, \hat{w})).
$$
so that both $(\hat{\theta}, \hat{\nu}, g_\theta Q_\theta^1(\hat{\nu}, 0))$ and $(\hat{\theta}, \hat{\nu}, \Psi_{\hat{\theta},0}^w(\hat{\nu}, 0))$ lie on a periodic solution of the $(\theta, \nu, w)$ system. Since by our nondegeneracy condition the periodic solution corresponding to $\hat{\nu}$ is locally unique, we get $g_\theta Q_\theta^1(\hat{\nu}, 0) = \Psi_{\hat{\theta},0}^w(\hat{\nu}, 0)$. So $g_\theta Q_\theta^1$ is a reversing symmetry of the periodic solution of the $(\theta, \nu, w)$ system with brake point $(\hat{\theta}, \hat{\nu}, \Psi_{\hat{\theta},0}^w(\hat{\nu}, 0))$.

4. Finally, we interpret the periodic solution of the $(\theta, \nu, w)$ system as a relative periodic solution of the original system. Let

$$\hat{p}_E = \hat{p}_E(\hat{\nu}) \simeq (\text{id}, \hat{\theta}, \hat{\nu}, \Psi_{\hat{\theta},0}^w(\hat{\nu}, 0), E),$$

where $E \approx 0$, and $\hat{\theta}$ is arbitrary in the nonreversible case and as above in the reversible case. Since the $\dot{E}$ equation of (3.16) does not depend on $E$ and because of (2.9), the point $\hat{p}_E(\hat{\nu})$ lies on a relative periodic orbit $P_E(\hat{\nu})$ of (3.1) with isotropy $\Gamma_{\hat{p}_E} = \hat{\Gamma}_p$ and spatio-temporal symmetry $\hat{\sigma}$ near $\sigma^t \gamma_p$. In the reversible case $L \neq \Lambda$, we see from (2.9) that $\rho \hat{p}_E = \hat{p}_E$ so that $G_{\hat{p}_E} = (\Gamma_{\hat{p}_E})_\rho$. The condition $\hat{\sigma} = \sigma^t \gamma_p \exp(\hat{\xi})$, where $\hat{\xi} \in z^N(\hat{G}_p) \cap g_{\mu_k}$ is small, follows from the fact that the vector field $f_G$ on the group is $G_{\rho}$-semiequivariant; see (2.10).

5. Example: Affine rigid bodies in ideal fluids. In this section, we illustrate how to apply the results of this paper to a specific symmetric Hamiltonian system. As our example we have chosen a finite dimensional model for the dynamics of a deformable body in an ideal irrotational fluid. The model extends the well-known Kirchhoff model for the motion of a rigid body in a fluid [18, 21, 3]. In this model, the configuration of the body, i.e., its position and orientation in $\mathbb{R}^3$, is given by the elements of the special Euclidean group $SE(3)$. The fluid motion outside the body is assumed to be irrotational, to have normal velocity at the surface of the body equal to that of the body, and to be stationary at infinity. It is therefore determined uniquely by the motion of the body itself, and the dynamics can be described by a Hamiltonian system on the cotangent bundle $T^*SE(3)$.

We extend this model by relaxing the assumption that the body is rigid to allow configurations that are obtained from orientation preserving linear deformations of a reference body. If the reference body is assumed to be a sphere, then the deformed configurations are always ellipsoids. We assume that the deformations preserve volume. The configuration space for this system is therefore the special affine group $SAff(3) = SL(3) \ltimes \mathbb{R}^3$ of $\mathbb{R}^3$, where $SL(3)$ is the group of invertible linear transformations of $\mathbb{R}^3$ with determinant 1, and the semidirect product is obtained from the natural action of $SL(3)$ on $\mathbb{R}^3$. The dynamics of the system are given by a Hamiltonian $H$ on $T^*SAff(3)$.

In addition to extending the Kirchhoff model for a rigid body in a fluid, this system also extends the “affine” or “pseudorigid” body model used in fluid dynamics and elasticity theory [8, 11, 49, 53, 54]. These models are usually invariant under a Galilean transformation group, and so the translational degrees of freedom can be ignored by using a coordinate system that moves with the center of mass [35]. This is not true for a body in a fluid that is translating relative to the fluid at infinity. In this case, the symmetry group is essentially noncompact. This is described in the next subsection.

We will use Theorem 4.9 to deduce the existence of some families of relative periodic orbits of this model in section 5.4. Since the underlying symmetry group is noncompact, the existing theories—which use compactness of the symmetry group—do not give these families
of relative periodic solutions. The solutions describe simple motions of deformable bodies in fluids.

In order to construct the relative periodic solutions, we start, in section 5.2, with a spherical equilibrium and study the dynamics in a neighborhood of this equilibrium. In section 5.3, we describe some families of nonlinear normal modes close to the equilibrium, and in section 5.4 we show how these normal modes persist to relative periodic orbits.

5.1. Symmetries and conserved quantities. In this subsection, we describe the symmetries and corresponding conserved quantities of our model of an affine rigid body in an ideal fluid. We assume that the reference body is spherically symmetric, which implies that $H$ is invariant under the action of SO(3) on $T^*\text{SAff}(3)$, which is induced from its natural action on the right of SL(3) (extended trivially to $\text{SAff}(3)$):

$$B.(S,s) = (SB^{-1}, s), \quad (S,s) \in \text{SAff}(3), \ B \in \text{SO}(3).$$

These are the “material” or “body” symmetries of the system. We also assume that the system is invariant under rotations and translations of $\mathbb{R}^3$, i.e., the natural action of SE(3) on $T^*\text{SAff}(3)$ induced from its action on the left of $\text{SAff}(3)$:

$$(A,a).(S,s) = (AS, a + As), \quad (S,s) \in \text{SAff}(3), \ (A,a) \in \text{SE}(3).$$

These are the “spatial” symmetries of the system. This assumption implies that there are no external forces such as gravity acting. In particular, the body is “neutrally buoyant” and has coincident centers of mass and buoyancy. It is natural also to assume that the system is invariant under the action of the inversion symmetry $-\text{id}$ in O(3) acting simultaneously on the left and right of $\text{SAff}(3)$. Denoting the diagonally embedded inversion operator in O(3) by $\kappa$, we have

$$\kappa.(S,s) = (S,-s), \quad (S,s) \in \text{SAff}(3), \ \kappa = (-\text{id}, -\text{id}) \in \text{O}(3) \times \text{O}(3).$$

Note that the action of $-\text{id}$ on the left or right alone does not preserve $\text{SAff}(3)$. Together the body and spatial symmetries and reflection $\kappa$ generate a semidirect product $\Gamma = \mathbb{Z}_2 \ltimes (\text{SO}(3) \times \text{SE}(3))$.

Finally, we will also assume that the system is invariant under the usual time reversal symmetry operation acting on $T^*\text{SAff}(3)$. Using left translations in $\text{SAff}(3)$, we identify $T^*\text{SAff}(3)$ with $\text{SAff}(3) \times \text{saff}(3)^*$, where $\text{saff}(3) = T_{(\text{id},0)}\text{SAff}(3) = \text{sl}(3) \oplus \mathbb{R}^3$. Then the action of the time reversal symmetry becomes

$$\rho.((S,s),(\mu_S,\mu_s)) = ((S,s),(-\mu_S,-\mu_s)), \quad (S,s) \in \text{SAff}(3), \ (\mu_S,\mu_s) \in \text{saff}(3)^*.$$  

The full group of time preserving and time reversing symmetries is $G = \Gamma \times \mathbb{Z}_2$.

It is a straightforward exercise to write down the conserved quantities associated to these symmetries; see [1]. In body coordinates $T^*\text{SAff}(3) \cong \text{SAff}(3) \times \text{saff}(3)^*$, the momentum generated by the material symmetry group SO(3) (acting from the right) is

$$J_R(S,s,\mu_S,\mu_s) = -\pi(\mu_S),$$
where \( \pi : \mathfrak{sl}(3)^* \rightarrow \mathfrak{so}(3)^* \) is the natural projection dual to the inclusion \( \mathfrak{so}(3) \subset \mathfrak{sl}(3) \). The momentum map for the spatial symmetry group \( \text{SE}(3) \) is

\[
\mathbf{J}_L(S, s, \mu_S, \mu_s) = \Pi(\text{Ad}^*_{(S, s)})(\mu_S, \mu_s),
\]

where \( \Pi : \text{saff}(3)^* \rightarrow \mathfrak{se}(3)^* \) is the natural projection. The two components of the momentum \( \mathbf{J}_L \) can be interpreted as angular and linear impulses of the body-fluid system (see, for example, [37]). The momentum \(-\mathbf{J}_R\) is the angular impulse in body coordinates.

### 5.2. The spherical equilibrium.

In this subsection, we describe the dynamics near a spherical equilibrium. So assume that the spherical configuration with zero-momentum \( p = ((\text{id}, 0), (0, 0)) \) in \( \text{SAff}(3) \times \text{saff}(3)^* \) is an equilibrium configuration. This has conserved momenta \( \mu = (\mu_L, \mu_R) = (J_L, J_R) = (0, 0) \), and so \( G_\mu = G \). The isotropy subgroup is \( G_p = O(3)_D \times \mathbb{Z}^2_2 \), where \( O(3)_D = \mathbb{Z}^2_2 \times SO(3)_D \) and \( SO(3)_D = \{ (\gamma, (\gamma, 0)) \in SO(3) \times \text{SE}(3) : \gamma \in SO(3) \} \) is the diagonally embedded copy of \( SO(3) \) in \( SO(3) \times \text{SE}(3) \). Let \( so(3)_D \) denote the Lie algebra of \( SO(3)_D \). A complement \( \mathfrak{m}_\mu \) to \( \mathfrak{g}_p \) in \( \mathfrak{g}_\mu = \mathfrak{g} \) is provided by \( so(3)_{AD} \oplus \mathbb{R}^3 \), where \( so(3)_{AD} = \{ (-\xi, (\xi, 0)) \in so(3) \oplus se(3) : \xi \in so(3) \} \) is the antidiagonal embedding of \( so(3) \) in \( so(3) \oplus se(3) \). Note that \( so(3)_{AD} \) is not a Lie subalgebra since \( [so(3)_{AD}, so(3)_{AD}] \subset so(3)_D \).

The symplectic normal space \( N_1 \) to the group orbit through \( p \) can be identified with \( V \oplus V^* \), where \( V \) is the 5-dimensional subspace of \( \mathfrak{sl}(3) \subset \text{saff}(3) \) consisting of symmetric traceless matrices and \( V^* = \ker \Pi = \text{ann}(se(3)) \) is the dual space in \( \text{saff}(3)^* \). We choose the standard symplectic structure \( \omega(w_1, w_2) = -3(\text{tr}(w_1 w_2)) \), where \( w_i = u_i + iv_i \), \( i = 0, 1 \), on \( V \oplus V^* \). The group \( \Gamma_p = O(3)_D \) acts symplectically on \( V \oplus V^* \) by conjugation of matrices. Note that \( \kappa \) acts trivially. An equivariant momentum map for this action is given by

\[
(5.1) \quad \mathbf{L}_{N_1}(w) = vu - uv, \quad w = (u, v) \in V \oplus V^*
\]

(see Lemma 5.6 of [37]).

In [50], we have presented an analogue of Theorem 3.3 for relative equilibria. In this case, there is no phase \( \theta \), the equations (3.14) of Theorem 3.3 are time-independent, and \( N (\tilde{N}) \) is a slice (extended slice) transverse to the relative equilibrium. Hence the dynamics near the group orbit of equilibria through \( p \), considered as a relative equilibrium, is given by a system of ordinary differential equations on the extended slice

\[
\tilde{N} = \mathfrak{g}_p^* \oplus N_1 \cong so(3)^* \oplus se(3)^* \oplus V \oplus V^*
\]

of the form

\[
\dot{\zeta} = \text{ad}^*_{D \mathcal{J}_\kappa \mathcal{H}(\zeta, w)}(\zeta), \quad \dot{w} = J_{N_1} D_w \mathcal{H}(\zeta, w),
\]

where \( \mathcal{H} \) is the function on \( \tilde{N} \) obtained by writing the Hamiltonian \( H \) in body coordinates and \( J_{N_1} \) is the chosen symplectic structure on \( V \oplus V^* \). We have used the fact that \( \mu = (0, 0) \) is split to obtain these equations. Taking \( \zeta = (\zeta_R, \zeta_L, \zeta_T) \), with \( \zeta_R \in so(3)^* \) and \( (\zeta_L, \zeta_T) \in se(3)^* \), the
The ζ equation takes the more concrete form (see, e.g., [50])

\[
\begin{align*}
\dot{\zeta}_R &= \zeta_R \times \frac{\partial h}{\partial \zeta_R}, \\
\dot{\zeta}_L &= \zeta_L \times \frac{\partial h}{\partial \zeta_L} + \zeta_T \times \frac{\partial h}{\partial \zeta_T}, \\
\dot{\zeta}_T &= \zeta_T \times \frac{\partial h}{\partial \zeta_L}, \\
\dot{w} &= J_{N_1} \frac{\partial h}{\partial w}.
\end{align*}
\]

(5.2)

To get the equation on the slice \(N\), we could use the analogue of Theorem 3.5 for relative equilibria in [50]. But instead of computing the expression \(\text{ad}_\zeta^{-1}\) that occurs in (3.16) of Theorem 3.5, we prefer to project (5.2) directly from the extended slice \(\tilde{N}\) onto the slice \(N\). In order to do this, we first write the differential equations for \(\zeta_{AD}\) and \(\zeta_D\):

\[
\begin{align*}
\dot{\zeta}_{AD} &= \zeta_D \times \frac{\partial h}{\partial \zeta_{AD}} + \zeta_{AD} \times \frac{\partial h}{\partial \zeta_D} - \zeta_T \times \frac{1}{2} \frac{\partial h}{\partial \zeta_T}, \\
\dot{\zeta}_D &= \zeta_D \times \frac{\partial h}{\partial \zeta_D} + \zeta_{AD} \times \frac{\partial h}{\partial \zeta_{AD}} + \zeta_T \times \frac{1}{2} \frac{\partial h}{\partial \zeta_T}, \\
\dot{\zeta}_T &= \zeta_T \times \frac{1}{2} \left( \frac{\partial h}{\partial \zeta_D} - \frac{\partial h}{\partial \zeta_{AD}} \right).
\end{align*}
\]

(5.3)

To obtain the equations on the slice \(N = N_0 \oplus N_1 \cong \text{so}(3)^*_{AD} \oplus V \oplus V^*\), we set \(\zeta = \nu + L_{N_1}(w)\), where \(\nu \in N_0 = (g_\mu / g_p)^* = \text{so}(3)^*_{AD}, L_{N_1}(w) \in g_\nu^* = \text{so}(3)^*_D\), and \(h(\zeta, w) = h(\nu, w)\). Setting \(\nu_{AD} = \zeta_{AD}, \nu_T = \zeta_T\) so that \(\nu = (\nu_{AD}, \nu_T) \in N_0\) and using \(\zeta_D = L_{N_1}(w)\) and that \(h = h(\nu, w)\) is independent of \(\zeta_D\) give

\[
\begin{align*}
\dot{\nu}_{AD} &= -\frac{\partial h}{\partial \nu_{AD}} \times L_{N_1}(w) + \frac{1}{2} \frac{\partial h}{\partial \nu_T} \times \nu_T, \\
\dot{\nu}_T &= \frac{1}{2} \frac{\partial h}{\partial \nu_{AD}} \times \nu_T, \\
\dot{w} &= J_{N_1} \frac{\partial h}{\partial w},
\end{align*}
\]

(5.4)

where all the partial derivatives of \(h\) are evaluated at \((\nu_{AD}, \nu_T, w)\). These equations are semiequivariant with respect to the action of \(G_p = \text{O}(3)_D \times \mathbb{Z}_2^p\) on \(N_0 \oplus N_1\). It would be an interesting exercise to compute their relative equilibria for the Hamiltonians \(h\) describing the motion of the body in the fluid. However, we do not attempt to give a systematic analysis of these equations here. Instead we will describe just some of the families of periodic orbits which bifurcate from the spherical equilibrium in the next subsection.

5.3. Nonlinear normal modes. In this subsection, we describe some families of periodic orbits near the spherical equilibrium.

The solutions of (5.4) leave invariant the subset defined by \(\nu = 0, L_{N_1} = 0\). The Hamiltonian \(h(0, w)\) is \(\text{O}(3)_D \times \mathbb{Z}_2^p\)-invariant, the action factoring through that of \(\text{SO}(3)_D\). Families of periodic orbits that typically bifurcate from spherically symmetric linearly stable equilibria of such Hamiltonians are described and illustrated in [36, 37]. Section 5 of [37] treats the irreducible symplectic representation of \(\text{SO}(3)\) on \(V \oplus V^*\), where \(V\) is the space of symmetric
traceless \((3,3)\)-matrices, though without taking time reversibility into account. As can be seen from Table 4 of [37], there are three different symmetry types which have \(L_{N_1} = 0\) (and there are two more families of “rotating wave” normal modes with \(L_{N_1} \neq 0\) nearby which we will not consider). Ignoring the \(\mathbb{Z}_2^3\) symmetry group that acts trivially on \(N_1\) but incorporating the time reversing symmetries, these three families have symmetry group triples \((L_n, G_p, \Gamma_p)\) isomorphic to

I. \((O(2) \times \mathbb{Z}_2^3, O(2) \times \mathbb{Z}_2^3, O(2))\),

II. \((D_4 \times \mathbb{Z}_2^3, D_2 \times \mathbb{Z}_2^3, D_2)\),

III. \((O^\rho, D_4^\rho, D_2)\).

Here \(D_2\) is the subgroup of \(SO(3)\) consisting of rotations by \(\pi\) about each of three mutually perpendicular axes. The subgroup \(D_4\) is generated by \(D_2\) together with rotations by \(\pi\) about axes in the plane of, and bisecting, two of the \(D_2\) axes. The group \(D_4^\rho\) is the subgroup of \(SO(3)_D \times \mathbb{Z}_2^3\) obtained by composing the additional rotations by \(\pi\) in \(D_4 \setminus D_2\) with \(\rho\). The group \(O\) is the subgroup of order 24 in \(SO(3)\) consisting of all rotations which preserve a cube. It can be generated by \(D_4\) together with an element of order 3 corresponding to a rotation about a diagonal of the cube. The subgroup \(O^\rho\) in \(SO(3)_D \times \mathbb{Z}_2^3\) is similar, but with \(D_4\) replaced by \(D_4^\rho\). Finally, \(O(2)\) is the subgroup of \(SO(3)\) consisting of all rotations about one axis and rotations by \(\pi\) about each of the perpendicular axes. Note that the kernels of the “sign” homomorphisms \(\chi : L_n \to \mathbb{Z}_2\) in the three cases are, respectively, \(O(2), D_4,\) and \(T,\) the group of all rotations which preserve a regular tetrahedron.

In the first case, the spatio-temporal symmetry \(\sigma\) is trivial, and \(k = n = 1\). In the second case, \(\sigma\) can be taken to be one of the rotations by \(\pi\) in \(D_4\) that does not lie in \(D_2\). In this case, \(k = n = 2\). In the third case, \(\sigma\) can be chosen to be a rotation by \(2\pi/3\) about a diagonal of the cube, and \(k = n = 3\).

Since for these periodic solutions the reversing isotropy subgroups \(G_p\) and isotropy subgroups \(\Gamma_p\) do not coincide, all of them are reversible. By construction they all have zero-momentum, i.e., \(J_L = J_R = 0\), and all can be described as “pulsating cubes.” At all times the body is ellipsoidal (which is why \(\Gamma_p\) always contains \(D_2 \times \mathbb{Z}_2^3\)), and its principal axes have fixed directions in both body and space. However, the lengths of the principal axes vary periodically in different ways. In the first case, the qualitative behavior is determined by the fact that the ellipsoid is always axisymmetric. In the second case, the longest axis switches periodically between two of the three, and the length of third axis varies with twice the period and a much smaller amplitude than the other two. The spatio-temporal symmetry \(\sigma\) corresponds to rotating by \(\pi\) about an axis bisecting the two principal axes with large amplitude variations. In the third case, the role of the longest principal axis is taken by each of the three in turn, with a \(2\pi/3\) phase shift between them. The spatio-temporal symmetry corresponds to rotating the body by \(2\pi/3\) about an axis trisecting the three principal axes. We will refer to them as the “axisymmetric,” “square,” and “cubic” oscillations, respectively.

We describe the \((\theta, \nu, w)\) equations for each of these periodic oscillations in turn. In the last two cases, the isotropy subgroup \(G_p\) is finite, and so \(N_0 = g^* = so(3)^* \oplus se(3)^*\) with its natural \(\chi\)-coadjoint action of \(L_n\). In both cases, the symplectic normal space is a two dimensional semisymplectic representation of \(L_n\). These representations can be read from Table 6 of [37]. For the square case, it is given by the nontrivial representation of \(D_4\) on \(\mathbb{C}\) with kernel \(D_2\). The time reversal symmetry \(\rho\) acts by conjugation on \(\mathbb{C}\). In the cubic case, it
is the representation of $\mathbb{O}^\rho$ that factors through the two dimensional irreducible representation of $\mathbb{O}^\rho/\mathbb{D}_2 \cong \mathbb{D}_3$. The $(\theta, \nu, w)$ equations in both cases have the form (5.2) with $\zeta$ replaced by $\nu$, $h$ an $L_\alpha$-invariant function of $(\nu_R, \nu_L, \nu_T, w, \theta)$ and the addition of the equation $\dot{\theta} = 1$.

For the axisymmetric oscillations, $g_p = so(2)_D$, and so

$$N_0 = (g/g_p)^* = (so(3)_D/so(2)_D)^* \oplus so(3)_{AD} \oplus (\mathbb{R}^3)^*.$$  

Table 6 in [37] shows that the symplectic normal space $N_1$ is the four dimensional irreducible symplectic representation of $O(2)$ on $\mathbb{C}^2$ with kernel $\mathbb{D}_2$ and $\rho$ acting by conjugation. It can be identified with the subspace of $sl(3) \otimes \mathbb{C} \subset saff(3) \otimes \mathbb{C} \cong saff(3) \oplus saff(3)^*$ consisting of symmetric traceless matrices of the form

$$A = \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b \in \mathbb{C}.$$  

In these coordinates, the momentum map ("vibrational angular momentum") $L_{N_1} : N_1 \rightarrow so(2)_D$ is $L_{N_1}(A) = \frac{1}{2}(AA - A\bar{A})$ (see Lemma 5.6 of [37]). The Hamiltonian $h$ is a function of $\nu_{AD} \in so(3)_{AD}$, $\nu_D \in (so(3)_D/so(2)_D)^*$, $\nu_T \in (\mathbb{R}^3)^*$, $A$, $\bar{A}$, and $\theta$. The $N_0$ part of the slice equations is easily obtained from (5.3) by replacing

$$\zeta_D \mapsto (\nu_D, L_{N_1}(w)), \quad \frac{\partial h}{\partial \zeta_D} \mapsto \left( \frac{\partial h}{\partial \nu_D} \right)$$  

and by projecting the $\dot{\zeta}_D$ equation to $\nu_D$. To these must be added the equations

$$\dot{\theta} = 1, \quad \dot{A} = -2i \frac{\partial h}{\partial A}, \quad \text{where} \quad A \simeq (a, b) \in \mathbb{C}^2.$$  

The second of these equations is the $w$ equation written in appropriate complex coordinates.

For all three cases, the component $M_1 : N_1 \rightarrow N_1$ of the linearizations at the periodic orbits will be equal to the "reduced Floquet operators" computed in [37] in terms of coefficients in the Taylor series expansion of the Hamiltonian at the spherical equilibrium. The results given there, combined with Corollary 4.5 and the fact that $A\bar{A}$ is always spectrally stable for compact and Euclidean groups, imply that the cubic oscillations are always spectrally stable (since the representations of $\Gamma_p$ and $G_p$ on $N_1$ are cyclospectral), while typically either the axisymmetric oscillations or the square oscillations are spectrally stable but not both.

### 5.4. Relative periodic orbits

In this final subsection, we use Theorem 4.9 to describe some relative periodic orbits that will typically bifurcate from the square and cubic oscillations of the previous subsection as $J_L$ and $J_R$ are perturbed away from 0. We assume that the original normal modes are nondegenerate in the sense required in Theorem 4.9, an assumption which is generically satisfied. In both of these cases, $\Gamma_p = \mathbb{D}_2 \times \mathbb{Z}_2^\ast$.

The coadjoint orbits $\Gamma \nu$ for the action of $\Gamma = \mathbb{Z}_2^\ast \times (\text{SO}(3) \times \text{SE}(3))$ on $g^* = so(3)^* \oplus se(3)^* = so(3)_R^* \oplus so(3)_L^* \oplus (\mathbb{R}^3)^*$ are given by $O_\nu = O_{\nu_R, \nu_L, \nu_T} = O_{\nu_R} \times O_{\nu_L, \nu_T}$, where

$$O_{\nu_R} = \{ \hat{\nu}_R \in so(3)^* : ||\nu_R|| = ||\hat{\nu}_R|| \};$$  

$$O_{\nu_L, \nu_T} = \begin{cases} \{ (\hat{\nu}_L, \hat{\nu}_T) \in se(3)^* : \hat{\nu}_T = 0, ||\hat{\nu}_L|| = ||\nu_L|| \} & \text{if } \nu_T = 0, \\ \{ (\hat{\nu}_L, \hat{\nu}_T) \in se(3)^* : \hat{\nu}_L, \hat{\nu}_T = \nu_L, \nu_T, ||\nu_L|| = ||\nu_T|| \} & \text{if } \nu_T \neq 0. \end{cases}$$
Thus $O_{\nu R}$ is either a point or a two-sphere, while $O_{\nu L, \nu T}$ is a point or a two-sphere or is diffeomorphic to the tangent bundle of a two-sphere.

The isotropy subgroups $\hat{\Gamma}_p$ of the actions of $\Gamma_p = D_2 \times Z_2^\ast$ on the coadjoint orbits $\Gamma \nu$ can be computed easily. The action of $Z_2^\ast$ on $so(3)_{\nu}^\ast$ is trivial, while that of $D_2$ has one dimensional fixed point subsets for each of its three $Z_2$ subgroups. It follows that if $\tau$ is a nonidentity element of $D_2$ and $\nu_R \neq 0$, the two-sphere $O_{\nu R}$ has precisely 2 points with isotropy subgroup $Z_2^\tau \times Z_2^\ast$. The same is true for the two-sphere coadjoint orbits $O_{\nu L, 0}$ in $se(3)_{\nu}^\ast$. So if $\nu_T = 0$ and $\nu_R \neq 0 \neq \nu_L$, then $O_{\nu}$ has precisely four points with isotropy subgroup $Z_2^\nu \times Z_2^\ast$. On the $(R^3)^*$ component of $se(3)^*$ the operation $\kappa$ acts by $-id$, while $D_2$ acts in the same way as on $so(3)^*$. The isotropy subgroups with one dimensional fixed point spaces for $\nu_T \in (R^3)^*$ are therefore equal to $Z_2^\tau \times Z_2^{\tau \ast}$, where $\tau$ and $\hat{\tau}$ are any two distinct nonidentity elements in $D_2$.

It follows that if $\nu_T \neq 0$ but $\nu_R = 0$ and $\nu_L, \nu_T = 0$, then $O_{\nu}$ has two points with isotropy group equal to $Z_2^\nu \times Z_2^{\tau \ast}$ lying in the $\{ (\nu_R, \nu_L) = 0 \}$-plane. If $\nu_T \neq 0$, $\nu_R = 0$, and $\nu_L, \nu_T \neq 0$, then $O_{\nu}$ has two points with isotropy group equal to $Z_2^\nu$, while if $\nu_T \neq 0$ and $\nu_R \neq 0$, then $O_{\nu}$ has four points with isotropy group equal to $Z_2^\nu$.

Summarizing, the subgroups $\hat{\Gamma}_p$ with zero dimensional fixed point sets $\nu \cap Fix_{\hat{\Gamma}_p}$ $(g^*)$ are given in the following table. In all these fixed point spaces, $\hat{\nu}_R$, $\hat{\nu}_L$, and $\hat{\nu}_T$ are parallel to each other.

\begin{center}
\begin{tabular}{|l|l|l|}
\hline
Orbit $\Gamma \nu$ & Isotropy $\hat{\Gamma}_p$ & Fixed point set \\
\hline
1. $\nu_T = 0$, $(\nu_R, \nu_L) \neq 0$ & $Z_2^\nu \times Z_2^\ast$ & $\hat{\nu}_T = 0$, $\hat{\nu}_R \mid | \hat{\nu}_L \mid | \tau$ \\
2. $\nu_T \neq 0$, $\nu_R = 0$, $\nu_L, \nu_T = 0$ & $Z_2^\nu \times Z_2^{\tau \ast}$ & $\hat{\nu}_T \mid | \tau$, $\hat{\nu}_R = \hat{\nu}_L = 0$ \\
3. $\nu_T \neq 0$, $\nu_R \neq 0$ or $\nu_L, \nu_T \neq 0$ & $Z_2^\nu$ & $\hat{\nu}_T \mid | \nu_R \mid | \nu_L \mid | \tau$ \\
\hline
\end{tabular}
\end{center}

In cases 1 and 3, the element $\tau$ is any of the nonidentity elements of $D_2 = SO(3)_D \cap \Gamma_p$, and in case 2, the elements $\tau$ and $\hat{\tau}$ are two different nonidentity elements in $D_2$. The notation $\nu \mid | \tau$ means that $\nu$ is parallel to the axis fixed by $\tau$.

By Theorem 4.9 the momentum $\hat{\mu}$ of the relative periodic orbits corresponding to a fixed point $\hat{\nu} = (\hat{\nu}_R, \hat{\nu}_L, \hat{\nu}_T)$ is given simply by $\hat{\mu} = \hat{\nu}$. The momentum isotropy subgroups $G_{\hat{\mu}}$ are the isotropy subgroups at $\hat{\mu}$ for the action of $G = Z_2^\nu \times Z_2^{\nu \ast} \times (SO(3) \times SE(3))$ on $g^* = so(3)^* \oplus se(3)^*$ and are easily calculated to be the following:

\begin{center}
\begin{tabular}{|l|l|}
\hline
Momentum $\hat{\mu}$ & Momentum isotropy $G_{\hat{\mu}}$ \\
\hline
(1a) $\hat{\mu}_T = 0$, $\hat{\mu}_R \neq 0$, $\hat{\mu}_L = 0$ & $Z_2^\nu \times (O(2)^{\ast}_{\nu} \times SE(3))$ \\
(1b) $\hat{\mu}_T = 0$, $\hat{\mu}_R \neq 0$, $\hat{\mu}_L = 0$ & $Z_2^\nu \times ((SO(3)_R \times O(2)^{\ast}_{\nu}) \times R^3)$ \\
(1c) $\hat{\mu}_T = 0$, $\hat{\mu}_R \neq 0$, $\hat{\mu}_L \neq 0$ & $Z_2^\mu \times ((O(2)_R \times O(2)_L)^{\rho} \times R^3)$ \\
(2) $\hat{\mu}_T \neq 0$, $\hat{\mu}_R = 0$, $\hat{\mu}_L = 0$ & $Z_2^{\mu \rho} \times (SO(3)_R \times O(2)^{\rho}_{\nu} \times R)$ \\
(3a) $\hat{\mu}_T \neq 0$, $\hat{\mu}_R = 0$, $\hat{\mu}_L \neq 0$ & $SO(3)_R \times (O(2)^{\mu}_{\nu} \times R)$ \\
(3b) $\hat{\mu}_T \neq 0$, $\hat{\mu}_R \neq 0$ & $(O(2)_R \times O(2)_L)^{\rho} \times R$ \\
\hline
\end{tabular}
\end{center}

In all cases, $\hat{\mu}_R$, $\hat{\mu}_L$, and $\hat{\mu}_T$ are parallel to each other. The group $O(2)^{\rho}_{\nu}$ is the subgroup of $SO(3)_R \times Z_2^{\mu \rho}$ consisting of all rotations about a fixed axis together with $\rho$ composed with rotations by $\pi$ about axes perpendicular to this fixed axis. The group $O(2)_L^{\rho}$ is the analogous subgroup of $SE(3) \times Z_2^{\mu \rho}$, and $(O(2)_R \times O(2)_L)^{\rho}$ is the group consisting of all rotations about a fixed axis in $SO(3)_R$, all rotations about the same fixed axis in $SO(3)_L$, together with $\rho$.
composed with simultaneous rotations by $\pi$ about axes perpendicular to this fixed axis in $SO(3)_R$ and $SE(3)_L$.

For each of the points with one of the isotropy subgroups $\hat{\Gamma}_p$, we can now compute $\hat{L} := L_n \cap G_p$. The results are shown in Table 5.1 for the square case and in Table 5.2 for the cubic case. The tables also give the symmetry data of the bifurcating relative periodic orbits: the new (reversing) isotropy $\hat{G}_p$, the new relative period $\ell$, the new spatio-temporal symmetry $\hat{\sigma}$, and the new drift $\hat{\xi}$, computed as in Theorem 4.9. In each case, $\hat{\sigma}$ is equal to $\sigma^\ell \gamma_p \exp(\hat{\xi})$, where $\sigma$ is the spatio-temporal symmetry for the original oscillation, $\ell \in \mathbb{N}$ is minimal such that $\sigma^\ell \gamma_p \in \hat{L}$ for some $\gamma_p \in \Gamma_p$, and $\hat{\xi} = (\tilde{\xi}_R, \xi_L, \tilde{\xi}_T) \approx 0$ must lie in the fixed point subspace of the $\chi$-dual action of $G_p$ on $so(3) \oplus se(3)$. For the square relative periodic orbit of type (i), we have $\ell = 2$, and so $\sigma^\ell = id = \gamma_p$. For the square relative periodic orbit of type (ii), we have $\ell = 1$, $\gamma_p$ is the rotation by $\pi$ about one of the principal axes undergoing large amplitude oscillations, and $\sigma^\ell \gamma_p = \tau^{\frac{1}{2}}$, i.e., a rotation by $\pi/2$ about the axis of $\tau$. For the cubic relative periodic orbits, $\ell = 3$ and $\sigma^\ell = id = \gamma_p$. In all cases, the forms of the possible $\hat{\xi}$'s are shown in the final columns of the tables.

**Table 5.1**

Symmetries of relative periodic orbits bifurcating from the square oscillations. The group $D_4^\rho$ is $Z_2^2 \times Z_2^\rho_{op}$, where $\tau$ and $\tilde{\tau}$ are two different nonidentity elements in $D_2 = SO(3) \cap \Gamma_p$. The group $D_4^\rho_{op}$ is generated by the rotation $\tau^{\frac{1}{2}}$ by $\pi/2$ about the $\tau$-axis and by $\tilde{\tau} \circ \rho$.

<table>
<thead>
<tr>
<th></th>
<th>$\Gamma_p$</th>
<th>$G_p$</th>
<th>$L$</th>
<th>$\ell$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\xi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (i)</td>
<td>$Z_2^2 \times Z_2^2$</td>
<td>$D_2^\rho \times Z_2^\rho_{op}$</td>
<td>$D_2^\rho \times Z_2^\rho_{op}$</td>
<td>2</td>
<td>$\exp(\hat{\xi})$</td>
<td>$\xi_R \parallel \xi_L \parallel \tau$, $\xi_T = 0$</td>
</tr>
<tr>
<td></td>
<td>(ii)</td>
<td></td>
<td></td>
<td>2</td>
<td>$\tau^{\frac{1}{2}} \exp(\hat{\xi})$</td>
<td></td>
</tr>
<tr>
<td>2. (i)</td>
<td>$Z_2^2 \times Z_2^{\rho_{op}}$</td>
<td>$D_2^\rho \times Z_2^{\rho_{op}}$</td>
<td>$D_2^\rho \times Z_2^{\rho_{op}}$</td>
<td>2</td>
<td>$\exp(\hat{\xi})$</td>
<td>$\xi_R = \xi_L = 0$, $\xi_T \parallel \tau$</td>
</tr>
<tr>
<td></td>
<td>(ii)</td>
<td></td>
<td></td>
<td>2</td>
<td>$\tau^{\frac{1}{2}} \exp(\hat{\xi})$</td>
<td></td>
</tr>
<tr>
<td>3. (i)</td>
<td>$Z_2^2$</td>
<td>$D_2^\rho$</td>
<td>$D_2^\rho$</td>
<td>2</td>
<td>$\exp(\hat{\xi})$</td>
<td>$\xi_R \parallel \xi_L \parallel \xi_T \parallel \tau$</td>
</tr>
<tr>
<td></td>
<td>(ii)</td>
<td></td>
<td></td>
<td>2</td>
<td>$\tau^{\frac{1}{2}} \exp(\hat{\xi})$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5.2**

Symmetries of relative periodic orbits bifurcating from the cubic oscillations; $\tau$ and $\tilde{\tau}$ are two different nonidentity elements in $D_2 = SO(3) \cap \Gamma_p$. The group $D_4^\rho_{op}$ is $Z_2^2 \times Z_2^\rho_{op}$, where $\tilde{\tau}$ is a rotation by $\pi$ about an axis perpendicular to the $\tau$-axis and inclined at an angle of $\pi/4$ to the $\tilde{\tau}$-axis. The group $D_4^\rho_{op}$ is generated by $D_2^\rho$ and $\tilde{\tau} \circ \kappa$.

<table>
<thead>
<tr>
<th></th>
<th>$\Gamma_p$</th>
<th>$G_p$</th>
<th>$L$</th>
<th>$\ell$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\xi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$Z_2^2 \times Z_2^2$</td>
<td>$D_2^\rho \times Z_2^\rho_{op}$</td>
<td>$D_2^\rho \times Z_2^\rho_{op}$</td>
<td>3</td>
<td>$\exp(\hat{\xi})$</td>
<td>$\xi_R \parallel \xi_L \parallel \tau$, $\xi_T = 0$</td>
</tr>
<tr>
<td>2.</td>
<td>$Z_2^2 \times Z_2^{\rho_{op}}$</td>
<td>$D_2^\rho_{op}$</td>
<td>$D_2^\rho_{op}$</td>
<td>3</td>
<td>$\exp(\hat{\xi})$</td>
<td>$\xi_R = \xi_L = 0$, $\xi_T \parallel \tau$</td>
</tr>
<tr>
<td>3.</td>
<td>$Z_2^2$</td>
<td>$D_2^\rho$</td>
<td>$D_2^\rho$</td>
<td>3</td>
<td>$\exp(\hat{\xi})$</td>
<td>$\xi_R \parallel \xi_L \parallel \xi_T \parallel \tau$</td>
</tr>
</tbody>
</table>

In the case of square oscillations, for each of the different spatial isotropy subgroups $\hat{\Gamma}_p$, the cases indicated by (i) and (ii) in Table 5.1 give two qualitatively distinct types of bifurcating relative periodic orbits. In the cases labelled by (i), the “angular velocities” $\xi_L$, $\xi_R$ and “linear velocity” $\xi_T$ are all aligned with one of the axes of the pulsating cube with large amplitude
oscillations, while in the cases labelled by (ii), they are aligned with the third axis with much smaller amplitude oscillations. In case (i), the relative period doubles, while in case (ii), the relative period remains approximately the same. For the relative periodic orbits of types 1(i) and 1(ii), the linear velocity is zero, but there may be both body and spatial rotations, and so the bifurcating relative periodic orbits are modulated rotating waves. For cases 2(i) and 2(ii), only the linear velocity is nonzero, and the body translates in space without rotating. For cases 3(i) and 3(ii), rotation and translation both occur. So in cases 2 and 3, the bifurcating relative periodic orbits are modulated travelling waves. All bifurcating relative periodic orbits are reversible.

The case of bifurcations from the cubic oscillations is completely analogous, except that now there is no distinction between the three principal axes of the pulsating cube, and so there is only one type of bifurcating relative periodic orbit. These may again have body and spatial rotations only, translation only, or all three, and the relative period always triples.

In future work, we will extend the bifurcation results used here and apply them to show that a number of other types of relative periodic orbits bifurcate from the square and cubic oscillations and to find relative periodic orbits bifurcating from the axisymmetric oscillations.

6. Proofs. This section is devoted to the proofs of the main theorems. The proofs build on the construction of coordinates near relative periodic orbits of general systems that we describe in section 6.1. In the subsequent subsections, we show how to adapt this bundle construction to Hamiltonian systems. First, in subsection 6.2, the symplectic structure of the tangent space decomposition at a point \( p \) on a relative periodic orbit is described. Then, in subsection 6.3, we analyze the linearization at a point \( p \) of the relative periodic orbit as this is needed for the construction of the bundle coordinates. In subsections 6.4 and 6.5, we present the adaptations of the bundle construction of subsection 6.1 to Hamiltonian systems. In subsections 6.6 and 6.7, we describe the symplectic structure of the bundle. Finally, in subsection 6.8, we derive the differential equations in bundle coordinates.

6.1. The bundle construction for general systems. In this section, we describe the construction of coordinates near relative periodic orbits of general systems. Most of this section summarizes results of [52, 55, 22].

As always, let \( \Gamma \) be algebraic, let \( p = \sigma^{-1}\Phi_1(p) \) lie on a relative periodic orbit of relative period 1, and let \( M = \sigma^{-1}D\Phi_1(p) \). Furthermore, let \( P \) be a \( G_p \)-equivariant projection from \( T_p\mathcal{M} \) to the \( G_p \)-invariant Poincaré section (or normal space) \( \mathcal{P} \) at \( p \) with kernel \( T_p\mathcal{P} = T \). According to [52, 55, 22], there is a smooth family \( N(\theta) \) of \( \Gamma_p \)-invariant Poincaré sections to \( \mathcal{P} \) at \( \Phi_\theta(p) \) such that \( N(0) = N \), \( N(\theta) \oplus T_{\Phi_\theta(p)}\mathcal{P} = T_{\Phi_\theta(p)}\mathcal{M} \), where \( T_{\Phi_\theta(p)}\mathcal{P} = \text{span}(f(\Phi_\theta(p)) \oplus g\Phi_\theta(p)) \), and

\[
N(\theta + 1) = \sigma N(\theta), \quad \rho N(\theta) = N(-\theta) \quad \text{for} \quad \rho \in G_p \setminus \Gamma_p.
\]

Let \( P(\theta) \) be the projection from \( T_{\Phi_\theta(p)}\mathcal{M} \) onto \( N(\theta) \) with kernel \( \ker P(\theta) = T_{\Phi_\theta(p)}\mathcal{P} \). Then \( P(\theta) \) is smooth in \( \theta \), \( P(\theta + 1) = \sigma P(\theta), \) \( P(0) = P \), and \( P(\theta) \) is \( G_p \)-semiequivariant:

\[
P(\theta) = g_\rho^{-1}P(\chi(g_\rho)\theta)g_\rho, \quad g_\rho \in G_p.
\]

Further, by [22, Lemma 5.1] (see Lemma 6.5 below), there is a \( \Gamma_p \)-equivariant homotopy
\( I_N(\theta) \in \text{GL}(N) \) depending smoothly on \( \theta \) and such that

\[
I_N(0) = \text{id}, \quad M_N I_N(\theta + 1) = I_N(\theta)Q_N^{-1}, \quad \rho I_N(\theta)\rho^{-1} = I_N(-\theta), \quad \rho \in G_p \setminus \Gamma_p,
\]

where \( M_N := PM|_N \) and \( Q_N^{-1} \) is twisted semiequivariant and has finite order \( 2n \).

The parametrization of a \( G \)-invariant neighborhood \( U \) of \( P \) in \( \mathcal{M} \) is then given by a submersion \( \tau : G \times N \times \mathbb{R} \rightarrow U \) defined by

\[
u = \tau(g, \theta, v) = g \exp(-\theta \xi) \psi(\Phi(\theta)(p), P(\theta)D\Phi(\theta)I_N(\theta)v),
\]

where \( \psi \) is a \( G \)-equivariant diffeomorphism from a neighborhood of \( P \) in its normal bundle to \( U \).

In this paper, we will construct the Poincaré sections \( N(\theta) \) in a slightly different way from the method used in [52, 55, 22]. We will show in section 6.5, Lemma 6.6, that there is a homotopy \( I(\theta) \in \text{GL}(T_p\mathcal{M}) \) which is \( G_p \)-semiequivariant:

\[
I(\theta) = g_p^{-1}I(\chi(g_p)\theta)g_p, \quad g_p \in G_p,
\]

and such that

\[
MI(\theta + 1) = I(\theta)Q^{-1}, \quad I(0) = \text{id},
\]

where \( Q = \text{diag}(Q_T, Q_N) \), \( Q_T \) is an orthogonal transformation of \( T \) of finite order \( 2n \), \( Q^{-1} \) is twisted semiequivariant, and \( I(\theta) \) has block structure

\[
I(\theta) = \begin{pmatrix} I_T(\theta) & I_D(\theta) \\ 0 & I_N(\theta) \end{pmatrix}
\]

with \( I_N(\theta) \) satisfying (6.1). We then define

\[
N(\theta) := D\Phi(\theta)I(\theta)N.
\]

This gives \( \Gamma_p \)-invariant Poincaré sections \( N(\theta) \) with the above properties, and we get

\[
D\Phi(\theta)I(\theta)|_N = P(\theta)D\Phi(\theta)I_N(\theta).
\]

**6.2. Symplectic structure of the tangent space decomposition.** Again, let \( p = \sigma^{-1}\Phi(\theta)(p) \) lie on a relative periodic orbit \( \mathcal{P} \), and let \( T = T_0 \oplus T_1 \oplus T_2 \) be the refinement of the \( G_p \)-invariant tangent space to \( \mathcal{P} \) at \( p \) given in (4.3). In this subsection, we show that there is a \( G_p \)-invariant Poincaré section \( N \subset T_p\mathcal{M} \) to \( \mathcal{P} \) at \( p \) such that the refinement \( N = N_0 \oplus N_1 \oplus N_2 \) defined in (3.6) holds true, and we discuss the symplectic structure of this decomposition of the tangent space \( T_p\mathcal{M} = T \oplus N \). Define the \( \omega \)-orthogonal complement of any subspace \( V \subset T_p\mathcal{M} \) to be

\[
V^\omega = \{ u \in T_p\mathcal{M} : \omega(u, v) = 0 \text{ for all } v \in V \}.
\]

**Lemma 6.1.** Let \( p \) lie on a relative periodic orbit with relative period different from zero. Then the following hold.
(a) The vector $f_H(p) \in T_2$ is $G_p$-semi-invariant and linearly independent of $T_pG_p$ and lies in $\ker DH(p) \cap \ker DJ(p)$.

(b) $DH(p)$ is linearly independent of the vectors $DJ_{\xi}(p)$, $\xi \in g$, and there is a $G_p$-invariant vector $v_E \in \ker DJ(p)$ with $DH(p)v_E \neq 0$ such that $T_pM = \text{span}(v_E) \oplus \ker DH(p)$.

(c) $\ker DJ(p) = (g_p)^{\omega}$, $\ker DH(p) = T_2^\omega$, $T \subset \ker DH(p)$, and $T \cap \ker DJ(p) = T_0 \oplus T_2$.

Proof. To prove part (a) note that $G_p$-semi-invariance of $f_H(p)$ follows from $G_p$-semiequivalence of $f_H$. Since $P$ is a relative periodic orbit and not a relative equilibrium, $T_pG_p$ and $f_H(p)$ are linearly independent. The Hamiltonian $H$ and the momentum $J$ are preserved by the Hamiltonian flow of (3.1), and so $f_H(p) \in \ker DH(p) \cap \ker DJ(p)$.

To prove part (b) observe that if $DH(p) = DJ_{\xi}(p)$ for some $\xi \in g$, then $p$ lies on a relative equilibrium, which we exclude. Hence there is some $v_E \neq 0$ with $v_E \in \ker DJ(p)$, but $DH(p)v_E \neq 0$. Since $\ker DH(p)$ has codimension 1 in $T_pM$, we conclude that $\ker DH(p)$ and $v_E$ span $T_pM$. We have $g_pv_E = \pm v_E$ for each $g_p \in G_p$ because $N_2$ is one dimensional and $G_p$-invariant. Since $H$ is $G_p$-invariant, $DH(p)g_p = DH(p)$ for all $g_p \in G_p$, and, therefore, $0 \neq DH(p)g_pv_E = DH(p)v_E$, which proves that $v_E$ is $G_p$-invariant.

The first two equations in part (c) follow from

\[(6.6) \quad \omega(f_H(p), v) = DH(p)v, \quad \omega(\xi, v) = DJ_{\xi}(p)v, \quad v \in T_pM, \quad \xi \in g.\]

By $G$-invariance of $H$ and part (a) we have $T \subseteq \ker DH(p)$, which proves the third equation of (c). Because of (a) we have $T_2 \subseteq \ker DJ(p)$, and by $G$-equivariance of $J$ we get $DJ(p)\xi = \xi J(p)$, which vanishes if and only if $\xi \in g_{\mu}$. This proves that $T_0 \subseteq \ker DJ(p)$ and $T_1 \cap \ker DJ(p) = \{0\}$.

The following proposition generalizes the usual Witt decomposition at group orbits to relative periodic orbits.

**Proposition 6.1.** Let $p \in M$ lie on a relative periodic orbit $P$ with relative period different from zero. Then there is a $G_p$-invariant Poincaré section $N$ to $P$ at $p$ such that the following are true.

(a) Equation (3.6) holds, and the spaces $T_i$, $N_i$, $i = 0, 1, 2$, are all $G_p$-invariant.

(b) The symplectic form $\omega$ on $T_pM$ restricts to symplectic forms $\omega_{T_0\oplus N_0}$ on $T_0 \oplus N_0$, $\omega_{T_1}$ on $T_1$, $\omega_{N_1}$ on $N_1$, and $\omega_{T_2\oplus N_2}$ on $N_2 \oplus T_2$. The actions of $G_p$ on these spaces are $\chi$-semisymplectic with respect to the restricted forms. Moreover,

\[\omega|_{T_0M} = \omega_{T_0\oplus N_0} + \omega_{T_2} + \omega_{N_1} + \omega_{T_2\oplus N_2}.\]

(c) $\ker DJ(p) = T_0 \oplus T_2 \oplus N_1 \oplus N_2$; $\ker DH(p) = T \oplus N_0 \oplus N_1$.

(d) Identify $g_{\mu}/g_p \cong T_0$ via the map $g \rightarrow T_pM$ given by $\xi \mapsto \xi p$. The symplectic form $\omega$, or, equivalently, the map $v \mapsto DJ(p)(\cdot)v$ (see (6.6)), defines a $G_p$-equivariant isomorphism between the induced $G_p$-action on $N_0$ and the $\chi$-coadjoint action on $T_0^* \cong (g_{\mu}/g_p)^*$. Similarly, $\xi \mapsto DJ_{\xi}(p)$ defines a $G_p$-equivariant isomorphism between $T_0$ and $N_0^*$ such that $N_0^* = DJ(p)(g_{\mu}/g_p)$ is the annihilator of $T \oplus N_1 \oplus N_2$. Under the first isomorphism, the symplectic form $\omega_{T_0\oplus N_0}$ becomes the natural symplectic form on $(g_{\mu}/g_p) \oplus (g_{\mu}/g_p)^*$:

\[\omega_{T_0\oplus N_0}(\xi_1, \nu_1), (\xi_2, \nu_2)) = \nu_2(\xi_1) - \nu_1(\xi_2).\]
(c) DJ(p) maps $T_\gamma$ isomorphically to $T_\mu(G\mu) \cong g/\mathfrak{g}_\mu$ and $\omega_{T_1}$ to the Kostant–Kirillov–Souriau (KKS) form $\omega_\mu$ (in body coordinates):

\[
\omega_{T_1}(\xi_1, p, \xi_2, p) = \omega_\mu(\xi_1, \xi_2) := \mu([\xi_1, \xi_2]),
\]

where $\xi_i \in \mathfrak{g}$, $i = 1, 2$, and $[,]$ is the Lie bracket on $\mathfrak{g}$.

(f) The symplectic form $\omega$ defines $G_p$-equivariant isomorphisms between $T_2^*$ and $N_2$ and between $T_2$ and $N_2^1$ such that $N_2^1 = \text{ann}(T \oplus N_0 \oplus N_1)$ is spanned by $DH(p)$. Under these isomorphisms, the symplectic form $\omega_{T_2\oplus N_2}$ becomes the standard symplectic structure $\omega_{T_2\oplus N_2}(\mathbf{1}, \mathbf{0}) = (E_1, \theta_1), (E_2, \theta_2)) = E_2\theta_1 - E_1\theta_2$.

**Proof.** The Witt decomposition near group orbits (see, for example, [4, 36]) gives $T_pM = \hat{T} \oplus \hat{N}$, where $\hat{T} = T_pG = T_0 \oplus T_1$ and $\hat{N} = \hat{N}_0 \oplus \hat{N}_1$. Here the symplectic normal space $\hat{N}_1$ to $Gp$ at $p$ is a $G_p$-invariant complement to $T_0$ in $\ker DJ(p)$, and the space $\hat{N}_0$ is a $G_p$-invariant complement to $T_1 + \ker DJ(p)$, which is chosen so that $\hat{N}_1 \oplus T_1 \subset \hat{N}_0^{-1}$. Now we show how to adapt this Witt decomposition to relative periodic orbits.

(a) We choose $\hat{N}_1$ to contain $T_2$ and $v_E$, which is possible by Lemma 6.1 (a), (b). Since $T_2 \subset \hat{N}_1 \subset \hat{N}_0^\omega$, we conclude from (6.6) that $\hat{N}_0 \subset \ker DH(p)$ and therefore define $\hat{N}_0 := \hat{N}_0$. The symplectic form $\omega_{\hat{N}_1}$ restricts to a symplectic form on $T_2 \oplus N_2$ because $\omega(f_H(p), v_E) = DH(p)v_E \neq 0$. Hence $N_1 := (T_2 \oplus N_2)^\omega \cap \hat{N}_1$ is also a symplectic space which is transverse to $T_2 \oplus N_2$ and, because of (6.6), satisfies $N_1 \subset \ker DH(p)$. With this construction, $\hat{N} = \hat{N}_1 \oplus T_2 \oplus N_2$, and (3.6) follows.

By definition $T_0$ and $T_1$ are $G_p$-invariant. By Lemma 6.1 (a), (b) the spaces $T_2$ and $N_2$ are $G_p$-invariant, and the above construction implies that $N_0$ and $N_1$ are also $G_p$-invariant.

(b) This follows from the usual Witt decomposition and the proof of (a).

(c) Because of Lemma 6.1 (c) and since $N_0 \oplus N_1 = \ker DH(p) \cap N_1$ holds. By definition $N \cap \ker DJ(p) = N_1 \oplus N_2$, which proves that $\ker DJ(p) = T_0 \oplus T_2 \oplus N_1 \oplus N_2$.

(d) From (c) we conclude that $T_0 \oplus T_2 \oplus N_1 \oplus N_2$ is annihilated by $N_0^* = DJ(p)g_\mu/g_\mu$. Now let $\eta \in \mathfrak{n}_\mu, \xi \in \mathfrak{g}_\mu$. Then

\[
DJ_\xi(p)\eta = (\eta J)(\xi)(p) = J([\eta, \xi])(p) = -(\xi J)(\eta)(p) = 0,
\]

which proves that $DJ(p)g_\mu/g_\mu$ annihilates $T_1$. The other statements follow from the usual Witt decomposition near group orbits.

(e) This follows from the usual Witt decomposition.

(f) That $N_2^*$ is spanned by $DH(p)$ follows from (6.6), and that it annihilates $T \oplus N_0 \oplus N_1$ follows from (c). □

### 6.3. Linearization along the relative periodic orbit.

The following three lemmas together with Proposition 4.1 prove Proposition 4.3 on the linearization near relative periodic orbits.

**Lemma 6.2.** Let $p \in M$, and let $\mathfrak{m}_\mu, \mathfrak{n}_\mu$ be $G_p$-invariant complements to $g_\mu$ in $g_\mu$ and to $g_\mu$ in $g$, respectively.

(a) Let $\gamma \in \Gamma_\mu(\Gamma_p)$. Then with respect to the decomposition $g = \mathfrak{m}_\mu \oplus \mathfrak{n}_\mu \oplus g_\mu$ the matrix
Ad_\gamma has the following block structure:

\[
\text{Ad}_\gamma = \begin{pmatrix}
\text{Ad}_\gamma & \pi_{\mathfrak{m}_\mu} \text{Ad}_\gamma |_{\mathfrak{n}_\mu} & 0 \\
0 & \pi_{\mathfrak{n}_\mu} \text{Ad}_\gamma |_{\mathfrak{n}_\mu} & 0 \\
\pi_{\mathfrak{g}_\mu} \text{Ad}_\gamma |_{\mathfrak{m}_\mu} & \pi_{\mathfrak{g}_\mu} \text{Ad}_\gamma |_{\mathfrak{n}_\mu} & \text{Ad}_\gamma |_{\mathfrak{g}_\mu}
\end{pmatrix}.
\]

Here \(\pi_{\mathfrak{m}_\mu}\), \(\pi_{\mathfrak{n}_\mu}\), and \(\pi_{\mathfrak{g}_\mu}\) are the projections from \(\mathfrak{g}\) to \(\mathfrak{m}_\mu\), \(\mathfrak{n}_\mu\), and \(\mathfrak{g}_\mu\) with kernels \(\mathfrak{n}_\mu \oplus \mathfrak{g}_\mu\), \(\mathfrak{m}_\mu \oplus \mathfrak{g}_\mu\), and \(\mathfrak{m}_\mu \oplus \mathfrak{n}_\mu\).

(b) If \(\sigma \in N_{\Gamma_\mu}(\Gamma_p)\) has the form \(\sigma = \alpha \exp(\xi)\), where \(\xi \in \mathfrak{m}_\mu \cap \mathfrak{z}(\sigma) \cap \mathfrak{z}(\Gamma_p)\), and \(\text{Ad}_\alpha\) leaves \(\mathfrak{m}_\mu\) invariant, then \(\overline{\text{Ad}_\sigma} = \text{Ad}_\alpha \exp(\overline{\text{ad}_\xi})\).

**Proof.** (a) is clear. To prove (b) note that since \(\sigma, \alpha \in N_{\Gamma_\mu}(\Gamma_p)\) and \(\exp(\xi \theta) \in N_{\Gamma_\mu}(\Gamma_p)\) for all \(\theta \in \mathbb{R}\) and \(\xi \in \mathfrak{LZ}_{\Gamma_\mu}(\Gamma_p)\), the representations of their adjoint actions on \(\mathfrak{g}\) have the block structure given in part (a). The statement then follows from the fact that \(\mathfrak{m}_\mu\) is \(\text{Ad}_\alpha\)-invariant. \(\blacksquare\)

**Lemma 6.3.** Let \(p\) lie on a relative periodic orbit with minimal period 1, and let \(M = \sigma^{-1} D\Phi_1(p)\). Then the following hold.

(a) \(N^*_2 = \text{span}\{D\mathfrak{H}(p)\}\) is a left eigenspace of \(M\) with eigenvalue 1.

(b) We have \(DJ_\xi(p) M = DJ_{\text{Ad}_\xi(p)}\) for each \(\xi \in \mathfrak{g}\) and therefore \(M^*|_{N^*_2} = \overline{\text{Ad}}_\sigma\), where \(N^*_2 = DJ(p)(\mathfrak{m}_\mu)\).

(c) The spaces \(T_0 \oplus T_2 \oplus N_1\) and \(T_0 \oplus T_2 \oplus N_1 \oplus N_2\) are \(M\)-invariant.

**Proof.** That \(D\mathfrak{H}(p)\) is a left eigenvector of \(M\) with eigenvalue 1 follows from the \(G\)-invariance and conservation of \(H\). The first statement of part (b) is a direct computation which we omit. For part (c) note that \(T\) and \(T_p(Gp)\) are \(M\)-invariant by Proposition 4.1. Therefore, and by the symplecticity of \(M\),

\[
T^\omega = (T_p(Gp) \oplus T_2)^\omega = \ker(DJ(p)) \cap T_2^\omega = T_0 \oplus T_2 \oplus N_1
\]

and

\[
(T_p(Gp))^\omega = \ker(DJ(p)) = T_0 \oplus T_2 \oplus N_1 \oplus N_2
\]

are also \(M\)-invariant. Here again we used Lemma 6.1 (c). \(\blacksquare\)

**Lemma 6.4.** Let \(M : T_p\mathcal{M} \to T_p\mathcal{M}\) be a linear map with block structure

\[
M = \begin{pmatrix}
A_0 & 0 & D_0 & D_1 & D_2 \\
0 & A_1 & 0 & D_3 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & M_{10} & M_1 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

(6.8)

with respect to the tangent space decomposition \(T_p\mathcal{M} = T_0 \oplus T_1 \oplus T_2 \oplus N_0 \oplus N_1 \oplus N_2\). Let \(JN_i\) and \(JT_1\) denote the \(N_1\) and \(T_1\) blocks of the skew-symmetric matrix \(J \in \text{GL}(T_p\mathcal{M})\) generating
the symplectic form \( \omega_p \); see Proposition 6.1. Then \( M \) is symplectic if and only if

\[
A_1^T J_{T1} A_1 = J_{T1},
\]

\[
A_{01}^T M_0 + A_1^T J_{T1} D_3 = 0,
\]

\[
D_0^T M_0 - M_0^T D_0 + D_3^T J_{T1} D_3 + M_{10}^T J_{N1} M_{10} = 0,
\]

\[
D_1^T M_0 + M_1^T J_{N1} M_{10} = 0,
\]

\[
M_1^T J_{N1} M_1 = J_{N1},
\]

\[
M_0 = A_0^T T,
\]

\[
\Theta_1 - M_{12}^T J_{N1} M_1 = 0,
\]

\[
\Theta_0 - D_2^T M_0 - M_{12}^T J_{N1} M_{10} = 0.
\]

The proof of this lemma is by direct computation.

6.4. Symplectic twisted semiequivariant linear maps. We will need the following lemma for the construction of symplectic homotopies near Hamiltonian relative periodic orbits in section 6.5. Note that it is shown in [16] that every semi-invariant symplectic form on a vector space has a semiequivariant complex structure \( J \) satisfying \( J^2 = -\text{id} \).

Lemma 6.5. Let \( G \) be a compact Lie group acting orthogonally and semisymplectically on a finite dimensional symplectic vector space \( V \) with complex structure \( J \) satisfying \( J^2 = -\text{id} \). Let \( M : V \to V \) be a twisted semiequivariant linear map with twist diffeomorphism \( \phi : G \to G \) of order \( k \). Then the following hold.

(a) There is a twisted semiequivariant orthogonal symplectic linear map \( A : V \to V \) such that \( A^{2k} = \text{id} \) and \( A^{-1} M = \exp(-\eta) \), where \( \eta \) is infinitesimally \( G \)-semiequivariant (\( \chi(g) g \eta = \eta g \) for all \( g \in G \)) and infinitesimally symplectic (\( \eta^T J + J \eta = 0 \)) and commutes with \( A \) and \( M \).

(b) We have \( A^{-1} = \exp(J_-) Q \), where \( J_- \) is infinitesimally \( G \)-semiequivariant and symplectic, commutes with \( A \), and is such that \( Q^k = \text{id} \). Moreover, there is a \( \Gamma \)-equivariant homotopy \( I(\theta) \) which is smooth in \( \theta \) and satisfies

\[
MI(\theta + 1) = I(\theta) Q^{-1}, \quad \rho I(\theta) \rho^{-1} = I(-\theta) \quad \text{for all } \theta \in \mathbb{R}, \rho \in G \setminus \Gamma.
\]

Proof. Part (a) is essentially Lemma 5.2 of [22]. It is easily checked that the matrices \( \exp(\theta \eta) \) defined there are symplectic if \( V \) is symplectic.

To prove (b) note that since \( A \) is symplectic and \( A^{2k} = \text{id} \), we have \( V = V_+ \oplus V_- \), where \( V_\pm \) are symplectic \( G \)-invariant subspaces of \( V \) such that \( A^k |_{V_+} = \text{id} \) and \( A^k |_{V_-} = -\text{id} \). Let \( J_- : V \to V \) be the matrix defined by \( J_- |_{V_+} = 0, J_- |_{V_-} = \frac{1}{k} J |_{V_-} \). Then \( J_- \) is infinitesimally \( G \)-semiequivariant and symplectic and \( \exp(k J_-) = A^k \). Moreover, \( AV_+ = V_+, AV_- = V_- \), and, since \( A \) is symplectic and orthogonal, \( AJ = JA \) so that \( [A, J_-] = 0 \). Defining \( Q = A^{-1} \exp(-J_-) \), we get \( Q^k = \text{id} \), which proves the first statement of part (b).

For \( \theta \in [0,1) \), define \( I(\theta) := \exp(c(\theta) \eta) \exp(c(\theta) J_-) \), where \( c : [0,1) \to \mathbb{R}_0^+ \) is a \( C^\infty \) monotonically increasing function with

\[
c(\theta) \equiv 0 \text{ for } 0 \leq \theta < \epsilon, \quad \epsilon < 1/2 \text{ fixed, } \quad c(1 - \theta) = 1 - c(\theta).
\]

Then \( I(1) = \exp(\eta) \exp(J_-) = M^{-1} AA^{-1} Q^{-1} = M^{-1} Q^{-1} \) so that we can smoothly extend the homotopy \( I(\theta) \) to \( \theta \in [n, n+1), n \in \mathbb{Z} \setminus \{0\} \), by setting \( I(\theta + n) = M^{-n} I(\theta) Q^{-n} \).
It remains to prove that $\rho I(-\theta)\rho^{-1} = I(\theta)$ for $\rho \in G \setminus \Gamma$. Let $\theta \in [0, 1)$. Then by definition

$$I(-\theta) = MI(1-\theta)Q = M \exp((1-c(\theta))\eta) \exp((1-c(\theta))J_-)Q$$

so that

$$\rho I(-\theta)\rho^{-1} = M^{-1} \exp((c(\theta) - 1)\eta) \exp((c(\theta) - 1)J_-)Q^{-1}$$

$$= M^{-1} \exp(-\eta)I(\theta) \exp(-J_-)Q^{-1} = A^{-1}I(\theta)A = I(\theta),$$

where we used that $[A, \eta] = [J_-, A] = 0$. Now let $\theta = n + \hat{\theta} \in [n, n+1)$, $n \in \mathbb{Z} \setminus \{0\}$. Then by definition $I(\theta) = M^{-n}I(\hat{\theta})Q^{-n}$ and $I(-\theta) = M^nI(-\hat{\theta})Q^n$ so that

$$\rho I(-\theta)\rho^{-1} = M^{-n}\rho I(-\hat{\theta})\rho^{-1}Q^{-n} = M^{-n}I(\hat{\theta})Q^{-n} = I(\theta).$$

\[\Box\]

**Remark 6.2.** Let $G$ be trivial. Since $\text{Sp}(V)$ is connected, we can always symplectically homotope any symplectic linear map $M$ to the identity, and so $Q = \text{id}$ for all $M \in \text{Sp}(V)$. However, in general, the homotopies cannot be chosen to be exponentials. For example, $M = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \text{Sp}(2)$ is not of the form $M = \exp(\eta)$ over the reals. However, if $A = -\text{id}$, there exists an exponential homotopy of $A^{-1}M$ to identity.

**6.5. Symplectic homotopies.** Let $p = \sigma^{-1}\Phi_1(p)$ lie on a relative periodic orbit $\mathcal{P}$ with momentum $\mu = J(p)$. This subsection deals with the proof of the following lemma, which is needed for the adaptation of the bundle structure near relative periodic orbits to the Hamiltonian context. It will be used in the proof of Theorem 6.3.

**Lemma 6.6.** Assume that $\sigma = \alpha \exp(\xi)$, where $\xi \in \mathfrak{z}(\sigma) \cap \mathfrak{z}(G_p) \cap \mathfrak{g}_\mu$, and $\alpha \in \Gamma_\mu$ has order $n$, and choose the $G_p$-semi-invariant complex structure $J_{N_1}$ on $N_1$ such that $J_{N_1}^2 = -\text{id}$. Then the homotopy $I(\theta) \in \text{GL}(T_p\mathcal{M})$ in (6.2), (6.4), and (6.5), which is $G_p$-semiequivariant in the sense of (6.3), can be chosen to be symplectic and such that the matrix $Q$ in (6.4) has the block structure

$$Q = \left( \begin{array}{cc} \text{Ad}_\alpha|_{\mathfrak{m}_\mu \oplus \mathfrak{n}_\mu} & 1 \\ 0 & Q_0 \\ Q_1 & 1 \end{array} \right) \in \text{O}(T_p\mathcal{M}) \cap \text{Sp}(T_p\mathcal{M}),$$

where $Q_0 = (\text{Ad}_\alpha|_{N_0})^{-1}$,

$$Q_1 \in \text{Sp}(N_1) = \{ A \in \text{GL}(N_1) \mid J_{N_1} = A^T J_{N_1} A \},$$

and $Q_1^{-1} \in \text{O}(N_1)$ is twisted semiequivariant of order $k$. Consequently, $Q^{-1}$ is twisted semiequivariant of order $n$.

Since $M = \sigma^{-1}\Phi_1(p)$ is twisted semiequivariant, by Lemma 6.5 there is a symplectic homotopy $I(\theta)$ such that (6.4) and (6.3) hold provided the complex structure $J$ on $\mathcal{M}$ is chosen.
such that \( J^2 = -\text{id} \). However, it is not clear that for the choice of homotopy of Lemma 6.5 the matrix \( Q \) has the form \( Q = \text{diag}(Q_T, Q_N) \) with \( Q_N = \text{diag}(Q_0, Q_1, 1) \), \( Q_0 = (\text{Ad}_p|_{N_0})^{-1} \). The above lemma states that there always exists a \( G_p \)-semiequivariant symplectic homotopy \( I(\theta) \) such that this can be achieved.

Since by Proposition 4.3 the subblock \( M_1 \) of \( M = \sigma^{-1}D\Phi_1(p) \) is twisted semiequivariant, by Lemma 6.5 there is a \( G_p \)-semiequivariant homotopy such that

\[
(6.19) \quad M_1 I_1(\theta + 1) Q_1 = I_1(\theta).
\]

Here \( Q_1 \in \text{Sp}(N_1) \cap \text{O}(N_1) \) has order \( k \), where \( k \) is the order of the twist diffeomorphism, and \( Q_1^{-1} \) is twisted semiequivariant. Using Lemma 6.5, we conclude that, if \( Q := \text{diag}(\text{Ad}_\alpha|_{m_\mu \oplus n_\mu}, 1, (\text{Ad}_p|_{N_0})^{-1}, Q_1, 1) \), then \( Q^{-1} \) is a symplectic twisted semiequivariant map of order \( n \).

For the construction of the homotopies in Lemma 6.6, we will first restrict ourselves to the nonreversible case, i.e., \( \Gamma_p = G_p \), and we will then extend the result to reversible relative periodic orbits.

### 6.5.1. Equivariant symplectic homotopies

In this subsection, we construct \( \Gamma_p \)-equivariant symplectic homotopies \( I(\theta) \) which satisfy the conditions of Lemma 6.6. In order to do this, we will rely heavily on Lemma 6.4. Proposition 4.3 shows that \( M = \sigma^{-1}D\Phi_1(p) \) has the required structure for Lemma 6.4 to apply. Moreover, since time is reparametrized such that \( \dot{\theta} \equiv 1 \), we have \( \Theta_i = 0 \), \( i = 0, 1, 2 \) in (6.8), which by Lemma 6.4 implies that \( D_2 = M_{12} = 0 \).

We look for a homotopy \( I(\theta) \) satisfying

\[
M I(\theta + 1) = I(\theta) Q^{-1}, \quad \text{with} \quad Q \quad \text{as in (6.18)}.
\]

Let \( I(1) = M^{-1}Q^{-1} \). Then \( I(1) \) is \( \Gamma_p \)-equivariant, symplectic, and given by

\[
I^{-1}(1) = \begin{pmatrix}
\overline{\text{Ad}}_{\exp(-\xi)} & \pi_{m_\mu} \overline{\text{Ad}}_{\exp(-\xi)}|_{n_\mu} & 0 & \text{Ad}_\alpha D_0 & \text{Ad}_\alpha D_1 & 0 \\
0 & \pi_{n_\mu} \overline{\text{Ad}}_{\exp(-\xi)}|_{n_\mu} & 0 & \text{Ad}_\alpha D_0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{\text{Ad}}_{\exp(\xi)} & 0 \\
0 & 0 & 0 & 0 & Q_1 M_{10} & Q_1 M_1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Here we used Lemma 6.2. This matrix is \( \Gamma_p \)-equivariantly and symplectically homotopic to the identity. To see this we first define

\[
\hat{I}(\theta) = \begin{pmatrix}
\overline{\text{Ad}}_{\exp(\theta \xi)} & \pi_{m_\mu} \overline{\text{Ad}}_{\exp(\theta \xi)}|_{n_\mu} & 0 & D_0(\theta) & 0 & 0 \\
0 & \pi_{n_\mu} \overline{\text{Ad}}_{\exp(\theta \xi)}|_{n_\mu} & 0 & D_3(\theta) & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \overline{\text{Ad}}_{\exp(-\theta \xi)} & 0 \\
0 & 0 & 0 & 0 & 0 & \hat{I}_1(\theta) \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

where \( \hat{I}_1(\theta) = \exp(\theta \eta) \exp(\theta J_-) \) and \( \eta, J_- \) are as in Lemma 6.5. The blocks \( D_0(\theta) \) and \( D_3(\theta) \) are determined by the two symplecticity conditions (6.10) and (6.11) of Lemma 6.4,

\[
(6.20) \quad (\pi_{m_\mu} \overline{\text{Ad}}_{\exp(\theta \xi)}|_{n_\mu})^T \overline{\text{Ad}}_{\exp(-\theta \xi)} + (\pi_{n_\mu} \overline{\text{Ad}}_{\exp(\theta \xi)}|_{n_\mu})^T J_T D_3(\theta) = 0
\]
and
\[(6.21) \quad D_0(\theta)^T \overline{\text{Ad}_{\exp(-\theta \xi)}} - \overline{\text{Ad}_{\exp(-\theta \xi)}} D_0(\theta) + D_3(\theta)^T J_T \dot{D}_3(\theta) = 0,\]

and by defining the symmetric part of \(D_0(\theta)^T \overline{\text{Ad}_{\exp(-\theta \xi)}}\) to be zero:
\[(6.22) \quad D_0(\theta)^T \text{Ad}_{\exp(-\theta \xi)}^s + \overline{\text{Ad}_{\exp(-\theta \xi)}} D_0(\theta) = 0.\]

Equations (6.21) and (6.22) are equivalent to
\[(6.23) \quad 2D_0(\theta)^T \text{Ad}_{\exp(-\theta \xi)}^s + D_3(\theta)^T J_T \dot{D}_3(\theta) = 0.\]

Equations (6.20) and (6.23) determine \(D_0(\theta)\) and \(D_3(\theta)\) uniquely.

By Lemma 6.4 the homotopy \(\hat{I}(\theta)\) is symplectic since (6.10)–(6.16) are satisfied, and a calculation using \(\xi \in g_a\) and \(\langle \xi_1, J_T \xi_2 \rangle = \mu([\xi_1, \xi_2])\) for \(\xi_1, \xi_2 \in n_a\) shows that (6.9) is satisfied.

The \(\Gamma_p\)-equivariance of \(\text{Ad}_{\exp(\theta \xi)}\) implies that if \(D_0(\theta)\) and \(D_3(\theta)\) are solutions of (6.20) and (6.23), then so also are \(\gamma_p D_0(\theta) \gamma_p^{-1}\) and \(\gamma_p D_3(\theta) \gamma_p^{-1}\) for \(\gamma_p \in \Gamma_p\). Since the solutions are unique, this means that \(D_0(\theta)\) and \(D_3(\theta)\) are \(\Gamma_p\)-equivariant, and hence so is the homotopy \(\hat{I}(\theta)\). Moreover, \(B = \hat{I}^{-1}(1) I(1)\) is unipotent and so symplectically and \(\Gamma_p\)-equivariantly homotopic to the identity by the homotopy \(\exp(\theta \log(B))\).

Now we define \(I(\theta) = \hat{I}(c(\theta)) \exp(\theta \log(B))\) for \(0 \leq \theta < 1\), where \(c : [0,1) \to \mathbb{R}_0^+\) is the same \(C^\infty\) monotonically increasing function satisfying (6.17) as in the proof of Lemma 6.5. Since by construction \(I(1) = M^{-1} Q^{-1}\), we get a smooth homotopy by defining \(I(\theta)\) for \(\theta = n + \hat{\theta} \in [n, n + 1], n \in \mathbb{Z} \setminus \{0\}\), as \(I(\theta) = M^{-n} \hat{I}(\hat{\theta}) Q^{-n}\). Thus we obtain a \(\Gamma_p\)-equivariant smooth symplectic homotopy \(I(\theta)\) such that (6.4) is satisfied for all \(\theta\).

Note that by construction the \(A_0, A_1, A_{01}\), and \(M_0\) blocks of \(I(\theta)\) and \(\hat{I}(c(\theta))\) coincide if we define
\[(6.24) \quad c(\theta + n) = c(\theta) + n \quad \text{for} \quad \theta \in [n, n + 1], \quad n \in \mathbb{Z}.\]

Moreover, the \(M_1\)-block of \(I(\theta)\) is given by the homotopy \(I_1(\theta)\) of (6.19), obtained from Lemma 6.5, since we chose the same reparametrization \(c(\theta)\) in the construction of both homotopies.

The \(D_3\)-blocks of \(I(\theta)\) and \(\hat{I}(c(\theta))\) coincide because they are uniquely defined by the corresponding \(A_1\) and \(M_0\)-blocks; see (6.10). The other blocks of \(I(\theta)\) and \(\hat{I}(\theta)\) are in general not related.

### 6.5.2. Reversible equivariant symplectic homotopies

In this subsection, we will extend the construction of symplectic homotopies of subsection 6.5.1 to the reversible case. So let \(G_p \neq \Gamma_p\), let \(I(\theta)\) be the \(\Gamma_p\)-equivariant symplectic homotopy satisfying (6.4) defined in subsection 6.5.1 above, and let \(\mu_G, \mu_{\Gamma_p}\) be the Haar measures of \(G_p\) and \(\Gamma_p\). Since \(G_p/\Gamma_p = \mathbb{Z}_2\) for any function \(f\) from \(G_p\) to a vector space, we have
\[
\int_{G_p} f(g_p) d\mu_G = \frac{1}{2} \int_{\Gamma_p} f(\gamma_p) d\mu_{\Gamma_p} + \frac{1}{2} \int_{\Gamma_p} f(\rho \gamma_p) d\mu_{\Gamma_p} \quad \text{for all} \quad \rho \in G_p \setminus \Gamma_p.
\]
As a consequence,

$$I^{av}(\theta) := \int_{G_p} g_p I(\chi(g_p)\theta) g_p^{-1} d\mu_{G_p} = \frac{1}{2} \left( I(\theta) + \rho(I(\theta))^{-1} \right) \quad \text{for} \quad \rho \in G_p \setminus \Gamma_p.$$  

This clearly defines a homotopy to the identity map, which is $G_p$-semiequivariant in the sense of (6.3).

We will now show that $I^{av}(\theta)$ satisfies (6.4). Let $\rho \in G_p \setminus \Gamma_p$. Then

$$M(\rho I(-(\theta + 1))\rho^{-1}) = \phi(\rho)M^{-1}I(-(\theta + 1))\rho^{-1},$$

where $\phi : G_p \to G_p$ is the twist diffeomorphism, and we used the fact that $M$ is twisted semiequivariant: $Mg_p = \phi(g_p) M \chi(g_p)$ for $g_p \in G_p$. Since $I(\theta)$ satisfies (6.4), we get

$$\phi(\rho)M^{-1}I(-(\theta + 1))\rho^{-1} = \phi(\rho)I(-\theta)Q\rho^{-1},$$

and because $Q^{-1}$ is twisted semiequivariant we altogether have

$$M(\rho I(-(\theta + 1))\rho^{-1}) = \phi(\rho)I(-\theta)(\phi(\rho))^{-1}Q^{-1} \quad \text{for} \quad \rho \in G_p \setminus \Gamma_p.$$  

Since

$$\int_{G_p} \phi(g_p) I(\chi(g_p)\theta) (\phi(g_p))^{-1} d\mu_{G_p} = \frac{1}{2} \left( I(\theta) + \rho(I(\theta))^{-1} \right) = I^{av}(\theta),$$

the homotopy $I^{av}(\theta)$ satisfies (6.4).

Note that the $A_0$, $A_1$, $A_{01}$, and $M_0$ blocks of $I^{av}(\theta)$ equal the corresponding blocks of $I(\theta)$ because these subblocks are given by $\text{Ad}_{\exp(c(\theta)\xi)}$ (the $A$-blocks) and $\overline{\text{Ad}}_{\exp(-c(\theta)\xi)}$ (the $M_0$-block) and are therefore $G_p$-semiequivariant since by (6.24) and (6.17) the function $c(\theta)$ satisfies $c(-\theta) = -c(\theta)$. Moreover, by construction we have $I_1^{av}(\theta) = I_1(\theta)$.

The homotopy $I^{av}(\theta)$ has the same block structure as $M$ since all $g_p \in \text{Sp}(T_p\mathcal{M})$ have the same block structure as $M$. As a consequence, $I^{av}(\theta)$ is invertible.

The problem is that $I^{av}(\theta)$ need not be symplectic in general. We modify it to obtain a $G_p$-semiequivariant (in the sense of (6.3)) symplectic homotopy $I^{rev}(\theta)$ with the same block structure as $M$. We prescribe the subblocks

$$I_{0}^{rev}(\theta) = \overline{\text{Ad}}_{\exp(-c(\theta)\xi)}^*, \quad I_{1}^{rev}(\theta) = I_{1}(\theta), \quad I^{rev}(\theta)|_{T_{0} \oplus T_{1}} = \text{Ad}_{\exp(c(\theta)\xi)}.$$

Here $I_{i}^{rev}(\theta)$ are the $M_{i}$-subblocks of $I^{rev}(\theta)$, $i = 0, 1$. We define the $M_{10}$-block $I_{10}^{rev}(\theta)$ of $I^{rev}(\theta)$ to be

$$I_{10}^{rev}(\theta) = I_{10}^{av}(\theta),$$

and we define the symmetric part of $\overline{\text{Ad}}_{\exp(-c(\theta)\xi)}I_{D_0}^{rev}(\theta)$ to be

$$(I_{D_0}^{rev}(\theta))^T \overline{\text{Ad}}_{\exp(-c(\theta)\xi)}^* + \overline{\text{Ad}}_{\exp(-c(\theta)\xi)} I_{D_0}^{rev}(\theta) = (I_{D_0}^{rev}(\theta))^T \overline{\text{Ad}}_{\exp(-c(\theta)\xi)}^* + \overline{\text{Ad}}_{\exp(-c(\theta)\xi)} I_{D_0}^{rev}(\theta).$$

(6.25)
The other blocks of $I^{\text{rev}}(\theta)$ are defined uniquely using the symplecticity conditions of Lemma 6.4. This gives a homotopy $I^{\text{rev}}(\theta)$ which is symplectic and smooth in $\theta$. Moreover, $I^{\text{rev}}(\theta)$ is $G_p$-semiequivariant in the sense of (6.3) with respect to the corresponding $G_p$-actions. This can be seen as follows. The subblocks $I^{\text{rev}}(\theta)|_{\gamma_0 @ T_1}$ and the $M_0$, $M_1$, and $M_{10}$ subblocks of $I^{\text{rev}}(\theta)$ are $G_p$-semiequivariant because they equal the corresponding subblocks of the $G_p$-semiequivariant homotopy $I^{\text{av}}(\theta)$. The $D_3$-subblock of $I^{\text{rev}}(\theta)$ is given by (6.10). Since we know that all terms of this equation except the $D_3$-term are $G_p$-semiequivariant and the $D_3$ subblock of $I^{\text{rev}}(\theta)$ is uniquely determined by this equation, the $D_3$ subblock of $I^{\text{rev}}(\theta)$ is also $G_p$-semiequivariant. Similarly, we see that the $D_1$ subblock of $I^{\text{rev}}(\theta)$, which is determined by (6.12), is $G_p$-semiequivariant. Finally, by (6.11) the antisymmetric part of the matrix $\overline{A}\overline{d}_{\exp(-c(\theta)\xi)}I_{D_0}^{\text{rev}}(\theta)$ is $G_p$-semiequivariant, and by (6.25) the same holds for the symmetric part of $\overline{A}\overline{d}_{\exp(-c(\theta)\xi)}I_{D_0}^{\text{rev}}(\theta)$. Hence $I_{D_0}^{\text{rev}}(\theta)$ is also $G_p$-semiequivariant.

Finally, we will show that $I^{\text{rev}}(\theta)$ satisfies (6.4). Due to the block structure of $M$, $Q$, and $I^{\text{rev}}(\theta)$, and since the $M_1$, $M_0$, and $M_{10}$-subblocks of $I^{\text{rev}}(\theta)$ are given by the corresponding subblocks of $I^{\text{av}}(\theta)$ and $I^{\text{rev}}(\theta)|_{\gamma_0 @ T_1} = I^{\text{av}}(\theta)|_{\gamma_0 @ T_1}$, we see that for these subblocks (6.4) is satisfied. Moreover, since both sides of (6.4) are symplectic and therefore all subblocks of both sides of (6.4) except for the symmetric part of the $M_D^T D_0$ matrices are determined by the corresponding $A$ and $M_1$, $M_0$ and $M_{10}$-subblocks by Lemma 6.6, we need only to check that the symmetric parts of the $M_D^T D_0$ matrices of both sides of (6.4) coincide.

The $M_0$ part of the right-hand side of (6.4) is given by $I_0^{\text{rev}}(\theta)Q_0^{-1} = \overline{A}\overline{d}_{\exp(-c(\theta)\xi)}Q_0^{-1}$, and the $D_0$ part of the right-hand side of (6.4) is $I_{D_0}^{\text{rev}}(\theta)Q_0^{-1}$. So twice the symmetric part of the $M_D^T D_0$ matrices of the right-hand side of (6.4) is

\[
\begin{align*}
(I_{D_0}^{\text{rev}}(\theta)Q_0^{-1})^T (I_{D_0}^{\text{rev}}(\theta)Q_0^{-1}) + (I_{D_0}^{\text{rev}}(\theta)Q_0^{-1})^T (\overline{A}\overline{d}_{\exp(-c(\theta)\xi)}Q_0^{-1}) \\
= Q_0^T (\overline{A}\overline{d}_{\exp(-c(\theta)\xi)}I_{D_0}^{\text{rev}}(\theta)) + (\overline{A}\overline{d}_{\exp(-c(\theta)\xi)}I_{D_0}^{\text{rev}}(\theta))^T Q_0^{-1} \\
= Q_0^T (\overline{A}\overline{d}_{\exp(-c(\theta)\xi)}I_{D_0}^{\text{av}}(\theta)) + (\overline{A}\overline{d}_{\exp(-c(\theta)\xi)}I_{D_0}^{\text{av}}(\theta))^T Q_0^{-1}.
\end{align*}
\]

Here we used definition (6.25) of the symmetric parts of the $M_D^T D_0$ matrices of $I_{D_0}^{\text{rev}}(\theta)$.

The $M_0$ part of the left-hand side of (6.4) is $(MT^{\text{rev}}(\theta + 1))_0 = M_0 I_0^{\text{rev}}(\theta + 1)$, and the $D_0$ part of the left-hand side of (6.4) is

\[
(MT^{\text{rev}}(\theta + 1))_{D_0} = \overline{A}\overline{d}_{\sigma}^{-1} I_{D_0}^{\text{rev}}(\theta + 1) + R(\theta + 1),
\]

where

\[
R(\theta) = \pi_m \overline{A}\overline{d}_{\sigma}^{-1} \mid_{n_m} I_{D_3}(\theta) + D_0 I_{10}^{\text{rev}}(\theta) + D_1 I_{10}^{\text{rev}}(\theta).
\]

So twice the symmetric part of the $M_D^T D_0$ matrices of the left-hand side of (6.4) is

\[
\begin{align*}
(M_0 I_0^{\text{rev}}(\theta + 1))^T (\overline{A}\overline{d}_{\sigma}^{-1} I_{D_0}^{\text{rev}}(\theta + 1) + R(\theta + 1)) + (\overline{A}\overline{d}_{\sigma}^{-1} I_{D_0}^{\text{rev}}(\theta + 1) + R(\theta + 1))^T (M_0 I_0^{\text{rev}}(\theta + 1)) \\
= (\overline{A}\overline{d}_{\exp(-c(\theta + 1)\xi)} I_{D_0}^{\text{rev}}(\theta + 1)^T + \overline{A}\overline{d}_{\exp(-c(\theta + 1)\xi)} I_{D_0}^{\text{av}}(\theta + 1)^T) + \tilde{R}(\theta + 1) \\
= (\overline{A}\overline{d}_{\exp(-c(\theta + 1)\xi)} I_{D_0}^{\text{rev}}(\theta + 1)^T + (\overline{A}\overline{d}_{\exp(-c(\theta + 1)\xi)} I_{D_0}^{\text{av}}(\theta + 1)^T) + \tilde{R}(\theta + 1),
\end{align*}
\]
where
\[ \tilde{R}(\theta) = (M_0 I_0^{\text{rev}}(\theta))^T R(\theta) + R(\theta)^T M_0 I_0^{\text{rev}}(\theta). \]
Here we again used (6.25). Since \( I^{\text{av}}(\theta) \) satisfies (6.4) and all parts of \( \tilde{R}(\theta) \) are determined by \( M \) and \( I^{\text{av}}(\theta) \), we conclude that the homotopy \( I^{\text{rev}}(\theta) \) satisfies (6.4).

6.6. Poisson structure of the \( \Gamma \)-reduced bundle. In this subsection, we describe the Poisson structure on the symmetry reduced bundle \( \mathcal{U}/\Gamma \) near a Hamiltonian relative periodic orbit \( \mathcal{P} \).

Define a bracket on the set of smooth functions on \( g_\mu^* \cong \text{ann}(n_\mu) \subset g^* \) by
\[ \{ f_1, f_2 \}(\zeta, w) = - (\mu + \zeta) \left( [j_\mu(\zeta) D_\zeta f_1(\zeta), j_\mu(\zeta) D_\zeta f_2(\zeta)] \right), \]
where \( j_\mu : g_\mu \oplus \text{ann}(n_\mu) \to g \) is as in (3.10) and the Lie bracket is on \( g \). It is straightforward to check that this is a Poisson bracket and equals the standard bracket on \( g_\mu^* \) if \( \mu \) is split (see also [50, section 5.1]).

Extend this bracket to a Poisson structure on \( g_\mu^* \oplus N_1 \) by defining
\[ \{ f_1, f_2 \}(\zeta, w) = \{ f_1, f_2 \}(\zeta, w) + \omega_{N_1}(J_{N_1} D w f_1(\zeta, w), J_{N_1} D w f_2(\zeta, w)). \]
A straightforward calculation using the \( L_n \)-invariance of \( n_\mu \) shows that this Poisson bracket is \( L_n \)-semi-invariant.

This extends to a Poisson structure on \( \tilde{N} = (g_\mu^* \oplus N_1) \oplus N_2 \) by making \( N_2 \) a space of Casimirs. Similarly, as the direct product of \( g_\mu^* \) and the symplectic manifolds \( N_1 \) and \( T^*(\mathbb{R}/n\mathbb{Z}) = \mathbb{R}/n\mathbb{Z} \times N_2 \), the space \( \mathbb{R}/n\mathbb{Z} \times (g_\mu^* \oplus N_1) \) is also naturally a Poisson space.

Let \( \iota \) denote the \( L_n \)-equivariant inclusion of \( g_\mu^* \) into \( g_\mu \), and define a map
\[ L_{R/n\mathbb{Z} \times \tilde{N}} : \mathbb{R}/n\mathbb{Z} \times \tilde{N} \to g_\mu^*, \quad L_{R/n\mathbb{Z} \times \tilde{N}}(\theta, \zeta, w, E) = L_{g_\mu^* \oplus N_1}(\zeta, w), \]
where
\[ L_{g_\mu^* \oplus N_1} : g_\mu^* \oplus N_1 \to g_\mu^*, \quad L_{g_\mu^* \oplus N_1}(\zeta, w) = - \tilde{P} \zeta + L_{N_1}(w), \]
and \( \tilde{P} \) is the \( L_n \)-equivariant projection from \( g_\mu^* \) to \( g_\mu^\iota \) dual to \( \iota \). These maps are \( L_n \)-equivariant and momentum maps for the \( L_n \)-action on the Poisson spaces \( \mathbb{R}/n\mathbb{Z} \times \tilde{N} \) and \( g_\mu^* \oplus N_1 \) (see [50, section 5.1]). It follows that the quotient variety
\[ \mathcal{U}/\Gamma = (\mathbb{R}/n\mathbb{Z} \times \tilde{N})/(\Gamma_p \times \mathbb{Z}_n) = L_{R/n\mathbb{Z} \times \tilde{N}}^{-1}(0)/(\Gamma_p \times \mathbb{Z}_n), \]
where
\[ L_{R/n\mathbb{Z} \times \tilde{N}}^{-1}(0) \cong \mathbb{R}/n\mathbb{Z} \times N, \]
has a natural Poisson structure. The group \( G_p/\Gamma_p \) is isomorphic to \( \mathbb{Z}_2 \) if \( G_p \) contains elements that act antisympetically on \( M \) and is trivial if it does not. In the first case, the action of the generator \( \rho \) of \( G_p/\Gamma_p \) on \( L_{R/n\mathbb{Z} \times \tilde{N}}^{-1}(0)/(\Gamma_p \times \mathbb{Z}_n) \) is “anti-Poisson.”

In section 2 of [55, 22], we proved that \( (\mathbb{R}/n\mathbb{Z} \times N)/(\Gamma_p \times \mathbb{Z}_n) \) is diffeomorphic as a set to a neighborhood of the relative periodic orbit \( \mathcal{P} \) in the orbit space \( M/\Gamma \). The above construction defines a Poisson structure on this neighborhood. It will follow from the proof below that this Poisson structure is isomorphic to that induced directly from \( M \) if we choose the homotopies \( I(\theta) \) occurring in the bundle construction of section 6.1 as in Lemma 6.6.
6.7. Symplectic structure of the bundle. In this section, we describe the symplectic structure of the bundle (2.8) near a Hamiltonian relative periodic orbit.

Let the symmetry group \( \Gamma \) be algebraic, and let \( \widetilde{M} \) denote the manifold

\[
\widetilde{M} = G \times \mathbb{R}/n\mathbb{Z} \times N,
\]

where \( N \) is the extended Poincaré section (see (3.11)). Define a smooth action of \( G \times L_n \) on \( \widetilde{M} \) by

\[
(g, g_p, i)(\tilde{g}, \theta, \zeta, w, E) = (g\tilde{g}^{-1}g_p^{-1}, \chi(g_p)(\theta + i), \chi(g_p)(\text{Ad}_{g_p}^*)^{-1}\zeta, g_pQ_1^iw, E),
\]

where \( g, \tilde{g} \in G, g_p \in G_p, \) and \( i \in \mathbb{Z}_n \). Define a two-form \( \tilde{\omega} \) on \( \widetilde{M} \) by

\[
\tilde{\omega}(g, \theta, \zeta, w, E) = \chi(g)(\tilde{\omega}_G + \tilde{\omega}_\mu + \tilde{\omega}_{N_1} + \tilde{\omega}_{T_2\oplus N_2}),
\]

where

1. \( \tilde{\omega}_G \) is the pullback of the natural symplectic form \( \omega_G \) on \( T^*G \cong G \times g^* \):

\[
\omega_G(g, \nu)((g\xi_1, \nu_1), (g\xi_2, \nu_2)) = \nu_2(\xi_1) - \nu_1(\xi_2) + \nu([\xi_1, \xi_2]),
\]

where \( g \in G, \nu, \nu_1, \nu_2 \in g^* \), and \( \xi_1, \xi_2 \in g \) (see [1, Proposition 4.4.1]) by the map \( (g, \theta, \nu, w, E) \mapsto (g, i_\nu) \), in which the inclusion \( i_\nu : g^*_\mu \rightarrow g^* \) is induced by the \( G_p \)-invariant complement \( n_\mu \) to \( g_\mu \) in \( g \);

2. \( \tilde{\omega}_\mu \) is the pullback of the KKS symplectic form (6.7) on the coadjoint orbit \( G\mu \) by \( (g, \theta, \nu, w, E) \mapsto \text{Ad}_{g^{-1}\mu}^*; \)

3. \( \tilde{\omega}_{N_1} \) is the pullback of the symplectic form \( \omega_{N_1} \) on \( N_1 \) by \( (g, \theta, \nu, w, E) \mapsto w; \)

4. \( \tilde{\omega}_{T_2\oplus N_2} \) is the pullback of the symplectic form \( \omega_{T_2\oplus N_2} \) on \( \mathbb{R}/n\mathbb{Z} \times \mathbb{R} \) by \( (g, \theta, \nu, w, E) \mapsto (\theta, E). \)

Then the form \( \tilde{\omega} \) is a symplectic form on a \( (G \times L_n) \)-invariant neighborhood of \( G \times \mathbb{R}/n\mathbb{Z} \times \{(0, 0, 0)\} \) in \( \widetilde{M} \). The action of \( G \) on this neighborhood is \( \chi \)-semisymplectic. The action (6.28) of \( L_n \), and, in particular, \( G_p \), is symplectic even though the \( G_p \)-action on the symplectic slice \( N_1 \) is semisymplectic with respect to the symplectic form \( \omega_{N_1} \).

A momentum map \( L_{\widetilde{M}} : \widetilde{M} \rightarrow g^*_p \) for the symplectic action of \( L_n \) on \( \widetilde{M} \) is given by

\[
L_{\widetilde{M}}(g, \theta, \nu, w, E) = L_{\mathbb{R}/n\mathbb{Z} \times \bar{N}}(\theta, \nu, w, E).
\]

The map \( L_{\widetilde{M}} \) is \( L_n \)-equivariant with respect to the action (6.28) on \( \widetilde{M} \) and the usual coadjoint action of \( L_n \) on \( g^*_p \). Because the action of \( L_n \) on \( \widetilde{M} \) is free, proper, and symplectic, we can reduce \( \widetilde{M} \) by it to obtain a natural symplectic structure \( \tilde{\omega}_0 \) on a \( G \)-invariant neighborhood \( \tilde{U}_0 \) of \( (G \times \mathbb{R}/n\mathbb{Z} \times \{(0, 0, 0)\})/L_n \) in the manifold

\[
\tilde{M}_0 = L_{\widetilde{M}}^{-1}(0)/L_n = (G \times L_{\mathbb{R}/n\mathbb{Z} \times \bar{N}}^{-1}(0))/L_n \cong (G \times \mathbb{R}/n\mathbb{Z} \times \bar{N})/L_n.
\]

The action of \( G \) on \( \tilde{M} \) drops to a \( \chi \)-semisymplectic action of \( G \) on \( \tilde{U}_0 \).
Let \( v = (\nu, w, E) \in N \). By Theorem 2.2 the differential equations on \( N \) in the new coordinates are of the form
\[
\dot{\theta} = f_\Theta(\theta, v), \quad \dot{v} = f_N(\theta, v),
\]
with \( f_\Theta(\theta, 0) \equiv 1 \). Hence \( \omega_1 \) with \( \tau^* \omega_1 = f_\Theta(\theta, v)\tau^* \omega \) is a symplectic form on a \( G \)-invariant neighborhood \( \mathcal{U} \) of \( \mathcal{P} \). Without loss of generality, we let \( \omega = \omega_1 \).

As shown in [55], \( \tilde{U}_0 \) is \( G \)-equivariantly diffeomorphic to a \( G \)-invariant neighborhood \( \mathcal{U} \) of the relative periodic orbit \( \mathcal{P} \) in \( \mathcal{M} \). The following theorem says that this diffeomorphism can be chosen to be a \( G \)-equivariant symplectomorphism with respect to the symplectic form \( \omega \) of \( \mathcal{M} \) and the symplectic form \( \tilde{\omega}_0 \) on \( \mathcal{M}_0 \). It is a generalization to relative periodic orbits of the local normal form for symplectic \( G \)-manifolds near group orbits obtained by Marle [29], Guillemin and Sternberg [15], and Bates and Lerman [4].

**Theorem 6.3.** There exists a \( G \)-equivariant symplectomorphism \( \Psi \) between a \( G \)-invariant open neighborhood of \( (G \times \mathbb{R}/n\mathbb{Z} \times \{0\})/L_n \) in \( \tilde{\mathcal{M}}_0 = (\mathcal{G} \times L^{-1}_0(0))/L_n \cong (G \times \mathbb{R}/n\mathbb{Z} \times N)/L_n \) and a \( G \)-invariant open neighborhood of \( \mathcal{P} \) in \( \mathcal{M} \).

**Proof.** Because of Proposition 6.1 we have \( \omega(\text{id}, 0, 0) = \tilde{\omega}_0(\text{id}, 0, 0) \) at \( p \cong (0, 0, 0) \). Moreover, since by Lemma 6.6 we can choose the \( G_p \)-semiequivariant homotopy \( I(\theta) \) occurring in the parametrization (6.2) of a neighborhood \( \mathcal{U} \) of the relative periodic orbit given in section 6.1 to be symplectic and such that the action of \( L_n \) on \( N \) is as in (6.28), we have that \( \tilde{\omega}_0 = \omega \) on \( \mathcal{P} \).

We now apply the semisymplesctic relative Darboux theorem [50, Theorem 5.3] (based on [15 and [4]) to conclude that there is a diffeomorphism \( \Psi \) defined on a neighborhood \( \mathcal{U} \) of \( \mathcal{P} \) in \( \mathcal{M} \) such that \( \tilde{\omega}_0 = \Psi^* \omega \). 

This proves Theorem 3.1.

**6.8. Skew product equations.** In this final subsection, we derive the skew product equations (3.14) near Hamiltonian relative periodic orbits. Again we reparametrize time so that \( \dot{\theta} \equiv 1 \). Let \( \hat{h}(\theta, \nu, w, E) = \hat{h}(\theta, \nu, w, E) \) for \( \zeta = \nu + \zeta_p, \zeta \in \mathfrak{g}^*_p, \zeta_p \in \mathfrak{g}^*_p \), and \( \nu \in (\mathfrak{g}_\mu/\mathfrak{g}_p)^* \). The vector field \( f_{\hat{h}} \) in the coordinates \( (g, \theta, \zeta, w, E) \subset G \times \mathbb{R}/n\mathbb{Z} \times (\mathfrak{g}_p^* \oplus N_1 \oplus N_2) \) is determined by the equation
\[
\tilde{\omega}(f_{\hat{h}}, (\hat{g}, \hat{\theta}, \hat{\zeta}, \hat{w}, \hat{E})) = D_{(\theta, \zeta, w, E)} \hat{h}(\theta, \zeta, w, E)(\hat{\theta}, \hat{\zeta}, \hat{w}, \hat{E}),
\]
where \( \hat{g} \in \mathfrak{g} \), and, by (6.29),
\[
\tilde{\omega}(f_{\hat{h}}, (\hat{g}, \hat{\theta}, \hat{\nu}, \hat{w}, \hat{E})) = -\hat{\zeta}(g^{-1}\hat{\dot{g}}) + \hat{\zeta}(g^{-1}\hat{\dot{g}}) + (\zeta + \mu)[g^{-1}\hat{\dot{g}}, g^{-1}\hat{\dot{g}}]
\]
\[+\omega_{N_1}(\hat{w}, \hat{w}) + \omega_{T_{\mathbb{Z}} \otimes N_1}((\hat{\theta}, \hat{E}), (\hat{\theta}, \hat{E})).\]
Comparing coefficients, we obtain the differential equations
\[
\dot{w} = J_{N_1} D_w \hat{h}, \quad \dot{E} = -D_\theta \hat{h}, \quad \dot{\theta} = D_E \hat{h},
\]
and, as in [50],
\[
\dot{g} = gj_\mu(\zeta)D_\zeta \hat{h}, \quad \dot{\zeta} = \text{ad}_{j_\mu(\zeta)}^* D_\zeta \hat{h}(\zeta + \mu).
\]
Since \( \dot{\theta} = 1 \), we have \( D_E \hat{h} \equiv 1 \) so that \( \hat{h} = \hat{\hat{h}} - E \) is independent of \( E \). This yields the equations of Theorem 3.3.

The equations of Theorem 3.5 are obtained as in [50].
Acknowledgment. C. Wulff thanks the University of Warwick for their hospitality during visits, when parts of this paper were written.

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