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LARGE DEVIATIONS FOR NONUNIFORMLY HYPERBOLIC SYSTEMS

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ABSTRACT. We obtain large deviation estimates for a large class of nonuniformly hyperbolic systems: namely those modelled by Young towers with summable decay of correlations. In the case of exponential decay of correlations, we obtain exponential large deviation estimates given by a rate function. In the case of polynomial decay of correlations, we obtain polynomial large deviation estimates, and exhibit examples where these estimates are essentially optimal.

In contrast with many treatments of large deviations, our methods do not rely on thermodynamic formalism. Hence, for Hölder observables we are able to obtain exponential estimates in situations where the space of equilibrium measures is not known to be a singleton, as well as polynomial estimates in situations where there is not a unique equilibrium measure.

1. INTRODUCTION

Large deviations theory concerns the probability of outliers in the convergence of Birkhoff averages. Quantitative estimates of these probabilities are used in engineering, information and statistical mechanics [8, 11]. Suppose ϕ is an observable on an ergodic dynamical system (T, X, μ). We are interested in the asymptotic behavior of $\mu(|\frac{1}{N}\phi_N - \bar{\phi}| > \epsilon)$ where $\phi_N = \sum_{j=0}^{N-1} \phi \circ T^j$ is the Nth Birkhoff sum and $\bar{\phi} = \int_X \phi \, d\mu$. The classical situation is that this quantity converges exponentially quickly and moreover

$$\lim_{N\to\infty} \tfrac{1}{N} \log \mu(|\tfrac{1}{N}\phi_N - \bar{\phi}| > \epsilon) = -c_\phi(\epsilon) \tag{1.1}$$

for small enough ϵ, where c_ϕ is strictly convex and vanishes only at 0. Such a function is called a *rate function* and is often characterised in terms of thermodynamic quantities. See [10] for the case of iid random variables.

Many authors have studied large deviations for dynamical systems with hyperbolicity. Uniformly hyperbolic (Axiom A) dynamical systems are covered entirely (for both discrete and continuous time) by the work of [18, 21, 28, 34, 35]. Moreover, when X is an Axiom A attractor and μ is an SRB measure, then μ can be replaced

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by Lebesgue measure in (1.1). For a general class of one-dimensional maps, Keller and Nowicki [17] obtain a large deviations result (1.1) for observables of bounded variation, again in terms of Lebesgue measure.

In addition, Kifer [18] and Young [35] formulated quite general large deviation principles for dynamical systems; for example Kifer obtained the upper bound half of (1.1) for uniformly partially hyperbolic dynamical systems. However, these results yield strong conclusions (in particular (1.1)) only if it is known that there is a unique equilibrium measure for the underlying map. More recently, Araújo and Pacifico [2] obtain large deviation results, in terms of Lebesgue measure, for continuous functions over nonuniformly expanding maps with nonflat singularities or criticalities and for certain partially hyperbolic nonuniformly expanding attracting sets. Araújo [1] has extended these results to obtain large deviation bounds for continuous functions on suspension semiflows over a nonuniformly expanding base transformation with nonflat singularities or criticalities (including semiflows modelling the geometric Lorenz flow and the Lorenz flow). Again, the results in [1, 2] yield strong conclusions only when there is a unique equilibrium measure.

We note also results on level 2 large deviation principles by [14] for Hölder observables on parabolic rational functions (see also [9]) and by [29] on upper bounds for preimages weighted by the derivative for certain dynamical systems with indifferent fixed points.

In this paper, we prove large deviation results for Hölder observables of nonuniformly hyperbolic systems modelled by Young towers. In contrast to the results mentioned above, we do not require that there is a unique equilibrium measure. Moreover, we obtain to our knowledge the first polynomial large deviations estimates. The general set up is that $T: M \to M$ is a nonuniformly hyperbolic system in the sense of Young [36, 37] with a return time function R that decays either exponentially [36], or polynomially [37]. In particular, $T: M \to M$ is modelled by a Young tower constructed over a "uniformly hyperbolic" base $Y \subset M$. The degree of nonuniformity is measured by the return time function $R: Y \to \mathbb{Z}^+$ to the base. Such systems are known to have an SRB measure μ absolutely continuous with respect to Lebesgue measure m^u on unstable manifolds. Let $\phi: M \to \mathbb{R}$ be a Hölder continuous observable. Our main results are:

(1) If $m^u(y \in Y : R(y) > n) = O(\gamma^n)$ for some $\gamma \in (0,1)$, then the limit $\sigma^2 = \lim_{N\to\infty} \frac{1}{N}\int_M (\phi_N - N\bar{\phi})^2\, d\mu$ exists, and if $\sigma^2 > 0$, then there is a rate function $c_\phi(\epsilon)$ such that (1.1) holds.
(2) If $m^u(y \in Y : R(y) > n) = O(n^{-(\beta+1)})$ for some $\beta > 1$, then for any $\delta > 0$ there exists a constant $C_{\phi,\delta}$ such that for any $\epsilon > 0$ and N sufficiently large

$$\mu(|\tfrac{1}{N}\phi_N - \bar{\phi}| > \epsilon) \le C_{\phi,\delta}\, \epsilon^{-2(\beta-\delta)} N^{-(\beta-\delta)}.$$

Moreover, for each $\delta > 0$ the constant $C_{\phi,\delta}$ depends continuously on $\|\phi\|_{C^\alpha}$.

Subject to conditions on the density, the SRB measure μ can be replaced by Lebesgue measure. In certain situations, we show that the upper bound in (2) is close to optimal. We obtain similar results for nonuniformly hyperbolic flows, but we do not obtain a rate function.

Remark 1.1. For the classes of systems discussed in this paper, it is well known that typically $\sigma^2 > 0$. Indeed, $\sigma^2 = 0$ only for Hölder observables lying in a closed subspace of infinite codimension.

Remark 1.2. Since we require $\beta > 1$ in (2), our results are restricted to cases where the CLT and related results are known to hold (see for example [23, 24]). An interesting problem is to investigate the case $\beta \in (0,1]$ which occurs in Example 1.3 below for $\alpha \in [\frac{1}{2}, 1)$. Other examples with $\beta = 1$ include Bunimovich-type stadia and certain classes of semidispersing billiards; see [3, 6, 22].

Example 1.3 (Intermittency-type maps)**.** Various authors, including [16, 20, 30, 37], have studied intermittency (Pomeau-Manneville) maps of the type $T : [0,1] \to [0,1]$ given by

$$Tx = \begin{cases} x(1 + 2^\alpha x^\alpha), & 0 \le x < \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} \le x < 1, \end{cases}$$

for $\alpha \in (0,1)$, where there is an indifferent fixed point at 0. The reference measure here is Lebesgue measure, and there is a unique ergodic invariant probability measure μ equivalent to Lebesgue. There is a Young tower with base $Y = [\frac{1}{2}, 1]$ and it follows from Hu [16] that the optimal return time decay rate is given by $\mathrm{Leb}(y \in Y : R(y) > n) \approx n^{-(\beta+1)}$ where $\beta = \frac{1}{\alpha} - 1$.

We restrict to $\alpha \in (0, \frac{1}{2})$ (so $\beta > 1$). Then by our results in Section 3, for $\delta > 0$,

(i) For any $\phi : [0,1] \to \mathbb{R}$ Hölder and $\epsilon > 0$, there exists a constant $C \ge 1$ such that $\mu(|\frac{1}{N}\phi_N - \bar{\phi}| > \epsilon) \le CN^{-(\frac{1}{\alpha}-1-\delta)}$, for all $N \ge 1$, and
(ii) For an open and dense set of Hölder observables $\phi : [0,1] \to \mathbb{R}$, and $\epsilon > 0$ sufficiently small, $\mu(|\frac{1}{N}\phi_N - \bar{\phi}| > \epsilon) > N^{-(\frac{1}{\alpha}-1+\delta)}$ for infinitely many N.

Furthermore, Hu [16] shows that $d\mu = g\, d\mathrm{Leb}$ where $g(x) \approx x^{-\alpha}$. Hence with respect to Lebesgue measure, $g^{-1} \in L^\infty$ and $g \in L^p$ for any $p < \alpha^{-1}$. By Hölder's inequality, for $\delta > 0$,

(iii) For any $\phi : [0,1] \to \mathbb{R}$ Hölder and $\epsilon > 0$, there exists a constant $C \ge 1$ such that $\mathrm{Leb}(|\frac{1}{N}\phi_N - \bar{\phi}| > \epsilon) \le CN^{-(\frac{1}{\alpha}-1-\delta)}$, for all $N \ge 1$, and
(iv) For an open and dense set of Hölder observables $\phi : [0,1] \to \mathbb{R}$, and $\epsilon > 0$ sufficiently small, $\mathrm{Leb}(|\frac{1}{N}\phi_N - \bar{\phi}| > \epsilon) > N^{-(\frac{1}{\alpha}+\delta)}$ for infinitely many N.

In other words, we have polynomial upper and lower bounds for large deviations for intermittency maps with $\alpha < \frac{1}{2}$, using either the invariant measure μ or Lebesgue measure to compute probabilities. For the invariant measure, our estimates are almost optimal. It remains an open problem to obtain sharp results for Lebesgue measure. As far as we are aware, all four estimates (i)–(iv) are new. (Bressaud [4, Lemma 2.2] obtains polynomial decay rates for piecewise linear observables of Wang maps [13] which are piecewise linear approximations of Pomeau-Manneville maps. Here, the analysis reduces to the iid situation. The authors are grateful to J.-R. Chazottes for pointing out reference [4].)

Example 1.4 (Planar periodic Lorentz gas)**.** The planar periodic Lorentz gas is a class of examples introduced by Sinaĭ [33]. The Lorentz flow is a billiard flow with unit speed on $\mathbb{T}^2 - \Omega$ where Ω is a disjoint union of strictly convex regions with C^3 boundaries. The phase space $M = (\mathbb{T}^2 - \Omega) \times S^1$ is three-dimensional and the flow preserves volume (so the invariant measure coincides with the reference measure).

The flow has a natural cross-section $X = \partial\Omega \times [-\pi/2, \pi/2]$ corresponding to collisions. The Poincaré map $T : X \to X$ is called the billiard map and preserves the Liouville measure $d\mu = \cos\theta\, dx\, d\theta$. The Lorentz gas has finite horizons if the time between collisions is uniformly bounded; otherwise it has infinite horizons.

Young [36] proved that the billiard map $T : X \to X$ has exponential decay of correlations in the finite horizon case, and Chernov [5] extended this result to infinite horizons. In both cases, the map is modelled by a Young tower with exponential tails. For typical Hölder observables $\phi : X \to \mathbb{R}$, we have $\sigma^2 > 0$ and by Theorem 4.1 there exists a rate function $c_\phi(\epsilon)$ such that (1.1) is satisfied.

In the finite horizon case, we obtain a large deviations result also for the continuous time Lorentz flow $f_t : M \to M$. Given $\phi : M \to \mathbb{R}$ Hölder, define $\phi_T = \int_0^T \phi \circ f_t \, dt$. Typically the variance is nonzero and by Theorem 5.1 there is a rate function $c_\phi(\epsilon)$ such that

$$\limsup_{T\to\infty} \tfrac{1}{T} \log \mathrm{Leb}(|\tfrac{1}{T}\phi_T - \bar{\phi}| > \epsilon) \le -c_\phi(\epsilon).$$

Example 1.5 (Dispersing Lorentz flows with vanishing curvature)**.** Chernov and Zhang [7] study a class of dispersing billiards where the billiard table has smooth strictly convex boundary with nonvanishing curvature, except that the curvature vanishes at two points. Moreover, it is assumed that there is a periodic orbit that runs between the two flat points, and that the boundary near these flat points has the form $\pm(1 + |x|^b)$ for some $b > 2$. The correlation function for the billiard map decays as $O((\ln n)^{\beta+1}/n^\beta)$ where $\beta = (b+2)/(b-2) \in (1,\infty)$. A byproduct of the proof is the existence of a Young tower with tails decaying as $O((\ln n)^{\beta+1}/n^{\beta+1})$. Hence for any $\phi : \mathbb{T}^2 - \Omega \to \mathbb{R}$ Hölder and $\delta > 0$, $\epsilon > 0$, there exists a constant $C \ge 1$ such that

$$\mathrm{Leb}(|\tfrac{1}{T}\phi_T - \bar{\phi}| > \epsilon) \le CT^{-(\frac{b+2}{b-2}+\delta)},$$

for all $T > 0$ by Theorem 5.3 (and similarly for the discrete time billiard map by Theorem 4.2). Moreover, Chernov and Zhang [7] anticipate that these results are sharp up to the logarithmic factor, in which case we obtain the corresponding lower bound $N^{-(\frac{b+2}{b-2}-\delta)}$ for the billiard map by Theorem 4.3.

Similarly, our results apply to all the examples described in Young [36], including large classes of one-dimensional maps and Hénon-like maps with SRB measure μ, for which we establish (1.1) for Hölder observables ϕ, and all the examples in Young [37] with $\beta > 1$, where we establish polynomial rates as in Examples 1.3 and 1.5.

The remainder of the paper is organised as follows. We first focus on (noninvertible) nonuniformly expanding maps, considering exponential tails in Section 2 and polynomial tails in Section 3. In Section 4, we consider nonuniformly hyperbolic systems. In Section 5, we consider nonuniformly hyperbolic flows.

2. Nonuniformly expanding maps with exponential tails

Let (X, d) be a locally compact separable bounded metric space with Borel probability measure m_0 and let $T : X \to X$ be a nonsingular transformation for which m_0 is ergodic. Let $Y \subset X$ be a measurable subset with $m_0(Y) > 0$, and let $\{Y_j\}$ be an at most countable measurable partition of Y with $m_0(Y_j) > 0$. We suppose that there is an L^1 *return time* function $R : Y \to \mathbb{Z}^+$, constant on each Y_j with value $R(j) \ge 1$, and constants $\lambda > 1$, $\eta \in (0,1)$, $C \ge 1$ such that for each $j \ge 1$,

(1) The induced map $F = T^{R(j)} : Y_j \to Y$ is a measurable bijection.
(2) $d(Fx, Fy) \ge \lambda d(x, y)$ for all $x, y \in Y_j$.

(3) $d(T^\ell x, T^\ell y) \le Cd(Fx, Fy)$ for all $x, y \in Y_j$, $0 \le \ell < r(j)$.
(4) $g_j = \frac{d((m_0|Y_j)\circ F^{-1})}{dm_0|Y}$ satisfies $|\log g_j(x) - \log g_j(y)| \le Cd(x,y)^\eta$ for all $x, y \in Y$.

Such a dynamical system $T : X \to X$ is called *nonuniformly expanding.* There is a unique T-invariant probability measure μ absolutely continuous with respect to m_0 (see for example [37, Theorem 1]).

Our main result in this section is the following.

Theorem 2.1. *Let $T : X \to X$ be a nonuniformly expanding map as above and assume that $m_0(y \in Y : R(y) > n) = O(\gamma^n)$ for some $\gamma \in (0, 1)$. Let $\phi : X \to \mathbb{R}$ be a Hölder-continuous function with mean $\bar\phi$.*

Then the limit $\sigma^2 = \lim_{N\to\infty} \frac{1}{N}\int_X (\phi_N - N\bar\phi)^2\, d\mu$ exists, and if $\sigma^2 > 0$, then there is a rate function $c : \mathbb{R} \to [0,\infty)$ such that

$$\lim_{N\to\infty} \tfrac{1}{N} \log \mu(|\tfrac{1}{N}\phi_N - \bar\phi| > \epsilon) = -c(\epsilon).$$

The proof of Theorem 2.1 contains three main steps: (i) reduction to a tower map $f : \Delta \to \Delta$, (ii) reduction to the case where the tower map is mixing, and (iii) application of an abstract result of [15] using the function space constructed in [36]. These steps are carried out in Subsections 2.1, 2.2 and 2.3.

2.1. **Reduction to a Young tower.** The nonuniformly expanding map $T : X \to X$ can be modelled by a Young tower $f : \Delta \to \Delta$ of the type studied by Young [37]. The by now standard details can be found for example in the proof of [23, Theorem 2.9]. Define $\Delta = \{(y, \ell) \in Y \times \mathbb{N} : 0 \le \ell \le R(y)\}/\sim$ where $(y, R(y)) \sim (Fy, 0)$. Define the tower map $f : \Delta \to \Delta$ by setting $f(y, \ell) = (y, \ell + 1)$ computed modulo identifications. The projection $\pi : \Delta \to X$, $\pi(y, \ell) = T^\ell y$ defines a semiconjugacy, $\pi \circ f = T \circ \pi$.

There is a unique invariant ergodic probability measure m equivalent to $m_0|Y$ for the induced map $F : Y \to Y$. We obtain an ergodic F-invariant probability measure on Δ given by $m_\Delta = m \times \nu/|R|_1$ where ν denotes counting measure, and hence an ergodic T-invariant probability measure $\mu = \pi_* m_\Delta$ on X.

If $x, y \in Y$, let $s(x, y)$ be the least integer $n \ge 0$ such that $F^n x$ and $F^n y$ lie in distinct partition elements in Y. If $x, y \in Y_j \times \{\ell\}$, then there exist unique $x', y' \in Y_j$ such that $x = f^\ell x'$ and $y = f^\ell y'$. Set $s(x, y) = s(x', y')$. For all other pairs x, y, set $s(x, y) = 0$. This defines a separation time $s : \Delta \times \Delta \to \mathbb{N}$ and hence a metric $d_\beta(x, y) = \beta^{s(x,y)}$ on Δ. Let $\mathrm{Lip}(\Delta)$ denote the Banach space of Lipschitz functions $\phi : \Delta \to \mathbb{R}$ with norm $\|\phi\|_{\mathrm{Lip}} = |\phi|_\infty + |\phi|_{\mathrm{Lip}}$ where $|\phi|_{\mathrm{Lip}} = \sup_{x\ne y} |\phi(x) - \phi(y)|/d_\beta(x, y)$. Given $\eta > 0$, we can choose $\beta \in (0, 1)$ (namely $\beta = 1/\lambda^\eta$) so that $\phi \circ \pi \in \mathrm{Lip}(\Delta)$ for all $\phi \in C^\eta(X)$.

Hence, we may reduce to the situation where the nonuniformly expanding map is given by the tower map $f : \Delta \to \Delta$ and the observable $\phi : \Delta \to \mathbb{R}$ lies in $\mathrm{Lip}(\Delta)$.

2.2. **Reduction to a mixing Young tower.** Let $k \ge 1$ be the greatest common divisor of the values of $R : Y \to \mathbb{Z}^+$. A Young tower is mixing if and only if $k = 1$; see [36, Lemma 5]. In this subsection, we show how to reduce the nonmixing case $k \ge 2$ to the mixing case. The only fact specific to Young towers that we use is the fact that the tower Δ is mixing up to a finite cycle: Δ is the disjoint union of k sets $\Lambda_1, \ldots, \Lambda_k$ cyclically permuted by f, and $f^k|\Lambda_i$ is mixing for $i = 1, \ldots, k$.

Moreover, each Λ_i has the structure of a (mixing) Young tower. Let $g = f^k|\Lambda_1$. Note that $m_\Delta(\Lambda_1) = \frac{1}{k}$ so $m_1 = km_\Delta|\Lambda_1$ is an ergodic invariant probability measure for $g : \Lambda_1 \to \Lambda_1$.

We show that large deviations for $f : \Delta \to \Delta$ are inherited from large deviations for $g : \Lambda_1 \to \Lambda_1$.

Given $\phi : \Delta \to \mathbb{R}$ with mean $\bar\phi$, define $\psi : \Lambda_1 \to \mathbb{R}$,

$$\psi = \phi^{(1)} + \phi^{(2)} \circ f + \cdots + \phi^{(k)} \circ f^{k-1}, \qquad \phi^{(i)} = \phi|\Lambda_i,$$

and set $\bar\psi = \int_{\Lambda_1} \psi \, dm_1$. It is easy to see that $\bar\psi = k\bar\phi$.

Proposition 2.2. *Assume that $\phi \in L^\infty(\Delta)$. Define $\psi : \Lambda_1 \to \mathbb{R}$ as above and let $\psi_N = \sum_{j=0}^{N-1} \psi \circ g^j$. Suppose that there is a rate function $c(\epsilon)$ such that*

$$\lim_{N\to\infty} \tfrac{1}{N} \log m_1(|\tfrac{1}{N}\psi_N - \bar\psi| > \epsilon) = -c(\epsilon).$$

Then

$$\lim_{N\to\infty} \tfrac{1}{N} \log m_\Delta(|\tfrac{1}{N}\phi_N - \bar\phi| > \epsilon) = -c(k\epsilon).$$

Proof. We give the details for $k = 2$. Write

$$\begin{aligned}
\phi_{2N} &= \sum_{j=0}^{2N-1} \phi \circ f^j = \sum_{j=0}^{2N-1} \phi \circ f^j|\Lambda_1 + \sum_{j=0}^{2N-1} \phi \circ f^j|\Lambda_2 \\
&= \sum_{j=0}^{N-1} \phi^{(1)} \circ g^j + \sum_{j=0}^{N-1} \phi^{(2)} \circ f \circ g^j + \sum_{j=0}^{N-1} \phi^{(2)} \circ g^j + \sum_{j=0}^{N-1} \phi^{(1)} \circ f \circ g^j \\
&= \sum_{j=0}^{N-1} \psi \circ g^j + \sum_{j=0}^{N-1} \phi^{(1)} \circ f \circ g^j + \sum_{j=0}^{N-1} \phi^{(2)} \circ g^j \circ f^2 + \chi \\
&= \psi_N + \psi_N \circ f + \chi,
\end{aligned}$$

where $\chi = \phi^{(2)} - \phi^{(2)} \circ g^N$. Hence

$$\begin{aligned}
m_\Delta(|\tfrac{1}{2N}(\phi_{2N} - \chi) - \bar\phi| > \epsilon) &= m_\Delta(|\tfrac{1}{N}(\phi_{2N} - \chi) - 2\bar\phi| > 2\epsilon) \\
&= \tfrac{1}{2} m_1(|\tfrac{1}{N}\psi_N - 2\bar\phi| > 2\epsilon) + \tfrac{1}{2} m_2(|\tfrac{1}{N}\psi_N \circ f - 2\bar\phi| > 2\epsilon) \\
&= m_1(|\tfrac{1}{N}\psi_N - \bar\psi| > 2\epsilon).
\end{aligned}$$

Since $|\chi|_\infty \le 2|\phi|_\infty$, it follows that

$$\lim_{N\to\infty} \tfrac{1}{N} \log m_\Delta(|\tfrac{1}{2N}\phi_{2N} - \bar\phi| > \epsilon) = \lim_{N\to\infty} \tfrac{1}{N} \log m_1(|\tfrac{1}{N}\psi_N - \bar\psi| > 2\epsilon) = -c(2\epsilon),$$

as required. □

2.3. **Proof of Theorem 2.1.** By Subsections 2.1 and 2.2, it suffices to prove Theorem 2.1 for $f : \Delta \to \Delta$ a mixing tower map and $\phi : \Delta \to \mathbb{R}$ a Lipschitz observable. Following Young [36], we define $\mathcal{B}$ to be a Banach space of weighted Lipschitz functions as follows. Let $\Delta_\ell = \bigcup \Delta_{j,\ell}$ where the union is over all j with $R(j) > \ell$. Let $\epsilon > 0$, $\beta \in (0,1)$. Given $v : \Delta \to \mathbb{C}$ measurable, define

$$\|v\|_\infty = \sup_\ell |v1_{\Delta_\ell}|_\infty \, e^{-\epsilon\ell}, \quad \|v\|_\beta = \sup_\ell |v1_{\Delta_\ell}|_\beta \, e^{-\epsilon\ell}, \quad \|v\| = \|v\|_\infty + \|v\|_\beta,$$

where

$$|v1_{\Delta_\ell}|_\beta = \sup_{\substack{x,y\in\Delta_\ell \\ x\neq y}} \frac{|v(x)-v(y)|}{\beta^{s(x,y)}}.$$

Now define $\mathcal{B}$ to be the Banach space of functions $v : \Delta \to \mathbb{C}$ with $\|v\| < \infty$. Let $\mathcal{B}^*$ denote the Banach space of bounded linear functionals. We have the following elementary result.

Proposition 2.3. *Provided ϵ is sufficiently small ($e^\epsilon\gamma < 1$ suffices),*

(a) *If $v \in \mathcal{B}$, then $\bar{v} \in \mathcal{B}$ (complex conjugation) and $|v| \in \mathcal{B}$.*
(b) *For all $x \in \Delta$, the maps $v \mapsto v(x)$ and $v \mapsto \int_\Delta v\, dm_\Delta$ lie in $\mathcal{B}^*$.* □

Let $P : L^1(\Delta) \to L^1(\Delta)$ be the transfer (Perron-Frobenius) operator given by $\int_\Delta v\, w \circ f\, dm_\Delta = \int_\Delta Pv\, w\, dm_\Delta$ for $v \in L^1(\Delta)$, $w \in L^\infty(\Delta)$. Note that $P1 = 1$. By ergodicity, the eigenvalue at 1 is simple. Since f is mixing, there are no further eigenvalues on the unit circle. Young [36] shows that P lies in the space $L_{\mathcal{B}}$ of bounded linear operators on $\mathcal{B}$. Moreover, viewed as such an operator, $\|P^n\|$ is bounded (so the spectral radius is 1) and P is quasicompact. To summarize:

Lemma 2.4. *The transfer operator P restricts to a bounded linear operator on $\mathcal{B}$ satisfying $P1 = 1$ and with spectral radius 1. Moreover $\|P^n\|$ is bounded, P is quasicompact, and the spectrum of P lies strictly inside the unit circle with the exception of a simple eigenvalue at 1.* □

Proposition 2.5. *If $v \in \mathcal{B}$ and $\phi \in \mathrm{Lip}(\Delta)$, then $\phi v \in \mathcal{B}$ and $\|\phi v\| \le \|\phi\|_{\mathrm{Lip}}\|v\|$.*

Proof. Compute that $|\phi v1_{\Delta_\ell}|_\beta \le |\phi|_\infty |v1_{\Delta_\ell}|_\beta + |\phi|_{\mathrm{Lip}}|v1_{\Delta_\ell}|_\infty$ so that

$$\|\phi v\|_\beta \le |\phi|_\infty \|v\|_\beta + |\phi|_{\mathrm{Lip}}\|v\|_\infty.$$

Similarly, $\|\phi v\|_\infty \le |\phi|_\infty \|v\|_\infty$ and the result follows. □

Corollary 2.6. *Let $\phi \in \mathrm{Lip}(\Delta)$, and define $P_z : \mathcal{B} \to \mathcal{B}$ by $P_z v = P(e^{z\phi}v)$. Then P_z is a bounded operator for all $z \in \mathbb{C}$, and the map $z \mapsto P_z$ from $\mathbb{C}$ to $L_{\mathcal{B}}$ is analytic on the whole of $\mathbb{C}$.*

Proof. Write $P_z = PM_z$ where $M_z v = e^{z\phi}v$. Since $P \in L_{\mathcal{B}}$, it suffices to consider M_z. By Proposition 2.5, $\|M_z\| \le e^{|z|\|\phi\|_{\mathrm{Lip}}}$ so that $M_z \in L_{\mathcal{B}}$ for all $z \in \mathbb{C}$. Moreover, $\frac{\partial}{\partial z} M_z v = \phi M_z v$ so that $\|\frac{\partial}{\partial z} M_z\| \le \|\phi\|_{\mathrm{Lip}} e^{|z|\|\phi\|_{\mathrm{Lip}}}$. Again, $\frac{\partial}{\partial z} M_z \in L_{\mathcal{B}}$ for all $z \in \mathbb{C}$, proving that $z \mapsto M_z$ is analytic on the whole of $\mathbb{C}$. □

We can now verify the hypotheses of an abstract result of Hennion and Hervé [15, Theorem E*, p. 84]. Once the hypotheses are verified we obtain the large deviation result for Lipschitz observables $\phi : \Delta \to \mathbb{R}$. The result for Hölder observables $\phi : X \to \mathbb{R}$ is then immediate.

We note that $f : \Delta \to \Delta$, m_Δ, P and ϕ in this section correspond to $\tau : E \to E$, ρ, Q and ξ in [15]. Condition (K1) and part (i) of condition (K2) in [15, p. 81] are valid by Proposition 2.3. The remainder of condition (K2) follows from Lemma 2.4. Condition (K2)(iv) in [15, p. 82] follows from Lemma 2.4 (specifically the fact that $P1 = 1$). Condition (K3) holds for all $m \ge 1$ (and $I_0 = \mathbb{R}$) by Corollary 2.6 and so we have verified condition $\mathcal{K}[m]$ for all $m \ge 1$. Finally condition $\widetilde{\mathcal{D}}$ in [15, p. 84] follows from Corollary 2.6 (for any θ_0).

3. Nonuniformly expanding maps with polynomial tails

We continue to assume that $T : X \to X$ is a nonuniformly expanding map as in Section 2, but we relax the condition on $m_0(R > n)$. Write $\|\phi\| = \|\phi\|_{C^\eta}$.

Theorem 3.1. *Let $T : X \to X$ be a nonuniformly expanding map as above and assume that $m_0(y \in Y : R(y) > n) = O(n^{-(\beta+1)})$ for some $\beta > 1$. Let $\phi : M \to \mathbb{R}$ be Hölder with mean $\bar\phi = 0$ (for convenience).*

Then for any ϵ, $\delta > 0$,

$$\mu(|\tfrac{1}{N}\phi_N| > \epsilon) = O(\epsilon^{-2(\beta-\delta)} N^{-(\beta-\delta)}).$$

More precisely, there is a constant $C \ge 1$ with the following property. Let $p = \beta - \delta$ and define $C_p = 4C^{1/p}\sum_{j\ge1} j^{-\beta/(\beta-\delta)}$. Write $\|\phi\| = \|\phi\|_{C^\eta}$. Then for all $\epsilon > 0$, $N \ge 1$,

$$\mu(|\tfrac{1}{N}\phi_N| > \epsilon) \le \{4p|\phi|_\infty(|\phi|_p + C_p\|\phi\|^{1/p}|\phi|_\infty^{1-1/p})\}^p \epsilon^{-2p} N^{-p}.$$

Proof. As in Section 2, we may suppose without loss that $T : X \to X$ is mixing.

We claim that

$$|\phi_N|_{2p}^{2p} \le \{4p|\phi|_\infty(|\phi|_p + C_p\|\phi\|^{1/p}|\phi|_\infty^{1-1/p})\}^p N^p. \tag{3.1}$$

By Markov's inequality,

$$\mu(|\tfrac{1}{N}\phi_N| > \epsilon) \le |\phi_N|_{2p}^{2p}\,\epsilon^{-2p}N^{-2p},$$

so the result follows from (3.1).

It remains to verify (3.1). The two main ingredients are a martingale inequality of Rio [31] recalled for convenience as Theorem A.1 in the appendix, and a result on decay of correlations by Young [37, Theorem 3] which states that

$$|\textstyle\int_X P^N\phi\,\psi\,d\mu| = |\int_X \phi\,\psi\circ T^N\,d\mu| \le C\|\phi\||\psi|_\infty\,/N^\beta, \tag{3.2}$$

for all $\psi \in L^\infty$.

Following [25], we substitute $\psi = \operatorname{sgn} P^N\phi$ into (3.2), yielding $|P^N\phi|_1 \le C\|\phi\|/N^\beta$. Hence

$$\textstyle\int_X |P^N\phi|^p\,d\mu \le (|P^N\phi|_\infty)^{p-1}|P^N\phi|_1 \le C\|\phi\||\phi|_\infty^{p-1}/N^\beta,$$

so that $|P^N\phi|_p \le C^{1/p}\|\phi\|^{1/p}|\phi|_\infty^{1-1/p}\,/N^{\beta/(\beta-\delta)}$. It follows that $\chi = \sum_{N\ge1} P^N\phi$ is summable in L^p and $|\chi|_p \le \frac{1}{4}C_p\|\phi\|^{1/p}|\phi|_\infty^{1-1/p}$.

Next, write $\phi = \psi + \chi\circ T - \chi$ where $\psi \in L^p(X)$. It is immediate from the definition of χ that $P\psi = 0$. Restricting P to an operator on $L^2(X)$, we have $P = U^*$ where $U : L^2(X) \to L^2(X)$ is the Koopman operator $U\phi = \phi\circ T$. Hence $PU = I$ and $UP = E(\cdot|T^{-1}\mathcal{M})$ where $\mathcal{M}$ is the underlying σ-algebra. In particular, $E(\psi|T^{-1}\mathcal{M}) = 0$ and so $E(\psi\circ T^j|T^{-(j+1)}\mathcal{M}) = 0$. (In other words, $\{\psi\circ T^j;\, j \ge 1\}$ is a sequence of reverse martingale differences.)

Passing to the natural extension [32], we obtain $\phi\circ T^j = \psi\circ T^j + \chi\circ T^{j+1} - \chi\circ T^j$ for $j = 0, \pm1, \pm2, \ldots$ (this is standard; see for example [12, Remark 3.12]). Then the sequence $\{\psi\circ T^{-j};\, j \ge 1\}$ is a sequence of (forward) martingale differences with respect to the filtration $\mathcal{F}_j = T^j\mathcal{M}$.

Let $X_j = \phi \circ T^{-j}$, $Z_j = \psi \circ T^{-j}$. Then for each $i \le \ell$,

$$\begin{aligned}\sum_{j=i}^{\ell} E(X_j|\mathcal{F}_i) &= \sum_{j=i}^{\ell} E(Z_j|\mathcal{F}_i) + E(\chi \circ T^{-i+1}|\mathcal{F}_i) - E(\chi \circ T^{-\ell}|\mathcal{F}_i) \\ &= E(Z_i|\mathcal{F}_i) + E(\chi \circ T^{-i+1}|\mathcal{F}_i) - E(\chi \circ T^{-\ell}|\mathcal{F}_i),\end{aligned}$$

and so $|\sum_{j=i}^{\ell} E(X_j|\mathcal{F}_i)|_p \le |\psi|_p + 2|\chi|_p \le |\phi|_p + 4|\chi|_p$. Hence $b_{i,n}$ defined as in Theorem A.1 satisfies $b_{i,n} \le |\phi|_\infty(|\phi|_p + C_p\|\phi\|^{1/p}|\phi|_\infty^{1-1/p})$, and (3.1) follows from Theorem A.1. □

Remark 3.2. We note related results of [19]. In particular, [19, Theorem 3.6] gives a polynomial estimate for L^p martingales and [19, Corollary 4.4] generalises to the case of an L^p martingale plus coboundary. They obtain the decay rate $O(N^{-p/2})$ and show that this is sharp for L^p martingales.

We obtain a stronger result because the underlying observable lies in L^∞ even though the decomposition into martingale plus coboundary holds only in L^p.

We note also that the decay rate in Theorem 3.1 is stronger than the optimal decay rate, namely $o(N^{-(p-1)})$, for iids in L^p (see [19, Proposition 2.6]).

3.1. **Lower bounds.** In this subsection, we show that Theorem 3.1 is essentially optimal for Young towers. Assume that $m_0(y \in Y : R(y) > N) \sim N^{-(\beta+1)}$. Let $D_N = \{(y,\ell) \in \Delta : R(y) \ge N\}$. Note that $m_\Delta(D_N) \sim N^{-\beta}$.

Proposition 3.3. *Suppose that $\phi \ge \bar{\phi} + \epsilon_0$ on D_{N_0} for some $N_0 \ge 1$, $\epsilon_0 > 0$. Then there exists $C > 0$ such that*

$$m_\Delta(\tfrac{1}{N}\phi_N - \bar{\phi} > \epsilon) \ge CN^{-\beta} \tag{3.3}$$

for all $N \ge N_0$ and $\epsilon \in (0,\epsilon_0)$.

Proof. Let $N \ge N_0$, $\epsilon \in (0,\epsilon_0)$. If $R(y) \ge N$ and $\ell < R(y) - N - 1$, then $\phi_N(y,\ell) \ge (\bar{\phi} + \epsilon_0)N > (\bar{\phi} + \epsilon)N$. Hence

$$m_\Delta(\tfrac{1}{N}\phi_N - \bar{\phi} > \epsilon) \ge m_\Delta\{(y,\ell) : R(y) \ge N,\, \ell < R(y) - N - 1\} \sim 1/N^\beta. \quad \square$$

To obtain an explicit counterexample, take N_0 so large that $m_\Delta(D_{N_0}) \le \frac{1}{2}$. Define $\phi \equiv 2$ on D_{N_0} and $\phi \equiv 0$ elsewhere (so $\bar{\phi} \le 1$). Then the counterexample holds with $\epsilon_0 = 1$.

Remark 3.4. Let ϕ, ϵ_0, N_0 be as in Proposition 3.3. Given $\rho \in (0,\epsilon_0/2)$, let $\epsilon_1 = \epsilon_0 - 2\rho$. Then any ϕ' with $|\phi' - \phi|_\infty < \rho$ satisfies (3.3) for all $N \ge N_0$, $\epsilon \in (0,\epsilon_1)$.

Theorem 3.5. *Given $\beta' > \beta$, there is an open and dense set of Lipschitz observables $\psi : \Delta \to \mathbb{R}$ with $\epsilon_0(\psi) > 0$ such that for all $\epsilon \in (0,\epsilon_0(\psi))$*

$$m_\Delta(|\tfrac{1}{N}\psi_N - \bar{\psi}| > \epsilon) > N^{-\beta'}$$

for infinitely many N. Moreover $\epsilon_0(\psi)$ may be taken as constant on a Lipschitz neighborhood of ψ.

Proof. Let $\mathcal{A}$ denote the set of Lipschitz observables ψ for which the conclusion of the theorem holds. If $\psi \in \operatorname{Int}\mathcal{A}$, then we are finished. Otherwise, we make an initial perturbation so that $\psi \notin \mathcal{A}$. We complete the proof by showing that there exists $\psi' \in \operatorname{Int}\mathcal{A}$ arbitrarily close to ψ.

Since $\psi \notin \mathcal{A}$, there exists $\beta' > 0$ such that for all $\alpha > 0$ there exist $\epsilon \in (0, \alpha)$ and $N(\alpha)$ such that for all $N > N(\alpha)$

$$m_\Delta(|\psi_N - N\bar{\psi}| > N\epsilon) \le N^{-\beta'}.$$

Hence for all $\epsilon > 0$ there exists $N(\epsilon)$ such that for all $N > N(\epsilon)$

$$m_\Delta(|\psi_N - N\bar{\psi}| > N\epsilon) \le N^{-\beta'}.$$

Let ϕ be the explicit counterexample of Proposition 3.3, i.e. $\phi \equiv 2$ on D_{N_0} and $\phi = 0$ elsewhere. To simplify the construction, we normalise so that $\bar{\phi} = 0$.

For small $\delta > 0$ define $\psi' = \psi + \delta\phi$. Compute that

$$\begin{aligned} m_\Delta(|\psi'_N - N\bar{\psi}'| > N\epsilon) &= m_\Delta(|\delta\phi_N + \psi_N - N\bar{\psi}| > N\epsilon) \\ &\ge m_\Delta(|\delta\phi_N| > 2N\epsilon, |\psi_N - N\psi| < N\epsilon) \\ &\ge m_\Delta(|\delta\phi_N| > 2N\epsilon) - m_\Delta(|\psi_N - N\psi| > N\epsilon) \\ &= m_\Delta(|\phi_N| > 2N\epsilon/\delta) - m_\Delta(|\psi_N - N\psi| > N\epsilon) \\ &\ge CN^{-\beta} - N^{-\beta'} \ge N^{-\beta'} \end{aligned}$$

for sufficiently large N. Hence $\psi' \in \mathcal{A}$. Moreover, by Remark 3.4, we can replace ϕ by ϕ' sufficiently close to ϕ. Hence $\psi' \in \operatorname{Int} \mathcal{A}$. □

Remark 3.6. For certain systems $T : X \to X$ modelled by Young towers we may find an open and dense set of Lipschitz observables $\phi : X \to \mathbb{R}$ which have polynomial lower bounds. In the notation of Proposition 3.3, let $X_N = \pi(D_N)$. Suppose that there exists an N_0 sufficiently large that $\mu(\overline{X_{N_0}}) = a_0 < 1$ (hence we require that X_{N_0} is not dense in X for some N_0). Choose $a_1 \in (a_0, 1)$ and define $\phi \equiv 1/a_1$ on $\mu(\overline{X_{N_0}})$ and $\phi \equiv 0$ elsewhere. Then $\bar{\phi} = a_0/a_1 < 1$. Smooth ϕ using a bump function so that $\phi : X \to \mathbb{R}$ is C^∞ with $\phi \equiv 1/a_1$ on $\overline{X_{N_0}}$ and $\bar{\phi} < 1$.

It is immediate that ϕ lifts to a Lipschitz observable $\phi_\Delta : \Delta \to \mathbb{R}$ such that $\phi_\Delta \equiv 1/a_1$ on D_{N_0} and $\overline{\phi_\Delta} < 1$. Hence the construction in Proposition 3.3 holds with $\epsilon_0 = 1/a_1 - \overline{\phi_\Delta} > 0$. Furthermore there exists $\rho > 0$ such that if $|\phi - \phi^{'}|_{\mathrm{Lip}} < \rho$ (Euclidean metric) then $\phi^{'}$ also lifts to a function $\phi^{'}_\Delta : \Delta \to \mathbb{R}$ satisfying the conditions of Proposition 3.3 for $\epsilon_0(\rho) > 0$. The statement and proof of openness and density proceeds exactly as in the case of functions on the tower.

The condition on X_N is satisfied in Example 1.3. For these maps $T : [0, 1] \to [0, 1]$, we have $X_N \cap (1/2, 1] = \emptyset$ for all $N > 2$.

4. Nonuniformly hyperbolic systems

Let $T : M \to M$ be nonuniformly hyperbolic in the sense of Young [36, 37], (alternatively, see [23, Section 3]). In particular, there is a "uniformly hyperbolic" subset $Y \subset M$ with partition $\{Y_j\}$ and an integrable return time function $r : Y \to \mathbb{Z}^+$ constant on partition elements such that, modulo uniformly contracting directions, the induced map $F = T^r : Y \to Y$ is uniformly expanding.

Using the induced map $F : Y \to Y$ and the return time function $r : Y \to \mathbb{Z}^+$, we can build a tower map $f : \Delta \to \Delta$ just as in Section 2.1 with semiconjugacy $\pi : \Delta \to M$ given by $\pi(y, \ell) = T^\ell y$. Moreover, the quotient tower map $\bar{f} : \bar{\Delta} \to \bar{\Delta}$ has all the structure of the Young tower in Section 2.1.

In particular, there is an absolutely continuous $\bar{f}$-invariant ergodic probability measure $m_{\bar{\Delta}}$. Furthermore, there is an f-invariant measure m_Δ on Δ such that the natural projection $\bar{\pi} : \Delta \to \bar{\Delta}$ is a measure-preserving semiconjugacy. The required SRB or physical measure is given by $\mu = \pi_* m_\Delta$. This is an ergodic T-invariant probability measure whose restriction to unstable manifolds is absolutely continuous with respect to Lebesgue measure m^u.

Theorem 4.1. *Let $T : M \to M$ be nonuniformly hyperbolic modelled by a Young tower, and suppose that $m^u(y \in Y : R(y) > n) = O(\gamma^n)$ where $\gamma \in (0,1)$. Let $\phi : M \to \mathbb{R}$ be Hölder with mean $\bar{\phi}$.*

Then the limit $\sigma^2 = \lim_{N\to\infty} \frac{1}{N}\int_X (\phi_N - N\bar{\phi})^2\, d\mu$ exists, and if $\sigma^2 > 0$, then there is a rate function $c(\epsilon)$ such that

$$\lim_{N\to\infty} \tfrac{1}{N} \log \mu(|\tfrac{1}{N}\phi_N - \bar{\phi}| > \epsilon) = -c_\phi(\epsilon).$$

Proof. Without loss, we may suppose that $\bar{\phi} = 0$. By [23, Lemma 3.2], we can write $\phi \circ \pi = \psi + \chi - \chi \circ f$ where $\psi, \chi \in L^\infty(\Delta)$, and ψ depends only on future coordinates. In particular, we can write $\psi : \bar{\Delta} \to \mathbb{R}$ where $\bar{\Delta}$ is the quotient tower. This is a nonuniformly expanding tower as in Section 2. By [23, Lemma 3.2] ψ is Lipschitz on $\bar{\Delta}$. Hence, by Theorem 2.1 there is a rate function $c(\epsilon)$ such that

$$\lim_{N\to\infty} \tfrac{1}{N} \log m_{\bar{\Delta}}(|\psi_N| > N\epsilon) = -c(\epsilon).$$

This result lifts to $\psi : \Delta \to \mathbb{R}$. Since $|\phi_N \circ \pi - \psi_N| \le 2|\chi|_\infty$, we obtain

$$\lim_{N\to\infty} \tfrac{1}{N} \log m_\Delta(|\phi_N \circ \pi| > N\epsilon) = -c(\epsilon).$$

Since π is measure-preserving, we obtain the required result on M. □

Theorem 4.2. *Let $T : M \to M$ be nonuniformly hyperbolic modelled by a Young tower, and suppose that $m^u(y \in Y : R(y) > n) = O(n^{-(\beta+1)})$ where $\beta > 1$. There is a constant $\widetilde{C} \ge 1$ with the following property.*

Let $\phi : M \to \mathbb{R}$ be Hölder with mean $\bar{\phi} = 0$ (for convenience) and norm $\|\phi\| = \|\phi\|_{C^\eta}$. Let $p = \beta - \delta$ where $\delta > 0$. Then for all $\epsilon > 0$, $N \ge 1$,

$$\mu(|\tfrac{1}{N}\phi_N - \bar{\phi}| > \epsilon) \le \{4p|\phi|_\infty(|\phi|_p + C_p\|\phi\|_\gamma^{1/p}|\phi|_\infty^{1-1/p})\}^p(\epsilon - \widetilde{C}\|\phi\|_\gamma/N)^{-2p}N^{-p},$$

where C_p is the constant in Theorem 3.1.

Proof. As in the previous result, we have $\phi \circ \pi = \psi + \chi - \chi \circ f$ where $\chi \in L^\infty(\Delta)$ and $\psi : \bar{\Delta} \to \mathbb{R}$ is Lipschitz. By Theorem 3.1,

$$m_\Delta(|\tfrac{1}{N}\psi_N| > \epsilon) \le \{4p|\psi|_\infty(|\psi|_p + C_p\|\psi\|_\theta^{1/p}|\psi|_\infty^{1-1/p})\}^p\epsilon^{-2p}N^{-p}.$$

Since $|\phi_N \circ \pi - \psi_N| \le 2|\chi|_\infty$,

$$\mu(|\tfrac{1}{N}\phi_N| > \epsilon) \le \{4p|\phi|_\infty(|\phi|_p + C_p\|\psi\|_\theta^{1/p}|\phi|_\infty^{1-1/p})\}^p(\epsilon - 2|\chi|_\infty/N)^{-2p}N^{-p}.$$

Moreover, it follows from the proof of [23, Lemma 3.2] that $|\chi|_\infty \le \widetilde{C}\|\phi\|$ and $\|\psi\|_\theta \le \widetilde{C}\|\phi\|$, completing the proof (with a modified choice of $\widetilde{C}$). □

Theorem 4.3. *Let $T : M \to M$ be nonuniformly hyperbolic modelled by a Young tower $\pi : \Delta \to M$, and suppose that $m^u(y \in Y : R(y) > n) = O(n^{-(\beta+1)})$ where $\beta > 1$. Let $M_N = \pi(D_N)$ where $D_N = \{(y,\ell) \in \Delta : R(y) \ge N\}$ and suppose that $\mu(\overline{M_{N_0}}) < 1$ for some N_0. Then for any $\beta' > \beta$, there is an open and dense set of Hölder observables $\psi : M \to \mathbb{R}$ with $\epsilon_0(\psi) > 0$ such that for all $\epsilon \in (0, \epsilon_0(\psi))$*

$$\mu(|\tfrac{1}{N}\psi_N - \bar{\psi}| > \epsilon) > N^{-\beta'}$$

for infinitely many N. Moreover $\epsilon_0(\psi)$ may be taken as constant on a Hölder neighborhood of ψ.

Proof. This follows from the same arguments used in the proof of Theorem 3.5 and Remark 3.6. $\square$

5. Nonuniformly hyperbolic flows

Let $f : X \to X$ be an invertible measure-preserving transformation with ergodic invariant measure μ. Suppose that $h : X \to \mathbb{R}^+$ is an L^∞ roof function and define the suspension flow $f_t : X^h \to X^h$ with invariant ergodic measure $\mu^h = \mu \times \text{Lebesgue}/\bar{h}$ where $\bar{h} = \int_X h\,d\mu$.

Let $\phi : X^h \to \mathbb{R}$ be an L^∞ observable with mean zero. Define $\phi_T = \int_0^T \phi \circ f_t\,dt$. We consider large deviation results for $\mu^h(|\phi_T| > T\epsilon)$. We assume large deviation results on X and deduce results on X^h.

Define the induced observable $\Phi = \phi_h = \int_0^h \phi \circ f_t\,dt : X \to \mathbb{R}$ also with mean zero. As usual, $\Phi_N = \sum_{j=0}^{N-1} \Phi \circ f^j$. Similarly, $h_N = \sum_{j=0}^{N-1} h \circ f^j$.

5.1. **Exponential case.**

Theorem 5.1. *Suppose that $f_t : X^h \to X^h$ is a suspension flow built over a map $f : X \to X$ with roof function $h \in L^\infty(X)$. Let $\phi \in L^\infty(X^h)$ be a mean zero observable and define $\Phi \in L^\infty(X)$ as above.*

Assume that there exist rate functions $c_\Phi(\epsilon)$, $c_h(\epsilon)$ such that

$$\lim_{N\to\infty} \tfrac{1}{N} \log \mu(|\tfrac{1}{N}\Phi_N| > \epsilon) = -c_\Phi(\epsilon), \tag{5.1}$$

$$\lim_{N\to\infty} \tfrac{1}{N} \log \mu(|\tfrac{1}{N} h_N - \bar{h}| > \epsilon) = -c_h(\epsilon). \tag{5.2}$$

Then there is a rate function $c(\epsilon)$ such that

$$\limsup_{T\to\infty} \tfrac{1}{T} \log \mu^h(|\tfrac{1}{T}\phi_T| > \epsilon) \le -c(\epsilon).$$

Define the lap number $n[x,T]$ to be the integer satisfying

$$h_{n[x,T]}(x) \le T < h_{n[x,T]+1}(x).$$

Proposition 5.2. *For $\delta > 0$ sufficiently small,*

$$\limsup_{T\to\infty} \tfrac{1}{T} \log \mu(|n[\cdot,T] - T/\bar{h}| \ge \delta T) \le -(1/\bar{h})(1+\delta\bar{h}) c_h(\delta\bar{h}^2(1+\delta\bar{h})^{-1}).$$

Proof. Write

$$\mu(|n[\cdot,T] - T/\bar{h}| \ge \delta T) = \mu(n[\cdot,T] \ge T_+) + \mu(n[\cdot,T] \le T_-),$$

where $T_\pm = (1/\bar{h})(1 \pm \delta\bar{h})T$.

Since $\mu(n[\cdot,T] \ge K) = \mu(h_K \le T)$, we compute that

$$\begin{aligned}\mu(n[\cdot,T] \ge T_+) = \mu(h_{T_+} \le T) &= \mu(h_{T_+} - T_+\bar{h} \le \bar{h}(1+\delta\bar{h})^{-1}T_+ - T_+\bar{h}) \\ &= \mu(h_{T_+} - T_+\bar{h} \le -\delta\bar{h}^2(1+\delta\bar{h})^{-1}T_+),\end{aligned}$$

and similarly

$$\mu(n[\cdot,T] \le T_-) = \mu(h_{T_-} - T_-\bar{h} \ge \delta\bar{h}^2(1-\delta\bar{h})^{-1}T_-).$$

Applying (5.2),

$$\limsup_{T\to\infty} \tfrac{1}{T} \log \mu(|n[\cdot,T] - T| \geq \delta T)$$
$$\leq -(1/\bar h)\min\{(1+\delta\bar h)c_h(\delta\bar h^2(1+\delta\bar h)^{-1}),(1-\delta\bar h)c_h(\delta\bar h^2(1-\delta\bar h)^{-1})\},$$

and the result follows from convexity of c_h. □

For $\epsilon > 0$, there is a unique $\delta < \epsilon/(|h|_\infty|\phi|_\infty)$ such that $(1/\bar h)c_\Phi((\epsilon-\delta|h|_\infty|\phi|_\infty)\bar h) = (1/\bar h)(1+\delta\bar h)c_h(\delta\bar h^2(1+\delta\bar h)^{-1})$. Define $c(\epsilon)$ to be this common value. Then $c(\epsilon)$ is a rate function (in particular $c(\epsilon) > 0$ for all $\epsilon \neq 0$ sufficiently small).

Proof of Theorem 5.1. For notational convenience, we restrict T to integer multiples of $\bar h$. Since $\phi \in L^\infty$, this is no loss of generality.

Let $\delta > 0$ and define $A = \{x : |n[x,T] - T/\bar h| \leq \delta T\}$. For $x \in A$, we have $|\Phi_{n[x,T]}(x) - \Phi_{T/\bar h}(x)| \leq \delta T|\Phi|_\infty \leq \delta TK$ where $K = |h|_\infty|\phi|_\infty$. Hence

$$\mu\Big(|\Phi_{n[x,T]}(x)| \geq \epsilon T\Big) \leq \mu\Big(|\Phi_{T/\bar h}| \geq (\epsilon - \delta K)\bar h(T/\bar h)\Big) + \mu(X - A).$$

By (5.1) and Proposition 5.2,

$$\limsup_{T\to\infty} \tfrac{1}{T}\log\mu(|\Phi_{n[\cdot,T]}| \geq \epsilon T)$$
$$\leq -(1/\bar h)\min\{c_\Phi((\epsilon-\delta K)\bar h),(1+\delta\bar h)c_h(\delta\bar h^2(1+\delta\bar h)^{-1})\}.$$

This holds for all $\delta > 0$ sufficiently small and hence by definition of c,

$$\limsup_{T\to\infty} \tfrac{1}{T}\log\mu(|\Phi_{n[\cdot,T]}| \geq \epsilon T) \leq -c(\epsilon).$$

Now write

$$\phi_T(x,u) = \int_u^{T+u} \phi\circ f_t(x,0)\,dt = \Big(\int_0^T + \int_T^{T+u} - \int_0^u\Big)\phi\circ f_t(x,0)\,dt,$$

for $x \in X$ and $u < h(x)$. Hence, $|\max_{u\in[0,h(x)]}\phi_T(x,u) - \phi_T(x,0)| \leq 2K$. Moreover, $|\phi_T(x,0) - \phi_{h_{n[x,T]}(x)}(x,0)| \leq K$ and $\phi_{h_n(x)}(x,0) = \Phi_n(x)$ for all n (cf. [26]) so we obtain

$$|\max_{u\in[0,h(x)]}\phi_T(x,u) - \Phi_{n[x,T]}(x)| \leq 3K$$

for all $x \in X$. It follows that

$$\limsup_{T\to\infty}\tfrac{1}{T}\log\mu(|\max_{u\in[0,h(x)]}\phi_T(x,u)| \geq \epsilon T) \leq -c(\epsilon).$$

Finally, if $E \subset X$ is measurable, define $\hat E \subset X^h$ to be $\hat E = \{(x,u) \in X^h : x \in E\}$. Then $\int_{X^h} 1_{\hat E}\,d\mu^h = (1/\bar h)\int_X h 1_E\,d\mu$ so that $\mu^h(\hat E) \leq (1/\bar h)|h|_\infty\mu(E)$. The result follows. □

5.2. **Polynomial case.**

Theorem 5.3. *Suppose that $f_t : X^h \to X^h$ is a suspension flow built over a map $f : X \to X$ with roof function $h \in L^\infty(X)$. Let $\phi \in L^\infty(X^h)$ be a mean zero observable and define $\Phi \in L^\infty(X)$ as above.*

Assume that there exist $C \geq 1$ and $p > 0$ such that

$$\mu(|\tfrac{1}{N}\Phi_N| > \epsilon) \leq C\epsilon^{-2p}N^{-p}, \qquad \mu(|\tfrac{1}{N}h_N - \bar h| > \epsilon) \leq C\epsilon^{-2p}N^{-p},$$

for all $\epsilon > 0$ and all N sufficiently large. Then there is a constant C' such that

$$\mu(|\tfrac{1}{T}\phi_T| > \epsilon) \le C'\epsilon^{-2p}T^{-p},$$

for all $\epsilon > 0$ and all T sufficiently large.

Proof. This is similar to the proof of Theorem 5.1. Define the lap number $n[\cdot, T]$ as before. Below, the value of C may change from line to line and depends on ϕ, h, and p but not on ϵ, δ, N, T. We compute that

$$\mu(|n[\cdot, T] - T/\bar{h}| \ge \delta T) \le C\delta^{-2p}T^{-p},$$

and hence that

$$\mu(|\Phi_{n[\cdot,T]}| \ge \epsilon T) \le C(\epsilon - \delta K)^{-2p}T^{-p} + C\delta^{-2p}T^{-p},$$

where $K = |\phi|_\infty |h|_\infty$. Taking $\delta = \epsilon/(K+1)$, we obtain

$$\mu(|\Phi_{n[\cdot,T]}| \ge \epsilon T) \le C\epsilon^{-2p}T^{-p}.$$

Hence

$$\mu(|\max_{u\in[0,h(x)]} \phi_T(x,u)| \ge \epsilon T) \le C\,\epsilon^{-2p}T^{-p}$$

for T sufficiently large and the result follows. □

Appendix A. Rio's inequality

The following inequality due to Rio [31, Theorem 2.5] is taken from [27, Proposition 7].

Theorem A.1. *Let $\{X_i\}$ be a sequence of L^2 random variables with filtration $\mathcal{F}_i$. Let $p \ge 1$ and define*

$$b_{i,n} = \max_{i\le u\le n} \|X_i \sum_{k=i}^{u} E(X_k|\mathcal{F}_i)\|_p.$$

Then

$$E|X_1 + \cdots + X_n|^{2p} \le \Big(4p\sum_{i=1}^{n} b_{i,n}\Big)^p.$$

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