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Transverse intersection of invariant manifolds in perturbed multi-symplectic systems

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A multi-symplectic system is a PDE with a Hamiltonian structure in both temporal and spatial variables. This paper considers spatially periodic perturbations of symmetric multi-symplectic systems. Due their structure, unperturbed multi-symplectic systems often have families of solitary waves or front solutions, which together with the additional symmetries lead to large invariant manifolds. Periodic perturbations break the translational symmetry in space and might break some of the other symmetries as well.

In this paper, periodic perturbations of a translation invariant PDE with a one dimensional symmetry group are considered. It is assumed that the unperturbed PDE has a three dimensional invariant manifold associated with a solitary wave or front connection of multi-symplectic relative equilibria. Using the momentum associated with the symmetry group, sufficient conditions for the persistence of invariant manifolds and their transversal intersection are derived. In the equivariant case, invariance of the momentum under the perturbation gives the persistence of the full three dimensional manifold. In the equivariant case, there is also a weaker condition for the persistence of a two dimensional submanifold with a selected value of the momentum. In the non-equivariant case, the condition leads to the persistence of a one dimensional submanifold with a seleceted value of the momentum and a selected action of the symmetry group. These results are applicable to general Hamiltonian systems with double zero eigenvalue in the linearisation due to continuous symmetry.

The conditions are illustrated on the example of the defocussing nonlinear Schrödinger equations with perturbations which illustrate the three cases. The perturbations are: a equivariant, periodic, Hamiltonian perturbation which keeps the momentum level sets invariant; a equivariant damped, driven perturbation; and a perturbation which breaks the rotational symmetry.

Keywords: Multi-symplectic systems; Symmetry; Invariant manifolds; Transverse intersection
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1. Introduction

A multi-symplectic system is a partial differential equation with a Hamiltonian or (pre-)symplectic structure in both spatial and temporal variables. Although existence of a multi-symplectic structure might seem to be overly restrictive, it turns out that many Hamiltonian PDEs have such structure. This can also be seen from the many multi-symplectic numerical methods that have been developed for various PDEs, for an overview see [1]. The concept of a multi-symplectic structure, in the sense used here, was introduced in Bridges [2–4], where it was shown to be a natural dynamical systems framework for analysing multi-dimensional patterns and their stability. The connection with the Cartan form requires starting with a Lagrangian formulation of the problem which after multiple Legendre transform leads to a similar higher-order form – but without distinguishing the role of the two forms. In the case of Hamiltonian PDEs associated with first-order field theories, this connection can be made precise (cf. Marsden & Shkoller [5]).

Because of the symplectic structure for both temporal and spatial variables in multi-symplectic systems, the study of perturbations of such systems has the potential to provide a starting point for understanding of so-called spatio-temporal chaos. Spatio-temporal chaos corresponds to incoherent behaviour in both temporal and spatial directions of the system and has been observed both in numerics and in real measurements. In particular, spatio-temporal chaos seems to have similarities to turbulence (see e.g. Bohr et al. [6]), which arises in areas such as fluid flows, plasmas, chemical and biological systems. More applications of spatio-temporal chaos can be found, for example, in the fibre-optic communications (see Garcia-Ojalvo &

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Roy [7] and references therein). Several model systems which exhibit spatio-temporal chaos can be written as perturbed multi-symplectic systems.

Many multi-symplectic systems have symmetries, which means that such systems usually possess families of special solutions such as (travelling) solitary waves or fronts. Perturbations can lead to the break-up of those solutions and subsequent formation of spatio-temporal chaos. In this paper, the focus is on the effects of spatially periodic perturbations on the invariant manifold associated with solitary waves/fronts of multi-symplectic systems with symmetry. Sufficient criteria will be derived for the persistence of this manifold, or parts of this manifold, under spatially periodic perturbations both equivariant and symmetry-breaking ones. Furthermore, intersections of the persisting manifolds will be investigated, as these can signify the existence of chaotic dynamics through a suitable modification of the Smale-Birkhoff theorem [8]. As an illustration of the general theory, we will analyse the effect of various spatially periodic perturbations of the nonlinear generalised Schrödinger (NLS).

The generalised NLS equation has the form

$$i\Psi_t = \Psi_{xx} + iv\,\Psi_x + W'(\mid \Psi \mid^2)\Psi, \tag{1}$$

where $\Psi(x,t)$ is a complex-valued function of $(x,t) \in \mathbb{R} \times \mathbb{R}$, v is a parameter often relating to a travelling wave frame and $W(\cdot) : \mathbb{R} \to \mathbb{R}$ can be any smooth function. The NLS equation can be written as a multi-symplectic system, as will be shown below. This presentation follows Bridges & Derks [9]. Using the notation $\Psi = q_1 + iq_2$ and $\Psi_x + i\frac{v}{2}\Psi = p_1 + ip_2$, the NLS equation (1) can be recast in the following multi-symplectic representation:

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z), \tag{2}$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^4,$$

and

$$S(Z) = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + \tfrac{v}{2}(p_1 q_2 - p_2 q_1) + \tfrac{v^2}{8}(q_1^2 + q_2^2) + \tfrac{1}{2}W(q_1^2 + q_2^2).$$

Thus the partial differential equation (1) is represented by the pair of skew-symmetric matrices $\mathbf{M}$ and $\mathbf{J}$, and the scalar function $S(Z)$ on $\mathbb{R}^4$.

The NLS equation (1) is equivariant under phase shifts, which are actions of the Lie group $\mathcal{G} = SO(2)$. In the multi-symplectic formulation (2), a phase shift corresponds to rotating the vectors (q_1, q_2) and (p_1, p_2) simultaneously and it will be denoted by G_θ.

Solutions often appearing in Hamiltonian or multi-symplectic systems are solitary wave or front solutions which are travelling wave solutions connecting two steady states of the PDE. In the case of the focusing NLS equation ($W'(|\Psi|^2) = k + |\Psi|^2$), localized bright solitons connect the trivial steady states. The defocussing NLS equation ($W'(|\Psi|^2) = k - |\Psi|^2$) has dark solitons which connect two non-trivial steady states (with different phases) [10]. The equivariance of the equations implies that there exists a full group orbit of steady states and solitary waves/fronts, obtained by letting the symmetry group act on the solution. The symmetry can also be used to generalise the steady states and solitary/wave front solutions. In classical Hamiltonian systems, equilibrium solutions are generalised by changing them into the solutions aligned with a group orbit travelling at some constant speed - the so-called, relative equilibria. Such solutions are represented as $G_{at+\theta}Z_0$. In a multi-symplectic system, a Hamiltonian structure is present in both time and space, so it is natural to generalise the definition of a relative equilibrium to be a solution of the form

$$Z(x,t) = G_{at+bx+\theta_0}Z^\infty \quad \text{or} \quad \psi(x,t) = e^{i(at+bx+\theta_0)}\psi^\infty$$

and a relative solitary wave or front solution to be a solution of the form

$$Z(x,t) = G_{at+bx+\theta_0}\widetilde{Z}(x), \quad \text{or} \quad \psi(x,t) = e^{i(at+bx+\theta_0)}\widetilde{\psi}(x)$$

where the wave shape $\widetilde{\psi}(x)$ ($\widetilde{Z}(x)$) converges exponentially fast to a relative equilibrium ψ^∞ (Z^∞) for $x \to \pm\infty$. The relative equilibria and relative solitary waves/fronts usually come in families, which can be parametrised by the wave speeds a and b, or equivalently, the value of the mass and the momentum of the wave shape. In case of the NLS equation, they are bright/dark solitons in a frame rotating both with time and space. More details can be found in section 6.

A general multi-symplectic system of the form

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z),$$

is invariant under translations. Often such a system has travelling solitary waves/fronts; they usually come in families, which can be parametrised by the wave speed. By going to a moving frame $(t, x) \to (t, x - vt)$, where v is the wave speed of the solitary wave, the solitary wave becomes stationary and the multi-symplectic form is preserved. The only change is in the symplectic matrices, and the equation in the moving frame is $\mathbf{M}Z_t + \mathbf{J}_v Z_x = \nabla S(Z)$, where $\mathbf{J}_v = \mathbf{J} - v\mathbf{M}$. Perturbations can be in a travelling wave form as well, for example, the driving of wind on waves in the ocean. In such case the perturbation will select the solitary wave with the appropriate wave speed. We will assume that a transformation to a moving frame has been made and the perturbation can be written as having only spatial and no temporal dependence.

So we will look at perturbed multi-symplectic systems of the form

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z) + \varepsilon F(Z, x), \quad t > 0, \quad x \in \mathbb{R},$$

where $Z \in \mathbb{R}^4$, $\mathbf{M}$ and $\mathbf{J}$ are skew-symmetric matrices with $\mathbf{J}$ invertible, $S : \mathbb{R}^4 \to \mathbb{R}$, $\varepsilon \in \mathbb{R}$, and $F : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is periodic in its second spatial argument. For $\varepsilon = 0$ (unperturbed multi-symplectic system) it is assumed that the system is invariant under the action G_θ of a one-dimensional Lie group $\mathcal{G}$ and that the unperturbed multi-symplectic system has a family of relative solitary waves/fronts of the form

$$Z(x,t) = G_{at+bx+\theta_0}\widetilde{Z}(x),$$

where $\widetilde{Z}(x)$ decays exponentially fast to a relative equilibrium at $\pm\infty$. Hence such solution is uniformly moving with the symmetry group in both space and time. By taking $a = b = 0$, a standard stationary solitary wave/front is obtained. These families of solutions form invariant manifolds in the unperturbed system. With Noether's Theorem, it follows that there is some $C : \mathbb{R}^4 \to \mathbb{R}$ such that $\mathbf{J}\frac{d}{d\theta}G_\theta Z = \nabla C(G_\theta Z)$. Any solution of the form $Z(x,t) = G_{at+bx+\theta_0}\widetilde{Z}(x)$ stays within the same C level set, both in time and space. Thus if $\varepsilon = 0$, C is a temporal and spatial constant of motion for solutions of this type.

In the next section, the multi-symplectic formalism will be introduced as a framework for studying persistence of invariant manifolds in perturbed systems, as well as their intersections. The methodology to study the intersection of the invariant manifolds is inspired by Melnikov's method [11], which was originally proposed to study time-periodic perturbations of a differential equation having a hyperbolic fixed point connected to itself by a homoclinic orbit. In this approach, after the persistence of perturbed invariant manifolds is established, the splitting of those manifolds is studied by evaluating a function that gives the signed distance between the manifolds to the leading order. Melnikov's method has been extended to hyperbolic systems in higher dimensions [12], as well as to the non-hyperbolic fixed points [13–16]. However, none of those extensions considers fixed points with a linearisation that has zero eigenvalues. If one considers systems with symmetry, then zero eigenvalues will appear generically in the linearisations. Our method will investigate the effects of symmetry breaking perturbations on Hamiltonian systems, i.e. also on saddle-centres characterized by a double zero eigenvalue due to a continuous symmetry. The complications in the analysis due to the double zero eigenvalue can be dealt with by using that the origin of the double eigenvalue is in a symmetry group.

Being originally used for detecting chaotic motion in the systems of ODEs, Melnikov's method has been also successfully used to analyse chaotic behaviour in the solutions of partial differential equations. One can mention the seminal paper by Holmes & Marsden [17], where a Melnikov-type analysis was applied to the problem of a linearly damped sinusoidally forced buckled beam. Other problems investigated with a Melnikov-type method include perturbed sine-Gordon equations [18], perturbed KdV-Burgers equations [19], and perturbed focussing nonlinear Schrödinger equations [20, 21]. All of those systems have hyperbolic fixed points (the symmetry in the focussing NLS equation acts in a degenerate way on the fixed points).

There are several steps involved in deriving Melnikov-type conditions for the intersection of perturbed invariant manifolds. The first of these steps is the proof of the persistence of the unperturbed periodic orbit under perturbation. While sometimes this persistence is just assumed [15], the traditional approach relies on the implicit function theorem to prove it. In this paper a Lyapunov-Schmidt type argument is used to prove the persistence of the periodic orbit in section 3, as the presence of the symmetry group leads to a double zero eigenvalue in the linearization matrix of the unperturbed system.

After the persistence of the periodic orbit is established, the next question is the persistence of its invariant manifolds. The symmetry group plays a very important role in this consideration as it underlines the geometry of the invariant manifolds. As a first step, in section 4 the spectrum of the linearised perturbed Poincaré map is derived and used to obtain an understanding of local invariant manifolds. Then this information is used to construct the global manifolds while taking into account the details of perturbed geometry for each kind of perturbation. Three types of perturbations are considered in this paper. The first type are equivariant perturbations which also preserve the level sets of C, the function related to the symmetry group by Noether's Theorem. Reduction to a level set of C transforms the problem to a standard case of hyperbolic fixed point, which is well understood. The second kind of perturbations are also equivariant but do not necessarily preserve the level sets of C. In this case, it is shown in section 5 how a Melnikov-type measurement can be performed to study the intersection between centre-stable and centre-unstable manifolds of the perturbed periodic orbits. Finally, symmetry-breaking perturbations are considered and for some of them it is demonstrated how the existence of a transversal intersection of perturbed invariant manifolds can be established via a Melnikov-type measurement.

To illustrate the theoretical results, the example of the defocussing NLS equation with several equivariant and symmetry-breaking perturbations is considered in section 6. The paper concludes with some final remarks and a discussion of the results.

2. The general problem and some preliminaries

Consider the following multi-symplectic system with a small spatially periodic perturbation:

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z) + \varepsilon F(Z, \omega x), \quad t > 0, \quad x \in \mathbb{R}, \tag{3}$$

with $Z \in \mathbb{R}^4$, $\mathbf{M}$ and $\mathbf{J}$ are skew-symmetric matrices with $\mathbf{J}$ invertible, $S : \mathbb{R}^4 \to \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $F : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is 2π-periodic in its second spatial argument and ω is a parameter such that $F(Z, \omega x)$ is $T = \frac{2\pi}{\omega}$ periodic in x.

For the unperturbed multi-symplectic system ($\varepsilon = 0$), it is assumed that the system is invariant under the action of a one-dimensional Lie group $\mathcal{G}$ which is generated by the generator $\xi(Z)$. The action is denoted by $G_\theta Z = \exp(\theta\xi(Z))$ and $\xi(Z) = \frac{d}{d\theta}\big|_{\theta=0} G_\theta Z$. By applying Noether's theorem to both (pre-)symplectic operators (see e.g. [2]), it follows that there exists a pair of functionals P, $C : \mathbb{R}^4 \to \mathbb{R}$ such that

$$\mathbf{M}\xi(Z) = \nabla P(Z), \quad \mathbf{J}\xi(Z) = \nabla C(Z). \tag{4}$$

It is assumed that all functionals S, P, and C are smooth (at least in C^r, $r \geq 3$).

A relative solitary wave is a solution of the form

$$Z(x,t) = G_{at+bx}Z_0(x),$$

where the shape of the solitary wave $Z_0(x)$ converges to some limit, say Z_0^∞, for $x \to \pm\infty$ and $G_{at+bx}Z_0^\infty$ is a relative equilibrium (note that the limits can be different at $\pm\infty$). Substitution of this expression into the unperturbed multi-symplectic system (3, with $\varepsilon = 0$) shows that the wave shape Z_0 will satisfy the Hamiltonian ODE

$$\mathbf{J}(Z_0)_x = \nabla H_0(Z_0; b), \quad \text{with} \quad H_0(Z_0; b) = S(Z_0) - aP(Z_0) - bC(Z_0). \tag{5}$$

In the analysis in this paper the parameter a will not vary, hence the dependence on a in functionals etc. will be suppressed.

The limits of the wave shape $Z(x)$ are fixed points of solitary wave equation and satisfy

$$\nabla S(Z_0^\infty) = a\nabla P(Z_0^\infty) + b\nabla C(Z_0^\infty). \tag{6}$$

This is the Euler-Lagrange equation for the critical point problem

$$\operatorname*{crit}_Z \{\, S(Z) - aP(Z) \mid C(Z) = c \,\}. \tag{7}$$

So solving the fixed point problem (6) is equivalent to solving the critical point problem (7).

Because of the equivariance of the Hamiltonian ODE (5) under the action of the symmetry group $\mathcal{G}$, all solitary wave solutions and fixed points come in families. If $Z_0(x)$ is a solitary wave shape or if Z_0^∞ is a fixed point, then $G_\theta Z_0(x)$ is a solitary wave shape and $G_\theta Z_0^\infty$ is a fixed point too for any $G_\theta \in \mathcal{G}$. We will consider solitary wave shapes which are heteroclinic connections between two points on the same group orbit, i.e., if $Z(x)$ is a solitary wave shape and $\lim_{x\to-\infty} = Z_0^\infty$, then $\lim_{x\to\infty} = G_\theta Z_0^\infty$ for some $G_\theta \in \mathcal{G}$. The following hypothesis guarantees the existence of such solitary wave solutions of the unperturbed multi-symplectic system and gives some non-degeneracy conditions.

Hypothesis 2.1

a There is an interval $\mathcal{C} \subset \mathbb{R}$, such that for all $c \in \mathcal{C}$, the critical point problem (7) has a solution $Z_0^\infty(c)$ and the corresponding Euler-Lagrange equation (6) has a Lagrange multiplier $b(c)$, both of which depend smoothly on c. Furthermore, $\frac{db(c)}{dc} \neq 0$ for any $c \in \mathcal{C}$.

b For every $c \in \mathcal{C}$, there is a solitary wave shape $Z_0(x;c)$ which depends smoothly on c and is a homo/heteroclinic solution of the Hamiltonian system

$$\mathbf{J}\frac{d}{dx}Z_0(x;c) = \nabla H_0\left(Z_0(x;c); b(c)\right),$$

with $\lim_{x\to-\infty} Z_0(x;c) = Z_0^\infty(c)$ and $\lim_{x\to+\infty} Z_0(x;c) = G_{\theta_\infty}Z_0^\infty(c)$, for some $G_{\theta_\infty} \in \mathcal{G}$. Moreover, the orbits $Z_0(x;c)$ approach the fixed points $Z_0^\infty(c)$, respectively, $G_{\theta_\infty}Z_0^\infty(c)$, exponentially fast with velocity $\lambda(c) > 0$, i.e., the limits $\lim_{x\to-\infty} \left(Z_0(x;c) - Z_0^\infty(c)\right)e^{-\lambda(c)x}$ and $\lim_{x\to\infty} \left(Z_0(x;c) - G_\theta Z_0^\infty(c)\right)e^{\lambda(c)x}$ exist. Define $V_0^\pm(c) = \lim_{x\to\infty} e^{\pm\lambda(c)x}(Z_0(x;c))_x$ for future use.

c At the fixed points $Z_0^\infty(c)$ the generator $\xi(Z_0^\infty(c))$ does not vanish for any $c \in \mathcal{C}$.

d For all $c \in \mathcal{C}$, the vectors $\nabla H_0(Z; b(c))$ and $\nabla C(Z)$ are pointwise linearly independent at any point $Z \in \mathbb{R}^4$ except for the fixed points $G_\theta Z_0^\infty(c)$ (since $\nabla H_0(G_\theta Z_0^\infty(c); b(c)) = 0$).

Using Noether's theorem, it follows that $C(Z)$ is a constant of motion for the Hamiltonian ODE (5), for any b. Hence, any solitary wave shape Z_0 has the property that $C(Z_0(x;c)) = C(Z_0^\infty(c)) = c$ for any $x \in \mathbb{R}$ and $c \in \mathcal{C}$. Using that the Hamiltonian H_0 is a constant of motion too, one can define the function $s_a(c)$

as follows:

$$s_a(c) = S(Z_0^\infty(c)) - aP(Z_0^\infty(c)) = S(Z_0(x;c)) - aP(Z_0(x;c)),$$

where $c = C(Z_0^\infty(c))$. Using the definition of $s_a(c)$ and the Euler-Lagrange equation (6) for $Z_0^\infty(c)$, it follows immediately that the Lagrange multiplier $b(c)$ is related to $s_a(c)$ by

$$b(c) = s_a'(c).$$

Furthermore, the linearisation of the Hamiltonian system about the fixed points Z_0^∞ can be determined with hypothesis **H2.1**.

Lemma 2.2: *For any c, the linearisation $\mathbf{J}^{-1}D^2H_0\left(Z_0^\infty(c);b(c)\right)$ around the point $Z_0^\infty(c)$ has the spectrum $\{0,0,\pm\lambda(c)\}$. The zero eigenvalue of this linearisation has the corresponding eigenvector $\xi(Z_0^\infty(c))$ and a generalised eigenvector $u(c)$ which is given by $u(c) = \frac{1}{s_a''(c)}\frac{dZ_0^\infty(c)}{dc}$. The hyperbolic eigenvalues correspond to the exponential decay of the heteroclinic solution $Z_0(x;c)$ to $Z_0^\infty(c)$ as x approaches $-\infty$, and of $G_{-\theta_\infty}Z_0(x;c)$ to $Z_0^\infty(c)$ as $x\to+\infty$, respectively.*

Proof: Using the invariance of the Hamiltonian $H(Z;b)$, it follows immediately that the generator $\xi(Z_0^\infty(c))$ is in the kernel of the operator $D^2H_0(Z_0^\infty(c);b(c))$. Differentiating the Euler-Lagrange equation $\nabla H_0\left(Z_0^\infty(c);b(c)\right)=0$ with respect to c and using that $b(c)=s_a'(c)$, we obtain

$$\mathbf{J}^{-1}D^2H_0\left(Z_0^\infty(c);b(c)\right)\frac{dZ_0^\infty(c)}{dc} = s_a''(c)\,\mathbf{J}^{-1}\nabla C\left(Z_0^\infty(c)\right) = s_a''(c)\,\xi(Z_0^\infty(c)). \tag{8}$$

Therefore, we can conclude that the zero eigenvalue of the operator $\mathbf{J}_v^{-1}D^2H_0\left(Z_0^\infty(b);b\right)$ has algebraic multiplicity two with the generalised eigenvector as given in the Lemma.

By the Hypothesis **H2.1**(b) there exist relative heteroclinic orbit $Z_0(x;c)$ which connects the equilibria $Z_0^\infty(c)$ and $G_{\theta_\infty}Z_0^\infty(c)$ and approaches these equilibria exponentially fast with the velocity $\lambda(c)$. This gives the eigenvalues $\pm\lambda(c)$ in the spectrum of the linearised operator. □

Remark 1: Note that the condition $\frac{db(c)}{dc}\neq 0$ in Hypothesis **H2.1**(a) implies that the relative equilibria are non-degenerate.

The solitary wave shapes can be associated with invariant manifolds in the unperturbed system. First of all, for a fixed value of c, one can define the one dimensional invariant manifold associated with the relative equilibria Z_0^∞:

$$\mathcal{M}_0^\infty(c) = \{G_\theta Z_0^\infty(c) \mid G_\theta\in\mathcal{G}\}$$

and the two-dimensional invariant manifold associated with the wave shape of the solitary wave or front Z_0:

$$\mathcal{M}_0(c) = \{G_\theta Z_0(x;c) \mid x\in\mathbb{R},\, G_\theta\in\mathcal{G}\}.$$

These manifold are sketched in Figure 1. All manifolds together form a three-dimensional invariant manifold

$$\mathcal{M}_0 = \bigcup_{c\in\mathcal{C}}\mathcal{M}_0(c).$$

It follows immediately from Lemma 2.2 and the invariance of $H(Z,b)$ that this manifold is the centre-stable manifold and the centre-unstable manifold of any fixed point $G_\theta Z_0^\infty(c)$.

In the following two sections the persistence of those invariant manifolds under the perturbation $F(Z,x)$ will be studied. To analyse the persistence for the perturbed system, a transformation to the moving frame

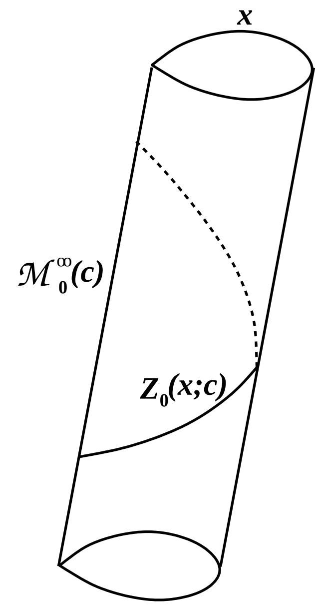

Figure 1. Geometry of the invariant two-dimensional manifold $\mathcal{M}_0(c)$ of the unperturbed system for a fixed value $c \in \mathcal{C}$. Different points on the group orbit $G_\theta Z_0^\infty(c)$ are connected by the relative heteroclinic orbits $Z_0(x; c)$.

$Z(x,t) = G_{at+bx} Z(x)$ is made and the resulting perturbed Hamiltonian system is

$$\mathbf{J} Z_x = \nabla H_0(Z; b) + \varepsilon D G^*_{at+bx}(Z)\, F(G_{at+bx} Z, x). \tag{9}$$

In case the perturbation F is equivariant, it holds $DG^*_{at+bx}(Z)\, F(G_{at+bx}Z, x) = F(Z, x)$. If F is not equivariant, it is natural to restrict the persistence questions to solutions with $a = 0 = b$. Using this, the non-autonomous system can be written as an autonomous system by using the suspended system

$$\begin{aligned} \mathbf{J} Z_x &= \nabla H_0(Z; b) + \varepsilon F(Z, \tau), \\ \tau_x &= \omega. \end{aligned} \tag{10}$$

Thanks to the smoothness assumption, this system has a smooth flow denoted by $\Phi_x^{\varepsilon,b} : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4 \times S^1$. The Poincaré map for this system is given by

$$\Pi^{\varepsilon,b} Z = \pi_1 \Phi_T^{\varepsilon,b}(Z, 0), \tag{11}$$

where $\pi_1 : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4$ denotes the projection onto the first factor.

By differentiating the flow map, it follows immediately that the linearised unperturbed Poincaré map $D\Pi^{0,b(c)}(Z_0^\infty(c))$ is the solution matrix at $x = T$ of the constant coefficient ODE $\mathbf{J} U_x = D^2 H_0(Z_0^\infty(c); b(c)) U$. In Appendix A it is shown that the linearisation of the unperturbed Poincaré map evaluated at the relative equilibria satisfies the following properties.

Lemma 2.3: *For any $c \in \mathcal{C}$, write $L_0(c) = D\Pi^{0,b(c)}\,(Z_0^\infty(c)) - I$, where I is the identity matrix. It holds*

$$L_0(c)\, \xi(Z_0^\infty(c)) = 0, \tag{12}$$

$$L_0(c)\, \frac{dZ_0^\infty(c)}{dc} = -s_a''(c)\, \frac{\partial}{\partial b}\bigg|_{b=b(c)} \Pi^{0,b}(Z_0^\infty(c)) = s_a''(c)\, T\, \xi(Z_0^\infty(c)), \tag{13}$$

Furthermore, the vector $\nabla C(Z_0^\infty(c))$ is orthogonal to the range of the operator $L_0(c)$.

3. Persistence of the relative equilibria

For any $c \in \mathcal{C}$ and b arbitrary, the relative equilibria $G_{(b(c)-b)x}Z_0^\infty(c)$ of the unperturbed system (5) correspond to relative periodic orbits of the form $(G_{(b(c)-b)x}Z_0^\infty(c), \omega x + \tau_0)$ in the suspended system (10), where τ_0 is an arbitrary parameter. We will denote the periodic orbits $(b(c) = b)$ by

$$p_0(c) = \{(Z_0^\infty(c), \tau) \mid \tau \in S^1\}.$$

Since periodic orbits of the suspended system correspond to fixed points of the Poincaré map, the points $Z_0^\infty(c)$ are fixed points of the unperturbed Poincaré map $\Pi^{0,b(c)}$.

The invariance of the unperturbed system under the group $\mathcal{G}$ implies that the relative periodic orbits come in families

$$\mathcal{G}p_0(c) = \{(G_\theta Z_0^\infty(c), \tau) \mid \tau \in S^1, G_\theta \in \mathcal{G}\}.$$

So, the two-dimensional invariant manifold $\cup_{c\in\mathcal{C}}\mathcal{M}_0^\infty(c)$ of relative equilibria in the unperturbed system (5) corresponds to a three-dimensional invariant manifold of relative periodic orbits in the unperturbed suspended dynamics (10) and a two-dimensional manifold of relative equilibria of the unperturbed Poincaré map (11).

The values of the C level sets are used to parametrize the relative equilibria in the unperturbed case. The dynamics of the C level sets plays an important role in the persistence arguments too. Using the suspended system (10), the relation $\mathbf{J}\xi(Z) = \nabla C(Z)$ and the invariance of H_0, it follows for a solution of the perturbed system

$$\frac{d}{dx}C(Z) = -\langle \xi(Z), \mathbf{J}Z_x\rangle = -\varepsilon\langle \xi(Z), F(Z,\tau)\rangle, \quad \tau = \omega x + \tau_0. \tag{14}$$

Hence, the C level sets are invariant under the perturbation if and only if $\langle F(Z,\tau), \xi(Z)\rangle = \langle \mathbf{J}^{-1}F(Z,\tau), \nabla C(Z)\rangle = 0$ for all Z and τ (i.e., $\mathbf{J}^{-1}F(Z,\tau)$ is tangent to the C-level sets).

Furthermore, fixed points of the Poincaré map correspond to periodic solutions of the suspended system, and any T-periodic solution satisfies

$$-\varepsilon\int_0^T \langle \xi(Z(x)), F(Z(x), \tau(x))\rangle\, dx = C(Z(T)) - C(Z(0)) = 0.$$

In other words, the periodic orbit $G_\theta p_0(c)$ can persist only if

$$\int_0^T \langle \xi(G_\theta Z_0^\infty(c)), F(G_\theta Z^\infty(c), \tau(x))\rangle\, dx$$

$$= \frac{1}{\omega}\int_0^{2\pi} \langle \xi(G_\theta Z_0^\infty(c)), F(G_\theta Z^\infty(c), \tau)\rangle\, d\tau = 0.$$

Therefore we introduce the following quantities

$$R(Z,\tau) = \langle F(Z,\tau), \xi(Z)\rangle = -\langle \mathbf{J}^{-1}F(Z,\tau), \nabla C(Z)\rangle,$$

and the average value of this quantity

$$\mathcal{R}(Z) = \frac{1}{2\pi}\int_0^{2\pi} R(Z,\tau)\, d\tau = \frac{1}{T}\int_0^T \langle F(Z,\omega x), \xi(Z)\rangle\, dx.$$

We have shown above that the C level sets are invariant under the perturbation if $R(Z,\tau) = 0$ for all Z and τ. Furthermore, if the perturbation is equivariant, then $\mathcal{R}(G_\theta Z) = \mathcal{R}(Z)$ for all θ and Z.

Theorem 3.1 : *A necessary condition for the existence of a smooth curve of (relative) periodic orbits $G_{at+b_\varepsilon x} Z_\varepsilon(x)$ with $Z_\varepsilon(0)$ converging to a relative equilibrium $G_\theta Z_0(c)$ for $\varepsilon \to 0$, is given by $\mathcal{R}(G_\theta Z_0(c)) = 0$. If the perturbation is not equivariant, then it is also necessary that $a = 0 = b_\varepsilon$ and $c = c_0$, where c_0 is such that $b(c_0) = 0$.*

Sufficient conditions for the persistence of the relative periodic orbits are given by one of the following three criteria:

(i) If F is equivariant and $R(Z,\tau) = 0$ for all values of Z and τ, then for any $c \in \mathcal{C}$, there is an $\varepsilon_0 > 0$ and smooth curves $\{b_\varepsilon(c), |\varepsilon| < \varepsilon_0\}$ and $\{Z_\varepsilon^\infty(c), |\varepsilon| < \varepsilon_0\}$ such that $Z_\varepsilon^\infty(c)$ are fixed points of the perturbed Poincaré map $\Pi^{\varepsilon, b_\varepsilon(c)}$ with $Z_\varepsilon^\infty(c) \to Z_0^\infty(c)$, $b_\varepsilon(c) \to b(c)$, for $\varepsilon \to 0$, and $C(Z_\varepsilon^\infty(c)) = c$. The equivariance implies the existence of a two-dimensional manifold of fixed points: $G_\theta Z_\varepsilon^\infty(c)$, for any $|\varepsilon| < \varepsilon_0$. Furthermore, for any $c \in \mathcal{C}$ and $G_\theta \in \mathcal{G}$, the perturbed PDE has relative periodic orbits of the form $Z(x,t) = G_{at+b_\varepsilon x+\theta} Z_\varepsilon(x)$, where $Z_\varepsilon(x)$ is periodic in x and $Z_\varepsilon(0) = Z_\varepsilon^\infty(c)$.

(ii) If F is equivariant and there is some $\hat{c} \in \mathcal{C}$ satisfying the selection criterion $\mathcal{R}(Z_0^\infty(\hat{c})) = 0$ and the non-degeneracy condition $\frac{d}{dc}|_{c=\hat{c}} \mathcal{R}(Z_0^\infty(c)) \neq 0$, then there is an $\varepsilon_0 \neq 0$ and a curve $\{(Z_\varepsilon^\infty, b_\varepsilon), |\varepsilon| < \varepsilon_0\}$ of the fixed points Z_ε^∞ of the perturbed Poincaré map $\Pi^{\varepsilon, b_\varepsilon}$. In the limit for $\varepsilon \to 0$ it holds $Z_\varepsilon^\infty \to Z_0^\infty(\hat{c})$, as well as $b_\varepsilon \to b(\hat{c})$. The equivariance implies the existence of a one-dimensional manifold of fixed points: $G_\theta Z_\varepsilon^\infty$, for any $|\varepsilon| < \varepsilon_0$. Furthermore, for any $G_\theta \in \mathcal{G}$ the perturbed PDE has relative periodic orbits of the form $Z(x,t) = G_{at+b_\varepsilon x+\theta} Z_\varepsilon(x)$, where $Z_\varepsilon(x)$ is periodic in x and $Z_\varepsilon(0) = Z_\varepsilon^\infty$.

(iii) If F is not equivariant, then take $b = 0 = a$ and $c = c_0$, where c_0 is such that $b(c_0) = 0$. If there is some $\hat{\theta}$ such that $\mathcal{R}(G_{\hat{\theta}} Z_0^\infty(c_0)) = 0$ and $\frac{d}{d\theta}|_{\theta=\hat{\theta}} \mathcal{R}(G_\theta Z_0^\infty(c_0)) \neq 0$, then there is an $\varepsilon_0 \neq 0$ and a curve $\{Z_\varepsilon^\infty, |\varepsilon| < \varepsilon_0\}$ of the fixed points of the perturbed Poincaré map $\Pi^{\varepsilon,0}$. In the limit for $\varepsilon \to 0$ it holds $Z_\varepsilon^\infty \to G_{\hat{\theta}} Z_0^\infty(\hat{c})$. Furthermore, the perturbed PDE has a spatially periodic solution $Z_\varepsilon(x)$ with $Z_\varepsilon(0) = Z_\varepsilon^\infty$.

The (relative) equilibria of the Poincaré map correspond to (relative) T-periodic solutions $Z_\varepsilon(x)$ of the perturbed Hamiltonian system (9). If ε is small, those (relative) periodic solutions $Z_\varepsilon(x)$ are order $\mathcal{O}(\varepsilon)$ near the persisting unperturbed (relative) equilibria.

The necessary conditions follow immediately from the observations before the Theorem. The proof of the sufficient conditions is given in the Appendix A.

From Theorem 3.1 it follows immediately that if F is equivariant and $R(Z,\tau) = 0$ for all Z and τ (case 1), then the full three-dimensional invariant manifold of invariant relative periodic orbits persists. If F is equivariant and $R(Z,\tau)$ has isolated roots (case 2), then a two-dimensional invariant manifold of invariant relative periodic orbits persists. Finally, if F is not equivariant, then one can expect at most the persistence of a one-dimensional manifold of periodic orbits.

Remark 2 : Any Hamiltonian equivariant perturbation automatically satisfies $R(Z,\tau) = 0$ for all values of Z and τ. Thus for such perturbations the fixed points of the Poincaré map persist for all admissible values of $c \in \mathcal{C}$.

4. Persistence of invariant manifolds

The invariant manifolds $\mathcal{M}_0^\infty(c)$, $\mathcal{M}_0(c)$ and $\cup_{c\in\mathcal{C}} \mathcal{M}_0(c)$ in the unperturbed system (5) can be related to invariant manifolds associated with the unperturbed periodic orbits $p_0(c)$ in the suspended system (10). The two-dimensional manifold $\mathcal{M}_0(c)$ corresponds to the three-dimensional manifold formed by the centre-stable manifold of the periodic orbit $p_0(c)$ restricted to a fixed level set of C and also to the three-dimensional manifold formed by the centre-unstable manifold of the periodic orbit $p_0(c)$ restricted to a fixed level set of C:

$$\mathcal{M}_0(c) \mapsto \mathcal{M}_0^{\rm ext}(c) = W^s(\mathcal{G}p_0(c)) = \{(G_\theta Z_0(x;c),\tau) \mid G_\theta \in \mathcal{G},\, x\in\mathbb{R},\, \tau\in S^1\};$$
$$\mathcal{M}_0(c) \mapsto \mathcal{M}_0^{\rm ext}(c) = W^u(\mathcal{G}p_0(c)) = \{(G_\theta Z_0(x;c),\tau) \mid G_\theta \in \mathcal{G},\, x\in\mathbb{R},\, \tau\in S^1\}.$$

Similarly, the three-dimensional manifold $\cup_{c\in\mathcal{C}}\mathcal{M}_0(c)$ corresponds to the four-dimensional centre-stable manifold of the periodic orbit $p_0(c)$ and also to the four-dimensional centre-unstable manifold of the periodic orbit $p_0(c)$:

$$\cup_{c\in\mathcal{C}}\mathcal{M}_0(c) \mapsto \mathcal{M}_0^{\rm ext} = \cup_{c\in\mathcal{C}} W^s(\mathcal{G}p_0(c)) = \{(G_\theta Z_0(x;c),\tau) \mid G_\theta \in \mathcal{G},\, x\in\mathbb{R},\, \tau\in S^1,\, c\in\mathcal{C}\};$$
$$\cup_{c\in\mathcal{C}}\mathcal{M}_0(c) \mapsto \mathcal{M}_0^{\rm ext} = \cup_{c\in\mathcal{C}} W^u(\mathcal{G}p_0(c)) = \{(G_\theta Z_0(x;c),\tau) \mid G_\theta \in \mathcal{G},\, x\in\mathbb{R},\, \tau\in S^1,\, c\in\mathcal{C}\}.$$

The tangent and normal spaces to those manifolds are described below.

Lemma 4.1: *For every $c\in\mathcal{C}$, at the point $P=(G_\theta Z_0(x;c),\tau)\in\mathcal{M}_0^{\rm ext}(c)$, the tangent space and normal space of this three dimensional manifold are*

$$T_P\mathcal{M}_0^{\rm ext}(c) = \operatorname{span}\left\{(\xi(G_\theta Z_0(x;c)),0)\,,\, \left(\tfrac{\partial}{\partial x}G_\theta Z_0(x;c),0\right)\,,\,(0,1)\right\}$$
$$N_P\mathcal{M}_0^{\rm ext}(c) = \operatorname{span}\left\{(\nabla C(G_\theta Z_0(x;c)),0)\,,\,(\nabla H_0(G_\theta Z_0(x;c);b(c)),0)\right\}$$

Similarly, the tangent space and normal space to the four dimensional manifold $\mathcal{M}_0^{\rm ext}$ are

$$T_P\mathcal{M}_0^{\rm ext} = \operatorname{span}\left\{(\xi(G_\theta Z_0(x;c)),0)\,,\, \left(\tfrac{\partial}{\partial x}G_\theta Z_0(x;c),0\right)\,,\, \left(\tfrac{\partial}{\partial c}G_\theta Z_0(x;c),0\right)\,,\,(0,1)\right\}$$
$$N_P\mathcal{M}_0^{\rm ext} = \operatorname{span}\left\{(\nabla H_0(G_\theta Z_0(x;c);b(c)),0)\right\}$$

Note that $\frac{\partial}{\partial x}G_\theta Z_0(x;c) = \mathbf{J}^{-1}\nabla H_0(G_\theta Z_0(x;c);b(c))$ and $\xi(G_\theta Z_0) = \frac{\partial}{\partial\theta}G_\theta Z_0 = \mathbf{J}^{-1}\nabla C(G_\theta Z_0)$.

Proof: For every $c\in\mathcal{C}$, the tangent space to the manifold $\mathcal{M}_0^{\rm ext}(c)$ at the point $(G_\theta Z_0(x;c),\tau)$ is spanned by $\xi(G_\theta Z_0(x;c)),0)$, $(\frac{\partial}{\partial x}G_\theta Z_0(x;c),0) = (\mathbf{J}^{-1}\nabla H_0(G_\theta Z_0(x;c);b(c)),0)$, and $(0,1)$ (note that $\xi(G_\theta Z_0(x;c)) = \frac{\partial}{\partial\theta}G_\theta Z_0(x;c))$. Since both C and H_0 are constants of motion for the dynamics of the unperturbed system, the three-dimensional manifold $\mathcal{M}_0^{\rm ext}(c) = W^s(\mathcal{G}p_0(c)) \equiv W^u(\mathcal{G}p_0(c))$ is embedded within the level set $C^{-1}(c)$ and within the level set $H_0^{-1}(s_a(c)-cb(c))$ in the five-dimensional ambient space. Thus, at every point $(G_\theta Z_0(x;c),\tau)$ in the invariant manifold, the vectors $(\nabla C(G_\theta Z_0(x;c)),0)$ and $(\nabla H_0(G_\theta Z_0(x;c);b(c)),0)$ are perpendicular to the invariant manifold $\mathcal{M}_0^{\rm ext}(c)$. Since both vectors are linearly independent if $x\neq\pm\infty$, the normal plane to the invariant manifold is spanned by those two vectors at each point of the manifold.

Similarly, for every point $(G_\theta Z_0(x;c),\tau)$ in the invariant four-dimensional manifold $\mathcal{M}_0^{\rm ext} = \cup_{c\in\mathcal{C}}W^s(\mathcal{G}p_0(c))$, the vector $(\frac{\partial}{\partial c}Z_0(x;c),0)$ is tangent to this manifold as well as the three vectors mentioned before. So the normal vector at a point of $\mathcal{M}_0^{\rm ext}$ is a linear combination of $(\nabla C(G_\theta Z_0(x;c)),0)$ and $(\nabla H_0(G_\theta Z_0(x;c)),0)$ which is perpendicular to $(\frac{\partial}{\partial c}Z_0(x;c),0)$. Differentiation of $H_0(Z_0(x,c);b(c)) = s_a(c)-b(c)c$ with respect to c shows that $\nabla H_0(Z_0(x,c);b(c))$ is orthogonal to $\frac{\partial}{\partial c}Z_0(x;c),0)$. Hence, $\nabla H_0(G_\theta Z_0(x,c);b(c))$ is orthogonal to the full four-dimensional invariant manifold $\mathcal{M}_0^{\rm ext}$ at each point $(G_\theta Z_0(x;c),\tau)$. □

Now the perturbed centre-stable/unstable manifolds of the persisting relative periodic orbits can be considered and compared to the corresponding unperturbed manifolds. The first step in this analysis is the spectrum of the perturbed Poincaré map at the persisting relative equilibria.

Lemma 4.2: *For the three cases in which persistence is proved, the linearised Poincaré map has the following spectrum:*

(i) If F is equivariant and $R(Z,\tau) = 0$ for all values of Z and τ, then for any $c \in \mathcal{C}$ and $|\varepsilon| < \varepsilon_0$, the spectrum of the linearised perturbed Poincaré map $D\Pi^{\varepsilon,b(c)}(Z_\varepsilon^\infty(c))$ is $(e^{\lambda_\varepsilon^- T}, 1, 1, e^{\lambda_\varepsilon^+ T})$, where $\lambda_\varepsilon^- < 0 < \lambda_\varepsilon^+$ and $\lim_{\varepsilon\to 0} \lambda_\varepsilon^\pm = \pm\lambda(c)$.

(ii) If F is equivariant and $\hat{c} \in \mathcal{C}$ is such that $\mathcal{R}(Z_0^\infty(\hat{c})) = 0$ and $\frac{d}{dc}|_{c=\hat{c}}\mathcal{R}(Z_0^\infty(c)) \neq 0$, then define $\mu_1 = -T\,\frac{d}{dc}|_{c=\hat{c}}\mathcal{R}(Z_0^\infty(c))$. For any $|\varepsilon| < \varepsilon_0$, the spectrum of the linearised perturbed Poincaré map $D\Pi^{\varepsilon,b_\varepsilon}(Z_\varepsilon^\infty)$ is $(e^{\lambda_\varepsilon^- T}, 1 + \varepsilon\mu_1 + \mathcal{O}(\varepsilon^2), 1, e^{\lambda_\varepsilon^+ T})$, where $\lambda_\varepsilon^- < 0 < \lambda_\varepsilon^+$ and $\lim_{\varepsilon\to 0}\lambda_\varepsilon^\pm = \pm\lambda(\hat{c})$. The eigenvector related to the eigenvalue bifurcating out of 1 is $\mathbf{v}_\varepsilon = \xi(Z_\varepsilon^\infty) + \varepsilon\,\frac{\mu_1}{s_a''(\hat{c})T}\,\frac{dZ_0^\infty}{dc}(\hat{c}) + \mathcal{O}(\varepsilon^2)$.

(iii) If F is not equivariant, then take $c = c_0$ and $b = 0 = a$. Let the group action $G_{\hat{\theta}}$ be such that $\mathcal{R}(G_{\hat{\theta}} Z_0^\infty(c_0)) = 0$ and $\frac{d}{d\theta}|_{\theta=\hat{\theta}}\mathcal{R}(G_\theta Z_0^\infty(c_0)) \neq 0$. Define μ_1 to be such that $\mu_1^2 = -s_0''(\hat{c})\,T^2\,\frac{d}{d\theta}|_{\theta=\hat{\theta}}\mathcal{R}(G_\theta Z_0^\infty(c_0))$. For any $|\varepsilon| < \varepsilon_0$, the spectrum of the linearised perturbed Poincaré map $D\Pi^{\varepsilon,0}(Z_\varepsilon^\infty)$ is $(e^{\lambda_\varepsilon^- T}, 1 \pm \sqrt{\varepsilon}\mu_1 + \mathcal{O}(\varepsilon), e^{\lambda_\varepsilon^+ T})$, where $\lambda_\varepsilon^- < 0 < \lambda_\varepsilon^+$ and $\lim_{\varepsilon\to 0}\lambda_\varepsilon^\pm = \pm\lambda(c_0)$. The eigenvectors related to the eigenvalues bifurcating out of 1 are $\mathbf{v}_\varepsilon^\pm = \xi(G_{\hat{\theta}} Z_0^\infty) \pm \sqrt{\varepsilon}\,\frac{\mu_1}{s_0''(c_0)T}\,\frac{dG_{\hat{\theta}} Z_0^\infty}{dc}(c_0) + \mathcal{O}(\varepsilon)$.

The proof of this lemma is given in the Appendix A.

The linearisation of the Poincaré map in the various cases gives the existence of local centre-stable and centre-unstable manifolds near the persisting periodic orbits. A description of those manifolds and extensions to global manifolds is given below:

Case 1 If F is equivariant and $R(Z,\tau) = 0$ for all values of Z and τ, then for any $c \in \mathcal{C}$ the fixed points $Z_\varepsilon^\infty(c)$ of the Poincaré map $\Pi^{\varepsilon,b_\varepsilon(c)}$ correspond to periodic orbits in (10), which will be denoted by $p_\varepsilon(c)$. Since $R(Z,\tau) = 0$ implies that $\frac{d}{dx}C(Z(x)) = 0$ for any solution $(Z(x), \omega x + \tau_0)$ of (10), the dynamics takes place within the level set $C^{-1}(c)$, where $c = C(Z(0))$. Within the four-dimensional level set $C^{-1}(c)$, there exist a three-dimensional local centre-stable manifold $G_\theta W_{loc}^s(p_\varepsilon(c))$ and a three-dimensional local centre-unstable manifold $G_\theta W_{loc}^u(p_\varepsilon(c))$, which are order ε close to the unperturbed three dimensional centre-stable/unstable manifold $G_\theta W^s(p_0(c)) = G_\theta W^u(p_0(c))$. Finite time global extensions of local perturbed manifolds are

$$\begin{aligned}\mathcal{M}_\varepsilon^s(c,Y) &= G_\theta W^s\left(p_\varepsilon(c)\right) = G_\theta \textstyle\bigcup_{-Y\le y\le 0} \Phi_y^{\varepsilon,b(c)} W_{loc}^s\left(p_\varepsilon(c)\right)\\ \mathcal{M}_\varepsilon^u(c,Y) &= G_\theta W^u\left(p_\varepsilon(c)\right) = G_\theta \textstyle\bigcup_{Y\ge y\ge 0} \Phi_y^{\varepsilon,b(c)} W_{loc}^u\left(p_\varepsilon(c)\right),\end{aligned}$$

For fixed Y, both manifolds are ε-close to the unperturbed manifold $\mathcal{M}_0^{\rm ext}(c)$. The manifolds can be parametrised by θ_0, τ_0 and x_0 as follows.

Fix c, θ_0, τ_0, and x_0. For $x \ge 0$, resp. $x \le 0$, let $(Z_\varepsilon^{s/u}(x; x_0, \tau_0, \theta_0, c), \omega x + \tau_0)$ be solutions of (10) (with $b = b_\varepsilon(c)$) in the stable/unstable manifold $\mathcal{M}_\varepsilon^{s/u}(Y,c)$, such that

$$Z_0^{s/u}(0; x_0, \tau_0, \theta_0, c) = G_{\theta_0} Z_0(-x_0; c)$$

and

$$\left[G_{-\theta_0} Z_\varepsilon^{s/u}(0; x_0, \tau_0, \theta_0, c) - Z_0(-x_0; c)\right] \in \operatorname{span}\left\{\nabla H(Z_0(-x_0; c)),\ \nabla C(Z_0(-x_0; c_0))\right\}.$$

The component in the direction of $\nabla C(Z_0(-x_0; c_0))$ is due to the curvature of the C-manifold, but both $Z_\varepsilon^{s/u}(0; x_0, \tau_0, \theta_0, c)$ and $Z_0(-x_0; c)$ are in the same C-levelset $C^{-1}(c)$. Hence if the component of $\left[G_{-\theta_0} Z_\varepsilon^{s/u}(0; x_0, \tau_0, \theta_0, c) - Z_0(-x_0; c)\right]$ in the direction of $\nabla H(Z_0(-x_0; c))$ vanishes, then automatically the component in the $\nabla C(Z_0(-x_0; c))$ direction vanishes too.

Case 2 If F is equivariant and $\hat{c} \in \mathcal{C}$ is such that $\mathcal{R}(Z_0^\infty(\hat{c})) = 0$ and $\frac{d}{dc}\big|_{c=\hat{c}}\mathcal{R}(Z_0^\infty(c)) \neq 0$, then denote the periodic orbits corresponding to the fixed points Z_ε^∞ of the Poincaré map $\Pi^{\varepsilon,b_\varepsilon}$ by p_ε. Without loss of generality it can be assumed that $\mu_1 > 0$ (if $\mu_1 < 0$, then read "stable" for all "unstable" and the other way around). In this case there is a four-dimensional local centre-unstable manifold $G_\theta W^u_{loc}(p_\varepsilon)$ and a three-dimensional local centre-stable manifold $G_\theta W^s_{loc}(p_\varepsilon)$. The tangent plane to the local centre-unstable manifold $G_\theta W^u_{loc}(p_\varepsilon)$ at Z^∞_ε is spanned by $(\xi(Z^\infty_\varepsilon),0)$, $(\xi(Z^\infty_\varepsilon)+\frac{\varepsilon\mu_1}{Ts''_a(\hat{c})}\frac{dZ^\infty_0}{c}(\hat{c}),0)+\mathcal{O}(\varepsilon^2)$, $(V^+_0,0)+\mathcal{O}(\varepsilon)$ and $(\mathbf{J}^{-1}[\nabla H_0(Z^\infty_\varepsilon)+\varepsilon F(Z^\infty_\varepsilon,\tau_0)],\omega)=(0,\omega)+\mathcal{O}(\varepsilon)$, thus this manifold is order $\mathcal{O}(\varepsilon)$ close to the extended manifold $\mathcal{M}^{\rm ext}_0$. Also, the tangent plane to the local centre-stable manifold $G_\theta W^s_{loc}(p_\varepsilon)$ at Z^∞_ε is spanned by $(\xi(Z^\infty_\varepsilon),0)$, $(V^-_0,0)+\mathcal{O}(\varepsilon)$ and $(\mathbf{J}^{-1}\nabla H_0(Z^\infty_\varepsilon)+\varepsilon F(Z^\infty_\varepsilon,\tau_0),\omega)$ and this manifold is order ε close to the manifold $\mathcal{M}^{\rm ext}_0(\hat{c})$.

The finite time global extensions of the local manifolds are

$$\begin{aligned}\mathcal{M}^u_\varepsilon(Y)&=G_\theta W^u(p_\varepsilon,Y)=G_\theta\bigcup_{0\le y\le Y}\Phi^{\varepsilon,b_\varepsilon}_y W^u_{loc}(p_\varepsilon)\\ \mathcal{M}^s_\varepsilon(Y)&=G_\theta W^s(p_\varepsilon,Y)=G_\theta\bigcup_{-Y\le y\le 0}\Phi^{\varepsilon,b_\varepsilon}_y W^s_{loc}(p_\varepsilon)\,.\end{aligned}\tag{15}$$

For fixed Y, the centre-unstable manifold $\mathcal{M}^u_\varepsilon(Y)$ is ε-close to the unperturbed four-dimensional manifold $\mathcal{M}^{\rm ext}_0$. The evolution of the C values of the solutions on the centre-unstable manifold $\mathcal{M}^u_\varepsilon(Y)$ is within an ε-neighbourhood of $\hat{c}$ as follows from (14). Using this, the centre-unstable manifold $\mathcal{M}^u_\varepsilon(Y)$ can be parametrised by θ, τ, c_1 and x in the following way: Fix θ_0, τ_0, c_1 and x_0 and for $x\ge 0$. For all ε small and $x\le 0$, let $(Z^u_\varepsilon(x;x_0,\tau_0,\theta_0,c_1),\omega x+\tau_0)$ be a solution of (10) (with $b=b_\varepsilon$) in the centre-unstable manifold $\mathcal{M}^u_\varepsilon(Y)$, such that

$$Z^u_0(x;x_0,\tau_0,\theta_0,c_1)=G_{\theta_0}Z_0(x-x_0;c)$$

and

$$[G_{-\theta_0}Z^u_\varepsilon(0;x_0,\tau_0,\theta_0,c_1)-Z_0(-x_0;c)]\in\operatorname{span}\{\nabla H_0(Z_0(-x_0,c);b(c))\}\tag{16}$$

with $c=\hat{c}+\varepsilon c_1$. In other words, the C-value of the point $Z_0(-x_0,\hat{c}+\varepsilon c_1)$ on the unperturbed manifold is chosen such that the vector which connects this point and the point $G_{-\theta_0}Z^u_\varepsilon(0;x_0,\tau_0,\theta_0,c_1)$ on the perturbed manifold is along the direction of $\nabla H_0(Z_0(-x_0,\hat{c}+\varepsilon c_1);b(\hat{c}+\varepsilon c_1))$.

Next define

$$y^u_1(x;x_0,\tau_0,\theta_0,c_1)=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}G_{-\theta_0}Z^u_\varepsilon(x;x_0,\tau_0,\theta_0,c_1).$$

Then, for $x\le 0$, $y^u_1(x;x_0,\tau_0,\theta_0,c_1)$ is a solution of the first variational equation

$$\begin{aligned}\frac{dy^u_1}{dx}=&\,\mathbf{J}^{-1}D^2H_0\left(Z_0(x-x_0;\hat{c});b(\hat{c})\right)y^u_1+\mathbf{J}^{-1}F\left(Z_0(x-x_0;\hat{c}),\omega x+\tau_0\right)\\ &+b_1\,\xi(Z_0(x-x_0;\hat{c})).\end{aligned}\tag{17}$$

where $b_1=\left.\frac{db_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0}$, thus $b_\varepsilon=b(\hat{c})+\varepsilon b_1+\mathcal{O}(\varepsilon^2)$. Furthermore, differentiating (16) with respect to ε shows that $y^u_1(0;x_0,\tau_0,\theta_0,c_1)-c_1\frac{\partial}{\partial c}Z_0(-x_0;c_0)$ is in the span of $\nabla H_0(Z_0(-x_0,\hat{c});b(\hat{c}))$. (Note that $Z^u_\varepsilon(x;x_0,\tau_0,\theta_0,c_1)$ has both fast and slow dynamics; the slow dynamics is due to the eigenvalue bifurcating out of 1. However, the slow dynamics does not play a role in the first order approximation as used in the next section to estimate the distance of the stable and unstable manifold.)

Similarly, for fixed Y, the centre-stable manifold $\mathcal{M}^s_\varepsilon(Y)$ is ε-close to the unperturbed three-dimensional manifold $\mathcal{M}^{\rm ext}_0(\hat{c})$ and can be parametrised by θ, τ, and x as follows. Fix θ_0, τ_0, and x_0 and for $x\ge 0$, let $(Z^s_\varepsilon(x;x_0,\tau_0,\theta_0),\omega x+\tau_0)$ be a solution of (10) (with $b=b_\varepsilon$) in the centre-stable manifold $\mathcal{M}^s_\varepsilon$, such that

$$Z^s_0(x;x_0,\tau_0,\theta_0)=G_{\theta_0}Z_0(x-x_0;\hat{c})$$

and

$$[G_{-\theta_0} Z_\varepsilon^s(0; x_0, \tau_0, \theta_0) - Z_0(-x_0; \hat{c})] \in \operatorname{span}\{\nabla H_0(Z_0(-x_0; \hat{c}); b(\hat{c})),\ \nabla C(Z_0(-x_0; \hat{c}))\}.$$

Define

$$y_1^s(x; x_0, \tau_0, \theta_0) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} G_{-\theta_0} Z_\varepsilon^s(x; x_0, \tau_0, \theta_0).$$

Then for $x \geq 0$, $y_1^s(x; x_0, \tau_0, \theta_0)$ is a solution of the first variational equation (17) and satisfies the initial condition that $y_1^s(0; x_0, \tau_0, \theta_0)$ is in the span of $\nabla H_0(Z_0(-x_0; \hat{c}); b(\hat{c}))$ and $\nabla C(Z_0(-x_0; \hat{c}))$.

Case 3 If F is not equivariant, then take $c = c_0$ and $b = 0 = a$, let the group action $G_{\hat{\theta}}$ be such that $\mathcal{R}(G_{\hat{\theta}} Z_0^\infty(c_0)) = 0$ and $\frac{d}{d\theta}\big|_{\theta=\hat{\theta}} \mathcal{R}(G_\theta Z_0^\infty(c_0)) \neq 0$ and denote the periodic orbits corresponding to the fixed points Z_ε^∞ of the Poincaré map $\Pi^{\varepsilon,0}$ by p_ε. Define μ_1 to be such that $\mu_1^2 = -s_0''(c_0)T^2 \frac{d}{d\theta}|_{\theta=\hat{\theta}} \mathcal{R}(G_\theta Z_0^\infty(c_0))$. If the orbits $Z_0(x)$ are heteroclinic, then we will need at least two values of $\hat{\theta}$ to persist and those values have to correspond to the fixed points of the heteroclinic orbit $G_\theta Z_0(x)$. Discrete symmetries in the perturbation can take care of this. Assuming that two such values exist, we denote $G_{\hat{\theta}^s} Z_0^\infty$ for the persisting solution creating the stable manifold and $G_{\hat{\theta}} Z_0^\infty$ for the persisting solution creating the unstable manifold. This implies that $\hat{\theta}^s = \hat{\theta} + \theta^\infty(c_0)$, where $\theta^\infty(c_0)$ is the angle such that $\lim_{x\to\infty} Z(x; c_0) = G_{\theta^\infty(c_0)} Z_0^\infty(c_0)$ (recall that $\lim_{x\to-\infty} Z(x; c_0) = Z_0^\infty(c_0)$).

Case 3a If $\mu_1^2 < 0$, then there is only one eigenvalue inside the unit circle and there are three eigenvalues outside the unit circle. Thus there is a two-dimensional local stable manifold $W_{loc}^s(p_\varepsilon)$ and a four-dimensional local unstable manifold $W_{loc}^u(p_\varepsilon)$ associated with the periodic orbit p_ε. As can be seen from the eigenvectors of the Poincaré map, the perturbed stable manifold $W_{loc}^s(p_\varepsilon)$ is order ε close to the two-dimensional unperturbed manifold $\mathcal{M}_0^{\rm ext}(c_0, \hat{\theta}) = \{(G_{\hat{\theta}} Z_0(x; c_0), \tau) \mid x \in \mathbb{R}, \tau \in S^1\}$ (since $\lim_{x\to\infty} G_{\hat{\theta}} Z_0(x; c_0) = G_{\hat{\theta}+\theta^\infty(c_0)} Z_0^\infty(c_0) = G_{\hat{\theta}^s} Z_0^\infty(c_0)$, this manifold forms the stable manifold of $G_{\hat{\theta}^s} Z_0^\infty(c_0)$). In first instance one might think that the perturbed unstable manifold $W_{loc}^u(p_\varepsilon)$ is order $\sqrt{\varepsilon}$ close to the unperturbed four-dimensional centre-unstable manifold $\mathcal{M}_0^{\rm ext}$. However, since the order $\sqrt{\varepsilon}$ correction to the eigenvector is $\frac{\mu_1}{s_0''(c_0)T} \frac{dG_{\hat{\theta}} Z_0^\infty}{dc}(c_0)$ and this vector is is tangent to $\mathcal{M}_0^{\rm ext}$, it follows that the perturbed unstable manifold $W_{loc}^u(p_\varepsilon)$ is order ε close to the unperturbed four-dimensional centre-unstable manifold $\mathcal{M}_0^{\rm ext}$ after all. For fixed Y, both manifolds can be extended to finite time global manifolds $\mathcal{M}_\varepsilon^{s/u}(Y)$ as in (15). The centre-unstable manifold will contain both fast and slow dynamics, while the stable manifold will only contain fast dynamics.

To describe solutions on the perturbed manifolds, similar parameterizations as in case 2 will be given. First the parametrization of the perturbed stable manifold $\mathcal{M}_\varepsilon^s$. Using that this manifold is order ε close to the unperturbed two-dimensional manifold $\mathcal{M}_0^{\rm ext}(c_0, \hat{\theta})$, the perturbed stable manifold can be parametrised by τ and x as follows. Fix τ_0 and x_0. For all $x \leq 0$, let $(Z_\varepsilon^s(x; x_0, \tau_0), \omega x + \tau_0)$ be a solution of (10) (with $b = 0$) in the stable manifold $\mathcal{M}_\varepsilon^s(Y)$, such that

$$Z_0^s(x; x_0, \tau_0) = G_{\hat{\theta}} Z_0(x - x_0; c_0)$$

and

$$\Big[G_{-\hat{\theta}} Z_\varepsilon^s(0; x_0, \tau_0) - Z_0(-x_0; c_0)\Big] \in \operatorname{span}\{\nabla H_0(Z_0(-x_0, c_0); 0),\ \nabla C(Z_0(-x_0, c_0)),\ \xi(Z_0(-x_0, c_0))\}.$$

Define

$$y_1^s(x;x_0,\tau_0) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} G_{-\hat\theta} Z_\varepsilon^s(x;x_0,\tau_0).$$

Then for $x \geq 0$, $y_1^s(x;x_0,\tau_0)$ is a solution of the first variational equation

$$\begin{aligned}\tfrac{dy_1^s}{dx} =& \mathbf{J}^{-1} D^2 H_0\left(Z_0(x-x_0;c_0);0\right)\, y_1^s \\ &+ DG_{-\hat\theta}(G_{\hat\theta} Z_0(x-x_0;c_0))\, \mathbf{J}^{-1} F\left(G_{\hat\theta} Z_0(x-x_0;c_0), \omega x+\tau_0\right)\end{aligned} \tag{18}$$

and $y_1^s(0;x_0,\tau_0)$ is in the span of the vectors $\nabla H_0(Z_0(-x_0,c_0);0)$, $\nabla C(Z_0(-x_0,c_0))$ and $\xi(G_{\hat\theta} Z_0(-x_0;c_0))$.

Next the solutions on the perturbed centre-unstable manifold will be parametrised. As in case 2, the perturbed centre-unstable manifold is order ε close to $\mathcal{M}_0^{\rm ext}$, hence this perturbed manifold can be parametrised by θ, τ, x and c. From the analysis above, it follows that the stable manifold stays ε-close to the solution $G_{\hat\theta} Z_0(x-x_0;c_0)$. So for the analysis of the intersection of the stable and centre-unstable manifold, we have to consider only order ε perturbations of the θ and c variables, i.e., $\theta = \hat\theta + \varepsilon\theta_1$ and $c = c_0 + \varepsilon c_1$.

Hence fix x_0, τ_0, θ_1 and c_1. For all ε small and $x \leq 0$, let $(Z_\varepsilon^u(x;x_0,\tau_0,\theta_1,c_1), \omega x+\tau_0)$ be a solution of (10) (with $b=0$) in the unstable manifold $\mathcal{M}_\varepsilon^u(Y)$ such that

$$Z_0^u(x;x_0,\tau_0,\theta_1,c_1) = G_{\hat\theta+\varepsilon\theta_1} Z_0(x-x_0;c)$$

and

$$\left[G_{-(\hat\theta+\varepsilon\theta_1)} Z_\varepsilon^u(0;x_0,\tau_0,\theta_1,c_1) - Z_0(-x_0;c)\right] \in \operatorname{span}\left\{\nabla H_0(Z_0(-x_0,c);0)\right\}$$

with $c = c_0+\varepsilon c_1$. As in case 2, define

$$y_1^u(x;x_0,\tau_0,\theta_1,c_1) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} G_{-(\hat\theta+\varepsilon\theta_1)} Z_\varepsilon^u(x;x_0,\tau_0,\theta_1,c_1).$$

Then for $x \leq 0$, $y_1^u(x;x_0,\tau_0,\theta_1,c_1)$ is a solution of the first variational equation (18) and $y_1^u(0;x_0,\tau_0) - c_1\frac{\partial}{\partial c} Z_0(-x_0;c_0)$ is in the span of the vector $\nabla H_0(Z_0(-x_0,c_0);0)$.

Case 3b If $\mu_1^2 > 0$, then there are two eigenvalues inside the unit circle and two eigenvalues outside the unit circle. Thus there is a three-dimensional local stable manifold $W_{loc}^s(p_\varepsilon)$ and a three-dimensional local unstable manifold $W_{loc}^u(p_\varepsilon)$, both of which are order $\sqrt{\varepsilon}$ close to the unperturbed three-dimensional manifold $\mathcal{M}_0^{\rm ext}(c_0)$. To avoid the term $\sqrt{\varepsilon}$ complicating the presentation, we define $\gamma^2 = \varepsilon$. On the stable/unstable manifolds, there is both slow and fast decaying dynamics. For fixed Y, both manifolds can be extended to finite time global manifolds $\mathcal{M}_\gamma^{s/u}(Y)$ as in (15) with $\varepsilon = \gamma^2$. These manifolds are also order $\mathcal{O}(\gamma) = \mathcal{O}(\sqrt{\varepsilon})$ close to the unperturbed three-dimensional manifold $\mathcal{M}_0^{\rm ext}(c_0)$ and can be parametrised by θ, τ, and x as follows.

Fix θ_0, τ_0, and x_0. For $x \geq 0$, resp. $x \leq 0$, let $(Z_\gamma^{s/u}(x;x_0,\tau_0,\theta_0), \omega x+\tau_0)$ be solutions of (10) (with $b=0$ and $\varepsilon=\gamma^2$) in the stable/unstable manifold $\mathcal{M}_\gamma^{s/u}(Y)$, such that

$$Z_0^{s/u}(0;x_0,\tau_0,\theta_0) = G_{\theta_0} Z_0(-x_0;c_0)$$

and

$$G_{-\theta_0} Z_\gamma^{s/u}(0;x_0,\tau_0,\theta_0) - Z_0(-x_0;c_0) \in \operatorname{span}\left\{\nabla H(Z_0(-x_0;c_0)),\ \nabla C(Z_0(-x_0;c_0))\right\}.$$

At $x = 0$, the perturbed solution $Z_\varepsilon^{s/u}(0; x_0, \tau_0, \theta_0)$ is close to $G_{\theta_0} Z_0(-x_0; c_0)$, while asymptotically this solution is close to $G_{\hat\theta} Z_0^\infty$ or $G_{\hat\theta^s} Z_0^\infty$. The change from $G_{\theta_0} Z_0^\infty$ to $G_{\hat\theta} Z_0^\infty$ (resp. from $G_{\theta_0+\theta^\infty} Z_0^\infty$ to $G_{\hat\theta^s} Z_0^\infty$) is due to the slow dynamics on the perturbed manifold. To capture this slow dynamics, we introduce the slow variable $\zeta = \gamma x$ and a function $\overline{\theta}^{s/u}(\zeta; \theta_0)$ to represent the slow dynamics in the direction of the symmetry group on the perturbed manifold. In other words, the function $\overline{\theta}^{s/u}(\zeta; \theta_0)$ takes care of the change from $G_{\theta_0} G_{\theta^\infty} Z_0^\infty / G_{\theta_0} Z_0^\infty$ to $G_{\hat\theta^s} Z_0^\infty / G_{\hat\theta} Z_0^\infty$ and has the property that $\overline{\theta}^{s/u}(0; \theta_0) = \theta_0$, $\lim\limits_{\zeta\to\pm\infty} \overline{\theta}^{s/u}(\zeta; \theta_0) = \hat\theta$ and the decay towards $\hat\theta$ is like $e^{-\mu_1|\zeta|}$. Finally, $\overline{\theta}^{s/u}$ has to be such that the expansion

$$Z_\gamma^{s/u}(x; x_0, \tau_0, \theta_0) = G_{\overline{\theta}^{s/u}(\gamma x;\theta_0)} \Big[Z_0(x - x_0; c_0) + \gamma\, y_1^{s/u}(x, \gamma x; x_0, \tau_0, \theta_0) + \gamma^2\, y_2^{s/u}(x, \gamma x; x_0, \tau_0, \theta_0) + \mathcal{O}(\gamma^3). \Big] \tag{19}$$

is uniform for $x \geq / \leq 0$. Hence the fast (x) and slow $(\zeta = \gamma x)$ behaviour on the stable/unstable manifolds comes back in first two arguments of $y_i^{s/u}(x, \zeta; \cdot)$. Then for $x \geq / \leq 0$ and with $\zeta = \gamma x$, $y_1^{s/u}(x, \zeta; x_0, \tau_0, \theta_0)$ is a solution of the first variational equation

$$\frac{\partial y_1^{s/u}}{\partial x} = \mathbf{J}^{-1} D^2 H_0 \left(Z_0(x - x_0; c_0); 0\right)\, y_1^{s/u} - (\overline{\theta}^{s/u})'(\zeta) \xi(Z_0(x - x_0; c_0)). \tag{20}$$

The definition of $Z_\gamma^{s/u}$ implies that the initial condition $y_1^{s/u}(0, 0; x_0, \tau_0, \theta_0)$ is in the span of $\nabla H_0(Z_0(-x_0; c_0))$ and $\nabla C(Z_0(-x_0; c_0))$. The asymptotic condition at $\pm\infty$ gives that $\lim\limits_{x\to\pm\infty} y_1^{s/u}(x, \gamma x; \cdot) = 0$, as the asymptotic solution p_ε does not depend on γ, but on $\varepsilon = \gamma^2$ and y_1 is related to an order γ correction, uniform in x.

One particular solution of the inhomogeneous linear problem (20) is (see Eq. (8))

$$y_{\rm part}(x, \zeta) = \frac{(\overline{\theta}^{s/u})'(\zeta)}{s_0''(c_0)} \frac{\partial Z_0}{\partial c}(x - x_0; c_0).$$

The bounded solutions of the homogeneous part of (20) are

$$\frac{\partial Z_0(x - x_0; c_0)}{\partial x} = \mathbf{J}^{-1} \nabla H_0(Z_0(x - x_0; c_0); 0), \quad \text{and} \quad \xi(Z_0(x - x_0; c_0)).$$

(The homogeneous ODE has one polynomially and one exponentially growing solution as well). Thus the first order approximation is of the form

$$y_1^{s/u}(x, \zeta; \cdot) = \frac{(\overline{\theta}^{s/u})'(\zeta)}{s_a''(c)} \frac{\partial Z_0}{\partial c}(x - x_0; c_0) + A(\zeta)\, \frac{\partial Z_0}{\partial x}(x - x_0; c_0)$$

$$+ B(\zeta)\, \xi(Z_0(x - x_0; c_0)).$$

Using the asymptotic condition $\lim\limits_{x\to\pm\infty} y_1^{s/u}(x, \gamma x; \cdot) = 0$, it follows that $B(\zeta)$, $(\overline{\theta}^{s/u})'(\zeta) \to 0$ for $\zeta \to \pm\infty$ as $\lim\limits_{x\to\pm\infty} \frac{\partial Z_0}{\partial c}(x - x_0; c_0) \neq 0$ and $\lim\limits_{x\to\pm\infty} \xi(Z_0(x - x_0; c_0)) \neq 0$. By considering the condition at $x = 0$ that $y_1(0, 0; x_0, \tau_0, \theta_0)$ is in the span of $\nabla H(Z_0(-x_0; c_0))$ and $\nabla C(Z_0(-x_0; c_0))$, it follows that an order γ perturbation is only possible in one direction (which is not surprising as this is also the case asymptotically). And if $\overline{\theta}^{s/u}(\zeta)$ is such that $(\overline{\theta}^{s/u})'(0) = 0$, then also $A(0) = 0 = B(0)$ and hence $y_1(x, 0; \cdot) = 0$, thus there is no order γ perturbation at $x = 0$.

The next order correction for $Z_\gamma^{s/u}$ is $y_2^{s/u}(x, \gamma x; x_0, \tau_0, \theta_0)$ and it satisfies the following variational equation for $x \geq / \leq 0$

$$\begin{aligned}\frac{\partial y_2^{s/u}}{\partial x} =& \mathbf{J}^{-1} D^2 H_0 \left(Z_0(x - x_0; c_0); c_0\right) y_2^{s/u} \\ &+ \mathbf{J}^{-1} D^3 H_0 \left(Z_0(x - x_0; c_0); c_0\right) (y_1^{s/u}, y_1^{s/u}) \\ &+ DG_{\overline{\theta}^{s/u}}(Z_0(x - x_0; c_0))^{-1} \mathbf{J}^{-1} F\left(G_{\overline{\theta}^{s/u}} Z_0(x - x_0; c_0), \omega x + \tau_0\right) \\ &- \frac{\partial y_1^{s/u}}{\partial \zeta} - (\overline{\theta}^{s/u})'(\zeta) D\xi(Z_0(x - x_0; c_0)) y_1^{s/u}.\end{aligned} \tag{21}$$

As before, at $x = 0$, the vector $y_2^{s/u}(0, 0; x_0, \tau_0, \theta_0)$ is in the span of $\nabla H(Z_0(-x_0; c_0))$ and $\nabla C(Z_0(-x_0; c_0))$.

Remark 3: In the cases 2 and 3(a), we could have introduced a similar description as in case 3(b) with both slow and fast variables explicit in the notation. This would have led to an approximation on the centre-unstable manifold which is uniformly valid for $x \leq 0$. However, this is not necessary to measure the distance between the persisting stable and centre-unstable manifolds.

5. Transverse intersection of the invariant manifolds

The transverse intersection of the persisting invariant manifolds $\mathcal{M}_\varepsilon$ will be determined by the following Melnikov-type function

$$M_H(\tau_0, \theta_0; c) = \int_{-\infty}^{\infty} \left\langle \nabla H_0(G_{\theta_0} Z_0(x; c); b(c)) \, , \, \mathbf{J}^{-1} F(G_{\theta_0} Z_0(x; c), \omega x + \tau_0)\right\rangle dx. \tag{22}$$

Theorem 5.1: *For the cases 1, 2 and 3(a), the following criteria are sufficient for the intersection of the persisting invariant manifolds:*

(i) If F is equivariant and $R(Z, \tau) = 0$ for all values of Z and τ, then the Melnikov function M_H does not depend on θ_0. If there is some $c \in \mathcal{C}$ for which there is a $\overline{\tau}_0(c)$ such that $M_H(\overline{\tau}_0(c), 0; c) = 0$ and $\frac{dM_H}{d\tau_0}(\overline{\tau}_0(c), 0; c) \neq 0$, then for $\varepsilon \neq 0$ sufficiently small the invariant manifolds $\mathcal{M}_\varepsilon^s(c)$ and $\mathcal{M}_\varepsilon^u(c)$ intersect transversely along a two-dimensional surface (within the level set $C^{-1}(c)$), which can be parametrised by τ and θ.

(ii) If F is equivariant and $\hat{c} \in \mathcal{C}$ is such that $\mathcal{R}(Z_0^\infty(\hat{c})) = 0$ and $\frac{d}{dc}|_{c=\hat{c}} \mathcal{R}(Z_0^\infty(c)) \neq 0$, then the Melnikov function M_H does not depend on θ_0. If there is some $\overline{\tau}_0$ such that $M_H(\overline{\tau}_0, 0; \hat{c}) = 0$ and $\frac{dM_H}{d\tau_0}(\overline{\tau}_0, 0; \hat{c}) \neq 0$, then for $\varepsilon \neq 0$ sufficiently small the invariant manifolds $\mathcal{M}_\varepsilon^s$ and $\mathcal{M}_\varepsilon^u$ intersect transversely along a two-dimensional surface which can be parametrised by τ and θ.

(iii) If F is not equivariant, then take $c = c_0$ and $b = 0 = a$. Let the group action $G_{\hat{\theta}}$ be such that $\mathcal{R}(G_{\hat{\theta}} Z_0^\infty(c_0)) = 0$ and $\frac{d}{d\theta}|_{\theta=\hat{\theta}} \mathcal{R}(G_\theta Z_0^\infty(c_0)) \neq 0$. Define μ_1 to be such that $\mu_1^2 = -s_0''(c_0) T^2 \frac{d}{d\theta}|_{\theta=\hat{\theta}} \mathcal{R}(G_\theta Z_0^\infty(c_0))$.

a) If $\mu_1^2 < 0$ and there is some $\overline{\tau}_0$ such that $M_H(\overline{\tau}_0, \hat{\theta}; c_0) = 0$ and $\frac{dM_H}{d\tau_0}(\overline{\tau}_0, \hat{\theta}; c_0) \neq 0$, then for $\varepsilon \neq 0$ sufficiently small the invariant manifolds $\mathcal{M}_\varepsilon^s$ and $\mathcal{M}_\varepsilon^u$ intersect transversely along a one-dimensional curve parametrised by τ.

The first case of the Theorem deals with invariant manifolds $C^{-1}(c)$. Within those invariant manifolds, the persistence of the intersection of the perturbed stable and one-dimensional perturbed unstable manifold has to be shown and this can be done along very similar lines as the standard theory. For the second and third case, the C level sets are not preserved anymore and the eigenvalues of the Poincaré map, which come out of the eigenvalue 1, play an essential role in the dimension and type of the persisting invariant manifolds, although the ultimate criterion for intersection remains unchanged. We will come back to case 3(b) after the proof of the Theorem.

Proof:

Case 1 In the case when F is equivariant and $R(Z,\tau) = 0$ for all Z and τ, the perturbed invariant manifolds $\mathcal{M}^s_\varepsilon(c,Y)$ and $\mathcal{M}^u_\varepsilon(c,Y)$ are three-dimensional manifolds within the four-dimensional level set $C^{-1}(c)$. As discussed in the previous section, these manifolds can be parametrised by x_0, τ_0 and θ_0 and the gradient of the Hamiltonian H_0 is the relevant normal vector to those manifolds, just as in the standard case without group action [22, 23]. The equivariance of the perturbation implies that $M_H(\tau_0,\theta_0;c)$ is independent of θ_0 and can be simplified to

$$M_H(\tau_0,0;c) = \int_{-\infty}^{\infty} \langle \nabla H_0(Z_0(x;c);b(c)), F(Z_0(x;c),\omega x+\tau_0)\rangle dx.$$

The signed distance between two points $Z^s(0;x_0,\tau_0,\theta_0)$ and $Z^u(0;x_0,\tau_0,\theta_0)$ on the perturbed invariant manifolds $\mathcal{M}^s_\varepsilon(c,Y)$ and $\mathcal{M}^u_\varepsilon(c,Y)$ can be found to be up to first order as

$$\begin{aligned} d_\varepsilon(x_0,\tau_0) &= \varepsilon \int_{-\infty}^{\infty} \langle \nabla H_0(Z_0(x;c);b(c)), F(Z_0(x;c),\omega x+\tau_0)\rangle dx + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon M_H(\omega x_0+\tau_0,0;c) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{23}$$

see [22,23] for details. An implicit function theorem argument gives that the distance $d_\varepsilon(x_0,\tau_0)$ vanishes whenever the Melnikov function $M_H(\tau,0;\hat{c})$ has a simple zero as a function of its first argument. Thus if M_H has a simple zero as a function of τ_0, then the perturbed invariant manifolds will intersect transversally (up to the action of the symmetry) within the level set $C^{-1}(c)$. Dimension counting shows that generically two three-dimensional manifolds in a four-dimensional ambient space intersect along some two-dimensional surface. Indeed, the perturbed manifolds $\mathcal{M}^s_\varepsilon(c,Y)$ and $\mathcal{M}^u_\varepsilon(c,Y)$ intersect along a surface which can be parameterised by τ_0 (or, equivalently x_0, see [22] for discussion) and θ_0. The parameter θ_0 is related to the invariance the problem with respect to the group action G_θ.

Case 2 The distance between the four-dimensional centre-unstable manifold $\mathcal{M}^u_\varepsilon(Y)$ and the three-dimensional centre-stable manifold $\mathcal{M}^s_\varepsilon(Y)$ has to be determined. In the previous section, we have seen that for every x_0, τ_0 and θ_0, the solution $Z^s_\varepsilon(x;x_0,\tau_0,\theta_0)$ on the three-dimensional centre-stable manifold is parametrised such that $G_{-\theta_0}Z^s_\varepsilon(0;x_0,\tau_0,\theta_0) - Z_0(-x_0;\hat{c})$ is of order ε and in the span of $\nabla H_0(Z_0(-x_0;\hat{c});b(\hat{c}))$ and $\nabla C(Z_0(-x_0;\hat{c}))$. Using that the unperturbed three-dimensional manifold $\mathcal{M}_0(\hat{c})$ is embedded in the four-dimensional manifold $\mathcal{M}_0^{\rm ext}$, it can be shown that there is some $c_1(\varepsilon)$ such that the difference $G_{-\theta_0}Z^s_\varepsilon(0;x_0,\tau_0,\theta_0) - Z_0(-x_0;\hat{c}+\varepsilon c_1)$ is in the direction of $\nabla H_0(Z_0(-x_0,\hat{c}+\varepsilon c_1);b(\hat{c}+\varepsilon c_1))$. Indeed, $G_{-\theta_0}Z^s_\varepsilon(0;x_0,\tau_0,\theta_0) - Z_0(-x_0;\hat{c})$ is in the span of $\nabla H_0(Z_0(-x_0;\hat{c});b(\hat{c}))$ and $\nabla C(Z_0(-x_0;\hat{c}))$, so it can be written as $\varepsilon(\alpha\nabla H_0(Z_0(-x_0;\hat{c});b(\hat{c}))+\beta\nabla C(Z_0(-x_0;\hat{c}))) + \mathcal{O}(\varepsilon^2)$ for some α and β. A straightforward expansion in ε shows that this implies that $c_1 = \beta\left|\frac{\partial Z_0(-x_0;\hat{c})}{\partial c}\right|^{-2} + \mathcal{O}(\varepsilon)$.

Therefore, this choice of c_1 in Z^u_ε gives that $G_{-\theta_0}Z^u_\varepsilon(0;x_0,\tau_0,\theta_0,c_1) - G_{-\theta_0}Z^s_\varepsilon(0;x_0,\tau_0,\theta_0)$ is parallel to $\nabla H_0(Z_0(-x_0,\hat{c}+\varepsilon c_1);b(\hat{c}+\varepsilon c_1))$. Define

$$d_\varepsilon(x_0,\tau_0,\theta_0) = \langle G_{-\theta_0}Z^u_\varepsilon(0;x_0,\tau_0,\theta_0,c_1) - G_{-\theta_0}Z^s_\varepsilon(0;x_0,\tau_0,\theta_0), \nabla H_0(Z_0(-x_0;c);b(c))\rangle$$

with $c = \hat{c}+\varepsilon c_1$. Then $d_\varepsilon(x_0,\tau_0,\theta_0)$ measures the distance between the stable and unstable manifolds and it satisfies

$$d_\varepsilon(x_0,\tau_0,\theta_0) = \varepsilon\langle y^u_1(0;x_0,\tau_0,\theta_0,c_1) - y^s_1(0;x_0,\tau_0,\theta_0), \nabla H_0(Z_0(-x_0;\hat{c});b(\hat{c}))\rangle + \mathcal{O}(\varepsilon^2).$$

To calculate the lowest order of this distance, define

$$\Delta^{u/s}(x; x_0, \tau_0, \theta_0) = \langle y_1^{u/s}(x; x_0, \tau_0, \theta_0, c_1), \nabla H_0(Z_0(x - x_0; \hat{c}); b(\hat{c}))\rangle.$$

Since $\nabla H_0(Z_0(x - x_0; \hat{c}); b(\hat{c}))$ decays exponentially fast to zero asymptotically (for $x \to \pm\infty$) and $y_1^{u/s}(x; x_0, \tau_0, c_1)$ are ε-derivatives of solutions on the centre-unstable/stable manifolds and thus at most polynomially growing asymptotically, it follows that $\lim\limits_{x\to\mp\infty} \Delta^{u/s}(x; x_0, \tau_0, \theta_0) = 0$.

Differentiating the functions $\Delta^{u/s}(x; x_0, \tau_0, \theta_0)$ with respect to x, using the differential equation (17) for $y_1^{s/u}$ and integrating gives

$$\Delta^u(0; x_0, \tau_0, \theta_0) - 0 = \int_{-\infty}^{0} \langle \mathbf{J}^{-1} F\left(Z_0(x - x_0; \hat{c}), \omega x + \tau_0\right), \nabla H_0(Z_0(x - x_0; \hat{c}); b(\hat{c}))\rangle\, dx.$$

and

$$-\Delta^s(0; x_0, \tau_0, \theta_0) = \int_{0}^{\infty} \langle \mathbf{J}^{-1} F\left(Z_0(x - x_0; \hat{c}), \omega x + \tau_0\right), \nabla H_0(Z_0(x - x_0; \hat{c}); b(\hat{c}))\, dx.$$

Here we used that for any Z, $D^2C(Z)\mathbf{J}^{-1}\nabla H_0(Z) = D^2H_0(Z)\mathbf{J}^{-1}\nabla C(Z)$, which follows from the invariance of H_0 under the flow of the symmetry group. Adding those two equalities gives

$$\Delta^u(0; x_0, \tau_0, \theta_0) - \Delta^s(0; x_0, \tau_0, \theta_0) = M_H(\omega x_0 + \tau_0, 0; \hat{c}).$$

Note that the right hand side of this expression does not depend on θ_0, which reflects that fact that the problem is equivariant under the group action G_θ. Using the expression just derived, one can write the distance between the perturbed centre-stable and centre-unstable manifolds as

$$d_\varepsilon(x_0, \tau_0, \theta_0) = \varepsilon M_H(\omega x_0 + \tau_0, 0; \hat{c}) + \mathcal{O}(\varepsilon^2).$$

An implicit function theorem argument gives that the distance $d_\varepsilon(x_0, \tau_0)$ vanishes whenever the Melnikov function $M_H(\tau, 0; \hat{c})$ has a simple zero as a function of its first argument. Hence if there is some $\overline{\tau}_0$ such that the Melnikov function $M_H(\tau, 0; \hat{c})$ has a simple zero at $\tau = \overline{\tau}_0$, then for any x_0 and τ_0 such that $\omega x_0 + \tau_0 = \overline{\tau}_0$, the stable and unstable manifolds have a non-trivial intersection. This intersection is two-dimensional, the equivariant group action G_θ parametrises one dimension and the periodic orbit parametrises the other dimension, as in case 1.

Case 3a If $\mu_1^2 < 0$, the distance between the two-dimensional stable manifold $\mathcal{M}_\varepsilon^s(Y)$ and the four-dimensional centre-unstable manifold $\mathcal{M}_\varepsilon^u(Y)$ has to be determined. The arguments will have a similar flavour as in case 2. In the previous section it is shown that for every x_0 and τ_0 the solution $Z_\varepsilon^s(x; x_0, \tau_0)$ on the two-dimensional centre-stable manifold is parametrised such that $G_{-\hat{\theta}} Z_\varepsilon^s(0; x_0, \tau_0) - Z_0(-x_0; c_0)$ is of order ε and in the span of $\nabla H_0(Z_0(-x_0; c_0); b(c_0))$, $\nabla C(Z_0(-x_0; c_0))$ and $\xi(Z_0(-x_0; c_0))$. As in case 2, using that the unperturbed two-dimensional manifold $\mathcal{M}_0^{\rm ext}(c_0, \hat{\theta})$ is embedded in the four-dimensional manifold $\mathcal{M}_0^{\rm ext}$, it follows that there are some $c_1(\varepsilon)$ and $\theta_1(\varepsilon)$ such that $G_{-\hat{\theta}+\varepsilon\theta_1} Z_\varepsilon^s(0; x_0, \tau_0) - Z_0(-x_0; c_0 + \varepsilon c_1)$ is in the direction of $\nabla H_0(Z_0(-x_0, c_0 + \varepsilon c_1); b(c_0 + \varepsilon c_1))$. Therefore these choices of c_1 and θ_1 give that $G_{-\hat{\theta}+\varepsilon\theta_1} Z_\varepsilon^s(0; x_0, \tau_0) - G_{-\hat{\theta}+\varepsilon\theta_1} Z_\varepsilon^u(0; x_0, \tau_0, \theta_1, c_1)$ is parallel to $\nabla H_0(Z_0(-x_0, c_0 + \varepsilon c_1); b(c_0 + \varepsilon c_1))$, so

$$d_\varepsilon(x_0, \tau_0) = \langle G_{-\theta} Z_\varepsilon^u(0; x_0, \tau_0, \theta_1, c_1), -G_{-\theta} Z_\varepsilon^s(0; x_0, \tau_0)\,,\ \nabla H_0(Z_0(-x_0; c); b(c))\rangle,$$

with $\theta = \hat{\theta} + \varepsilon\theta_1$ and $c = c_0 + \varepsilon c_1$, measures the distance between the stable and unstable manifolds.

This distance satisfies

$$d_\varepsilon(x_0,\tau_0) = \varepsilon\langle y_1^u(0;x_0,\tau_0,\theta_1,c_1) - y_1^s(0;x_0,\tau_0), \nabla H_0(Z_0(-x_0;c_0);b(c_0))\rangle + \mathcal{O}(\varepsilon^2).$$

To calculate the lowest order of this distance, define

$$\Delta^{u/s}(x;x_0,\tau_0) = \langle y_1^{u/s}(x;x_0,\tau_0,\theta_1,c_1), \nabla H_0(Z_0(x-x_0;c_0);b(c_0))\rangle.$$

Similar arguments as before give that

$$\Delta^u(0;x_0,\tau_0) - \Delta^s(0;x_0,\tau_0) = M_H(\omega x_0+\tau_0,\hat{\theta};c_0).$$

Thus the distance between the perturbed stable and centre-unstable manifolds is given by

$$d_\varepsilon(x_0,\tau_0) = \varepsilon M_H(\omega x_0+\tau_0,\hat{\theta};c_0) + \mathcal{O}(\varepsilon^2).$$

As before, this implies that the distance $d_\varepsilon(x_0,\tau_0)$ vanishes whenever the Melnikov function $M_H(\tau,\hat{\theta};c_0)$ has a simple zero as a function of its first argument. Hence the stable and unstable manifolds have a non-trivial one-dimensional intersection.

□

For the case 3(b), we have not yet been able to derive a sufficient criterion for the intersection of the invariant manifolds. Using the descriptions of the invariant manifolds in this case as derived in section 4, it follows that two measurements have to be taken in this case. One measurement is the distance between the stable and unstable manifolds in the direction of $\nabla H(G_{\theta_0}Z_0(-x_0,c_0),b(c_0))$. This measurement is similar to the ones presented before and would involve the Melnikov function $M_H(\omega x_0+\tau_0,\theta_0;c_0)$. The second measurement is the distance between the stable and unstable manifolds in the direction of $\nabla C(G_{\theta_0}Z_0(-x_0,c_0))$. Following the last vector along the unperturbed manifold to the fixed point, we see that its limit $\nabla C(G_{\theta_0}Z_0^\infty(c_0))$ does not vanish. Thus the behaviour at infinity will play a more important role and our estimates will have to be uniform in x. The slow variable ζ and function $\overline{\theta}^{s/u}(\zeta,\theta_0)$ have been introduced for this reason.

The function $\overline{\theta}^{s/u}(\zeta,\theta_0)$ describes the slow behaviour of the symmetry group near the persisting fixed point, apart from the condition $\overline{\theta}^{s/u}(0,\theta_0)=\theta_0$, the behaviour of this function near $\zeta=0$ is not prescribed. So we can take $\overline{\theta}^{s/u}(\zeta,\theta_0)$ to be such that $\frac{\partial}{\partial\zeta}\overline{\theta}^{s/u}(0,\theta_0)=0$. As we have seen in section 4, this implies that that $y_1(x,0;x_0,\tau_0,\theta_0)=0$. The differential equation (21) for $y_2(x,0;x_0,\tau_0,\theta_0)$ simplifies to the familiar form

$$\begin{aligned}\frac{\partial y_2^{s/u}}{\partial x}(x;0;\cdot) = {} & \mathbf{J}^{-1}D^2H_0\left(Z_0(x-x_0;c_0);c_0\right)\, y_2^{s/u}(x;0;\cdot)\\ & + DG_{\theta_0}(Z_0(x-x_0;c_0))^{-1}\mathbf{J}^{-1}F\left(G_{\theta_0}Z_0(x-x_0;c_0),\omega x+\tau_0\right)\end{aligned}$$

To estimate the difference $Z_\varepsilon^u(0;x_0,\tau_0,\theta_0) - Z_\varepsilon^s(0;x_0,\tau_0,\theta_0)$ in the direction of $\nabla C(G_{\theta_0}Z_0(-x_0;c_0))$, we define the quantity

$$d_\varepsilon^C(x_0,\tau_0,\theta_0) = \langle G_{-\theta_0}Z_\varepsilon^u(0;x_0,\tau_0,\theta_0) - G_{-\theta_0}Z_\varepsilon^s(0;x_0,\tau_0,\theta_0), \nabla C(Z_0(-x_0;c_0))\rangle$$

Since $y_1^{u/s}(0,0;x_0,\tau_0,\theta_0)=0$, in lowest order this is

$$d_\varepsilon^C(x_0,\tau_0,\theta_0) = \varepsilon\langle y_2^u(0,0;x_0,\tau_0,\theta_0) - y_2^s(0,0;x_0,\tau_0,\theta_0), \nabla C(Z_0(-x_0;c_0))\rangle + \mathcal{O}(\varepsilon\sqrt{\varepsilon}),$$

so we define

$$\Delta_C^{u/s}(x;x_0,\tau_0,\theta_0) = \langle y_2^{u/s}(x,0;x_0,\tau_0,\theta_0), \nabla C(Z_0(x-x_0;c_0))\rangle.$$

By using the differential equation for $y_2^{u/s}(x,0;\cdot)$ it can be shown that

$$\begin{aligned}
&\Delta_C^u(0,0;x_0,\tau_0,\theta_0) - \Delta_C^s(0,0;x_0,\tau_0,\theta_0) \\
&\quad = \int_{-\infty}^{\infty} \left[\, R(G_{\theta_0} Z_0(x;c_0), \omega x + \tau_0) - R(G_{\theta_0} Z_0^\infty(c_0), \omega x + \tau_0)\,\right] dx \\
&\qquad + \lim_{x\to-\infty} \tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Big[C(Z_\varepsilon^u(x;x_0,\tau_0,\theta_0)) - C(\Phi_x^{\varepsilon,0}(G_{\theta_0} Z_0^\infty, \tau_0)) \Big] \\
&\qquad - \lim_{x\to\infty} \tfrac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Big[C(Z_\varepsilon^s(x;x_0,\tau_0,\theta_0)) - C(\Phi_x^{\varepsilon,0}(G_{\theta_0} Z_0^\infty, \tau_0)) \Big] .
\end{aligned}$$

Although we expect that the integral in this expression acts as a Melnikov integral, we have not been able to prove this so far. However, if this is the case, then the two parameters τ_0 and θ_0 can be used to find simultaneous zeros for this Melnikov function and the Melnikov function $M_H(\omega x_0 + \tau_0, \theta_0; c_0)$.

6. Perturbed defocussing NLS equations

The methods developed in previous sections will now be used to analyse transversal intersections of persisting invariant manifolds for various perturbations of the defocussing nonlinear Schrödinger equation. The unperturbed version of this equation is presented in the Introduction together with its multi-symplectic formulation. The perturbed equation will have the form

$$i\Psi_t = \Psi_{xx} + iv\Psi_x + W'(|\Psi|^2)\Psi + \epsilon f(\Psi, \Psi_x, \omega x), \tag{24}$$

where $W'(|\Psi|^2) = k - |\Psi|^2$ (defocussing cubic NLS) and the perturbation $f = f_1 + if_2$. In the particular context of optical wave propagation in nonlinear waveguides, the perturbations in this equation describe various physical influences on the travelling waves such as cable losses, amplification, fibre birefringence and so on [24].

The perturbed NLS equation (24) can be rewritten in a perturbed multi-symplectic form (3), where $\mathbf{M}$, $\mathbf{J}$ and $S(Z)$ are defined in the introduction and the perturbation is multi-symplectified as

$$F(Z,x) = (f_1, f_2, 0, 0)^T.$$

As it was mentioned in the Introduction, the unperturbed NLS equation is equivariant with respect to a one-dimensional group $\mathcal{G} = SO_2$, whose action is given by

$$G_\theta(Z) = \mathbf{R}_\theta Z, \; \mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix},$$

with the generator $\xi(Z) = \frac{d}{d\theta}|_{\theta=0} G_\theta(Z) = (-q_2, q_1, -p_2, p_1)^T$ and Noether's functionals $P(Z)$ and $C(Z)$ given by

$$P(Z) = -\frac{1}{2}(q_1^2 + q_2^2), \quad \text{and} \quad C(Z) = q_1 p_2 - q_2 p_1.$$

Hence P corresponds to the mass density (amplitude) of Ψ and C is the momentum density.

The unperturbed defocussing NLS equation has dark solitons, which in the multi-symplectic formulation can be found as heteroclinic solutions of the Hamiltonian ODE (5) with the Hamiltonian

$$H_0(Z;b) = \tfrac{1}{2}(p_1^2+p_2^2) + \tfrac{1}{2}(k+a+\tfrac{v^2}{4})(q_1^2+q_2^2) - \tfrac{1}{4}(q_1^2+q_2^2)^2 + \tfrac{1}{2}(v+2b)(q_2p_1 - q_1p_2).$$

Following similar arguments as in [9], it can be shown that if the parameters a, b and v satisfy the condition $(2b+v)^2 < 4(k+a)+v^2$, then this Hamiltonian ODE (5) has fixed points $G_\theta Z_0^\infty(c)$, with

$$Z_0^\infty(c) = (\rho_0, 0, 0, \tfrac{1}{2}(v+2b)\rho_0)^T, \quad \text{with} \quad 4\rho_0^2 = 4(k+a)+v^2-(2b+v)^2, \tag{25}$$

and b and c are related by $2c = \rho_0^2(2b+v)$.

To ensure that the fixed points are saddle-centres, the following stronger condition on the parameters a, b and v is needed

$$3(2b+v)^2 < 4(k+a)+v^2. \tag{26}$$

If this condition is satisfied, then the corresponding front solution $Z_0(x;c)$ of the unperturbed NLS equation can be found as

$$Z_0(x;c) = \frac{1}{2\rho_0}\begin{pmatrix} 2\rho_0^2 - 4B^2\left(1+\tanh(Bx)\right) \\ 2(v+2b)B(1+\tanh(Bx)) \\ -4B^3\operatorname{sech}^2(Bx) - (v+2b)^2B(1+\tanh(Bx)) \\ 2B^2(v+2b)\operatorname{sech}^2(Bx) + (v+2b)\rho_0^2 - 2(v+2b)B^2(1+\tanh(Bx)) \end{pmatrix}$$

where

$$4B^2 = 2\rho_0^2 - (v+2b)^2 = \tfrac{1}{2}\left[4(k+a)+v^2-3(v+2b)^2\right]. \tag{27}$$

This front has a phase shift of $\theta_\infty = \arccos\left(1-\frac{4B^2}{\rho_0^2}\right)$ between $Z_0^\infty(c) = \lim\limits_{x\to-\infty} Z_0(x;c)$ and $G_{\theta_\infty}Z_0^\infty(c) = \lim\limits_{x\to\infty} Z_0(x;c)$. The front corresponds to the following dark soliton solution of the original unperturbed ($\varepsilon = 0$) NLS equation (24):

$$\Psi(x,t) = \frac{e^{i(at+bx+\theta_0)}}{\rho_0}\left[\rho_0^2 - 2B^2\left(1+\tanh Bx\right) + iB(v+2b)(1+\tanh Bx)\right].$$

It is easy to check that those solutions satisfy all criteria of Hypothesis (H1). In the following subsections, we will look at various perturbations and apply Theorem 5.1 to obtain results about intersecting invariant manifolds.

6.1. *Equivariant Hamiltonian perturbation*

The first example considers the symmetry-preserving Hamiltonian perturbation

$$f(\Psi,\Psi_x,\tau) = \left[m(\tau) + n(\tau)|\Psi|^2\right]\Psi, \tag{28}$$

where $m(\tau)$ and $n(\tau)$ are given real 2π-periodic functions. This kind of perturbation arises in the studies of wave propagation in weakly inhomogeneous and disordered media [25]. When written in a multi-symplectic form, the above perturbation (28) transforms into

$$F(Z,x) = \left(q_1\left[m(\tau)+n(\tau)(q_1^2+q_2^2)\right], q_2\left[m(\tau)+n(\tau)(q_1^2+q_2^2)\right], 0, 0\right)^T.$$

With Theorem 3.1 and Remark **2**, the persistence of all relative equilibria $G_\theta Z_0^\infty(c)$ as relative periodic orbits follows immediately for all values of b satisfying the unperturbed existence condition (26).

According to Theorem 5.1, the intersection of perturbed invariant manifolds is determined by the simple zeros of the Melnikov function as given in (22). Taking $m(\tau) = \mathcal{A}\cos\tau$ and $n(\tau) = \mathcal{B}\sin\tau$, this function can be evaluated to give

$$M_H(\tau_0, \theta_0; c) = \frac{\pi\omega^2}{3}\,\mathrm{csch}\left(\frac{\pi\omega}{2B}\right)\left[3\mathcal{A}\sin\tau_0 + 2\mathcal{B}\left(6\rho_0^2 - (4B^2 + \omega^2)\right)\cos\tau_0\right].$$

where B is given by (27) and ρ_0^2 by (25). From this expression it follows that at the points

$$\overline{\tau}_0 = -\arctan\frac{2\mathcal{B}(6\rho_0^2 - (4B^2 + \omega^2))}{3\mathcal{A}}$$

one has $M_H(\overline{\tau}_0, \theta_0; c) = 0$ and $\frac{dM_H}{d\tau_0}(\overline{\tau}_0, \theta_0; c) \neq 0$. Hence, with Theorem 5.1 we conclude that within each $C^{-1}(c)$ level set, the perturbed invariant manifolds $\mathcal{M}_\varepsilon^s(c)$ and $\mathcal{M}_\varepsilon^u(c)$, associated with the persisting periodic orbits, intersect transversely along a two-dimensional curve parametrised by θ_0 and τ_0.

6.2. *Equivariant damped-driven perturbation*

The second example considers the following equivariant non-Hamiltonian perturbation

$$f(\Psi, \Psi_x, \tau) = -i\frac{\Gamma}{2}\Psi + iA[1 - \delta\cos\tau]|\Psi|^2\Psi, \tag{29}$$

where $\Gamma > 0$, $A > 0$ and $\delta > 0$. The first term in the perturbation represents dissipation due to fibre losses in the optical fibre [24], while the second one is the term representing compensating amplifiers placed along the fibre. As the perturbation is multiplied by ε in the perturbed NLS equation, we can set $A = 1$ without loss of generality.

In the multi-symplectic form this perturbation is written as

$$F(Z, \tau) = \left(\frac{\Gamma}{2}q_2 - (1 - \delta\cos\tau)(q_1^2 + q_2^2)q_2, -\frac{\Gamma}{2}q_1 + (1 - \delta\cos\tau)(q_1^2 + q_2^2)q_1, 0, 0\right)^T.$$

The C level sets are not automatically preserved anymore and one has to obtain the conditions for the persistence of the fixed points first. Straightforward calculations show that the function $\mathcal{R}(Z)$ evaluated at the fixed points $Z_0^\infty(c)$ is given by

$$\mathcal{R}(Z_0^\infty(c)) = \rho_0^2(-\Gamma/2 + \rho_0^2).$$

Apart from the trivial solution $\rho_0 = 0$, this function vanishes whenever

$$\Gamma = 2\rho_0^2 = 2(a + k) + \frac{1}{2}v^2 - \frac{1}{2}(2b + v)^2, \quad \text{hence} \quad (2b + v)^2 = 4(a + k) + v^2 - 2\Gamma. \tag{30}$$

It can be easily checked that $d\mathcal{R}/dc \neq 0$ at fixed points satisfying the last relation. Thus Theorem 3.1 gives the existence of a family of curves of relative periodic orbits, coming out of $G_\theta Z_0^\infty(c(\hat{b}))$, where $\hat{b}$ satisfies (30) and G_θ is any element in $\mathcal{G}$. Note that in order for $\hat{b}$ to satisfy (30) and the existence condition (26), Γ has to be in the following interval

$$\tfrac{1}{3}\left[4(a + k) + v^2\right] < \Gamma < \tfrac{1}{2}\left[4(a + k) + v^2\right]. \tag{31}$$

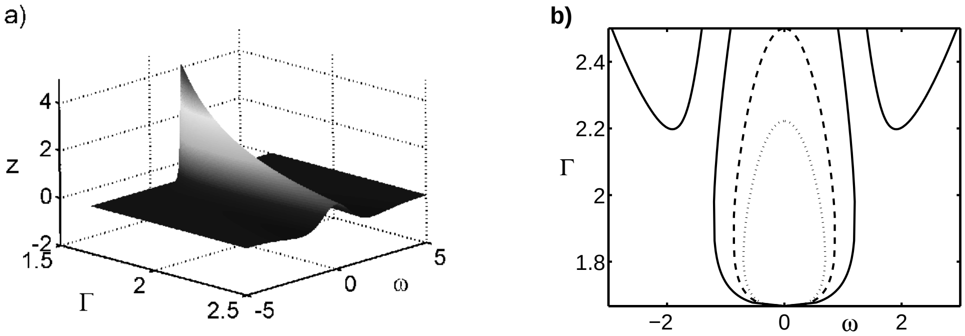

Figure 2. In both pictures $4(k+a)+v^2=5$. a) The plot of $z(\omega,\Gamma)$ as defined by (32). b) The bounded regions in the Γ-ω plane are the regions for which the perturbed invariant manifolds intersect transversely for $\delta=1$ (dotted line), $\delta=2$ (dashed line) and $\delta=10$ (solid line).

A direct calculation shows that the first order correction to the unit eigenvalue of the perturbed Poincaré map is equal to

$$\mu_1 = -T\frac{d}{dc}\Big|_{c=\hat{c}}\mathcal{R}(Z_0^\infty(c)) = T\frac{\Gamma(2\hat{b}+v)}{4B^2},$$

thus $\mu_1>0$. This means that the perturbed periodic orbit γ_ε has a three-dimensional centre-stable manifold and a four-dimensional centre-unstable manifold. To analyse possible intersections of these manifolds we compute the Melnikov function in the direction of the gradient of the Hamiltonian:

$$\begin{aligned} M_H(\tau_0,\theta_0;\hat{c}) = -\tfrac{B}{3}\sqrt{4(k+a)+v^2-2\Gamma}\ &\big[6\Gamma-8(k+a)-2v^2 \\ &+\tfrac{\delta\pi\omega}{2B}\ \mathrm{csch}\left(\tfrac{\pi\omega}{2B}\right)\ (8(k+a)+2v^2-2\omega^2-3\Gamma)\ \cos\tau_0\big]\,. \end{aligned}$$

where relation (30) for $\hat{b}$ gives that $4B^2 = 3\Gamma-4(k+a)-v^2$. This function will have simple zeros for some values of τ_0 provided that Γ satisfies (31) and ω, Γ and δ satisfy

$$z(\omega,\Gamma) = \frac{\pi\omega}{4B}\ \mathrm{csch}\left(\frac{\pi\omega}{2B}\right)\left|\frac{8(k+a)+2v^2-2\omega^2-3\Gamma}{3\Gamma-4(k+a)-v^2}\right| > \frac{1}{\delta}, \tag{32}$$

In Figure 2 we illustrate $z(\omega,\Gamma)$ as a function of Γ and ω, and also boundary of the region in the Γ-ω plane given by (32) for different values of δ, while taking $4(k+a)+v^2=5$ (the figure does not qualitatively change for other values of $4(k+a)+v^2$). Inside the bounded regions in Fig. 2(b), the perturbed invariant manifolds will intersect transversely. Note that if δ increases, two extra regions appear alongside the central region. These regions will never merge as $z=0$ at $\Gamma=\frac{2}{3}(\omega^2-4(k+a)-v^2)$ and $\Gamma=\frac{1}{3}(4(k+a)+v^2)$ with $\omega\neq 0$.

6.3. *Symmetry-breaking perturbation*

As an example of possible changes in the dynamics which occur when the system is exposed to the influence of symmetry-breaking perturbations, we consider a particular case given by

$$f(\Psi,\Psi_x,\tau) = \beta\frac{\overline{\Psi}^2}{|\Psi|^2} + iA[1-\delta\cos\tau]|\Psi|^2\Psi. \tag{33}$$

Here, the first term breaks rotational symmetry of the unperturbed system, while the second corresponds to the compensating amplifiers as in the previous example. Again, we can set $A=1$ without loss of generality as the perturbation is multiplied by ε in the perturbed NLS equation.

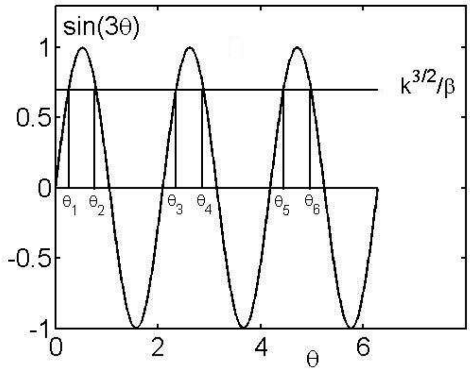

Figure 3. The points θ_i on the unperturbed group orbit persist under the perturbation (33).

In a multi-symplectic form this perturbation can be written as

$$F(Z,\tau) = \left(\beta\frac{q_1^2-q_2^2}{q_1^2+q_2^2} - q_2(q_1^2+q_2^2)[1-\delta\cos\tau], -2\beta\frac{q_1q_2}{q_1^2+q_2^2} + q_1(q_1^2+q_2^2)[1-\delta\cos\tau, 0, 0\right)^T.$$

As the symmetry is broken, we will take $a=b=0$ implying that the existence condition becomes $v^2<2k$ and

$$c_0 = \frac{1}{2}kv, \quad \rho_0^2 = k, \quad 4B^2 = 2k - v^2, \quad \text{and} \quad \theta_\infty = \arccos\left(\frac{v^2}{k}-1\right).$$

Theorem 3.1 implies that the angles $\hat\theta$, for which the fixed points $G_{\hat\theta}Z_0^\infty(c_0)$ persist, are the roots of

$$\mathcal{R}(G_{\hat\theta}Z_0^\infty(c_0)) = k^{1/2}(k\sqrt{k} - \beta\sin 3\hat\theta).$$

This shows that for $\beta > k\sqrt{k}$, there are six persisting fixed points with angles given by

$$\sin 3\hat\theta = \frac{k\sqrt{k}}{\beta}. \tag{34}$$

We have illustrated this relation in Figure 3. Thus there are six curves of periodic orbits in the perturbed system. The first order correction to the unit eigenvalues of the perturbed Poincaré map associated with those periodic, can be found as

$$\mu_1^2 = -s_0''(c_0)\,T^2\,\frac{d}{d\theta}\Big|_{\theta=\hat\theta}\mathcal{R}(G_\theta Z_0^\infty(c_0)) = 3\sqrt{k}\beta\, s_0''(c_0)\,T^2\,\cos 3\hat\theta.$$

This implies that the sign of μ_1^2 is alternating between the six successive roots $\hat\theta$ of (34). As $s_0''(c_0) = -\frac{v^2(14k^2+2v^2)}{16(2k-v^2)} < 0$, the first of these roots $\hat\theta_1 \in [0,\pi/6]$ is characterised by $\mu_1^2<0$, as is $\hat\theta_3 \in [\pi/2, 2\pi/3]$. The angular distance between these two roots is $2\pi/3$. The invariant manifolds of the persisting fixed points have to coincide. This implies that the angular shift between the surviving fixed points should equal the phase shift of the unperturbed heteroclinic orbit as it goes from $-\infty$ to $+\infty$. From this condition, one concludes that

$$\theta_\infty = \arccos\left(\frac{v^2}{k}-1\right) = \frac{2\pi}{3}, \quad \text{hence} \quad v^2 = \frac{k}{2}, \tag{35}$$

which fixes the relation between the velocity v and the squared radius k of the unperturbed orbit.

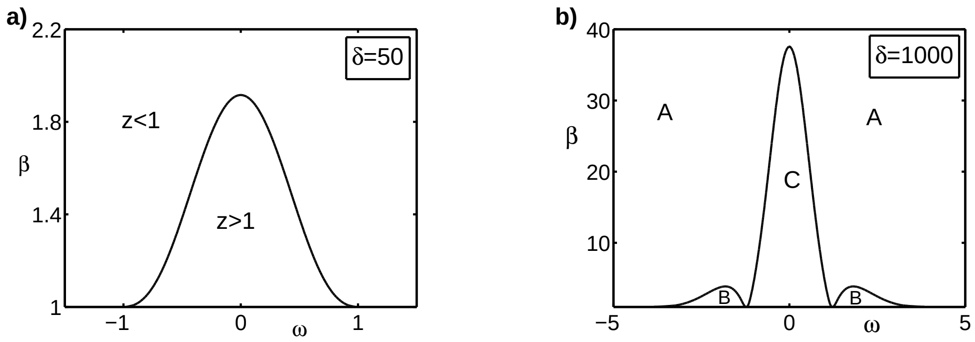

Figure 4. Boundary of the parameter region when the perturbed invariant manifold intersect transversely for $k = 1$. a) $\delta = 50$, the manifolds intersect in the region where $z > 1$. b) $\delta = 1000$, the manifolds intersect transversely in the regions B and C, and do not intersect in the region A.

Remark 1 : If one had taken the symmetry breaking part of f to be linear (e.g. $\overline{\psi}$) or like $|\psi|^2\overline{\psi}$, then the angular distance between two fixed points with equal μ_1^2 would have to be π. This is not possible as the angular distance between the two fixed points of the heteroclinic orbit is always less than π.

So for $\beta > k\sqrt{k}$, $\hat{\theta} = \frac{1}{3}\arcsin\left(\frac{k\sqrt{k}}{\beta}\right)$, and $v^2 = k/2$, we will investigate the intersection of two dimensional centre-stable and four dimensional centre-unstable manifolds of the perturbed fixed points evolving out of $\hat{\theta}$ and $\hat{\theta} + \frac{2\pi}{3}$. The Melnikov function is

$$\begin{aligned} M_H(\tau_0, \hat{\theta}; c_0) = \tfrac{1}{2} \int_{-\infty}^{\infty} \beta \Big[& \cos 3\hat{\theta}[(q_1^2 - q_2^2)(2p_1 + vq_2) - 2q_1q_2(2p_2 - vq_1)] \\ & + \sin 3\hat{\theta}((q_1^2 - q_2^2)(2p_2 - vq_1) + 2q_1q_2(2p_1 + vq_2))\Big] / (q_1^2 + q_2^2) \\ & + [1 - \delta\cos(\omega x + \tau_0)]|q|^2[v|q|^2 - 2C(Z_0)]\, dx \end{aligned}$$

where q_i and p_i are the components of $Z_0(x; c_0)$ and $|q|^2 = q_1^2 + q_2^2$. Straightforward but lengthy calculations show that the above integral can be evaluated as

$$\begin{aligned} M_H(\tau_0, \theta_0; c_0) = \tfrac{k^2}{48} \Bigg[& 54\sqrt{3} - 48\pi + 6\sqrt{\tfrac{3}{2}} - \sqrt{2\left(\tfrac{\beta^2}{k^3} - 1\right)}(52\pi - 15\sqrt{3}) \\ & - \sqrt{\tfrac{3}{2}} \tfrac{\delta\pi\omega}{kB} \operatorname{csch}\left(\tfrac{\pi\omega}{2B}\right) \left(3k - 2\omega^2\right) \cos\tau_0 \Bigg], \end{aligned}$$

where the expression (35) for v^2 gives that $B = \frac{1}{2}\sqrt{3k/2}$. This function will have simple zeros for some values of τ_0 provided $\beta > k\sqrt{k}$ and that ω, β and δ satisfy

$$z(\beta, \omega, \delta) = \frac{2\delta}{k\sqrt{k}} \left| \frac{\left(3k - 2\omega^2\right) \pi\omega \operatorname{csch}\left(\pi\omega\sqrt{\frac{2}{3k}}\right)}{54\sqrt{3} - 48\pi + 6\sqrt{\frac{3}{2}} - (52\pi - 15\sqrt{3})\sqrt{2\left(\frac{\beta^2}{k^3} - 1\right)}} \right| > 1. \tag{36}$$

In Figure 4 we show the boundary in β-ω plane for different values of δ as determined by the relation (36). For δ small, there is a single boundary separating the areas with $z > 1$ (the invariant manifolds intersecting transversely there) and those with $z < 1$. As δ increases, the boundary develops a dimple (see Fig. 4b), and inside this dimple we also have $z > 1$, so that the perturbed invariant manifolds intersect transversely in the regions B and C. As one increases δ further, the region B grows, and simultaneously the boundary between the regions A and C moves up. The regions B and C will always be disjoint as $z = 0$ at $\omega^2 = \frac{3k}{2}$.

The general conclusion is that in all three cases of different perturbations of the NLS equation, we have been able to indentify conditions on parameters which guarantee transversal intersection of some perturbed invariant manifolds. As the complexity of perturbations increases, more and more structure of the original system is destroyed. This inevitably leads to the decrease in dimensions of persisting invariant manifolds and as a consequence to a decrease in the dimension of their transversal intersection. It is worth noting that in the case of equivariant Hamiltonian perturbation, the Melnikov condition does not impose any restrictions on the parameters of the perturbation, and therefore a high-dimensional transversal intersection of perturbed invariant manifolds is guaranteed for any parameter values. When the nonlinear Schrödinger equation considered in this section is used to model optical wave propagation in waveguides, the description of different dynamical regimes in terms of perturbation parameters, as given above, is particularly useful when devising optimal schemes of signal amplification or chaotic communications. Finally, we would like to mention that Theorem 5.1 provides sufficient conditions for the transversal intersection of perturbed invariant manifolds (transversal in the parameter along the original heteroclinic orbit), and hence does not imply the persistence of a heteroclinic connection. One reason for the disappearance of the heteroclinic orbit is that under the perturbations, the geometry of the phase space changes, and some parts of the original structure (such as symmetry and/or Hamiltonian structure) disappear. Instead, it is more likely that the results about transversal intersection of perturbed invariant manifolds can be used to analyse the existence of spatio-temporally chaotic dynamics in the perturbed system.

7. Conclusions

In this paper we have derived sufficient criteria for the bifurcation (persistence) of relative equilibria into relative periodic orbits in multi-symplectic PDEs with spatially periodic perturbations, both equivariant and symmetry-breaking ones. The persistence of the periodic orbits is established by using a Lyapunov-Schmidt type argument to take into account the presence of the symmetry group. The symmetry group plays also a crucial role in the description of perturbed (centre-)stable and (centre-)unstable manifolds associated with those relative periodic orbits. These descriptions are used to show that a Melnikov integral gives a sufficient condition for the transversal interestion (up to symmetries) of those manifolds in case of symmetric perturbations.

In case of symmetry-breaking perturbations, one gets either a four-dimensional unstable manifold and a two-dimensional stable manifold or both stable and unstable manifolds are three-dimensional. In the first case, one Melnikov integral can be used to derive a sufficient condition for the transversal intersection of the stable and unstable manifolds. In the second case, an additional condition is needed and it is currently a work in progress to show that a second Melnikov function can be used for this.

It is worth noting that qualitatively the method developed in this paper can be considered in its own merit for Hamiltonian systems with saddle-centres characterized by a double zero eigenvalue. This is a degenerate case of the saddle-centre with non-hyperbolic eigenvalue being on the imaginary axis [15, 16]. The double zero eigenvalue presents certain complications in the analysis, which can be dealt with by using that the origin of the double eigenvalue is in a symmetry group.

Finally, we mention again the problem of spatio-temporal chaos. In the context of celestial mechanics some results about temporal chaos in non-dissipative systems have already been obtained. These results are concerned with the explicit construction of a horseshoe “polynomial” in time. This procedure was developed for area-preserving perturbations by Burns & Weiss [26], and has been also applied to the systems with non-hyperbolic fixed points [27, 28]. The techniques developed in this paper provide a description of the intersection of invariant manifolds in perturbed multi-symplectic systems with symmetry. The intersection of invariant manifolds might initiate spatio-temporal chaos in such systems, although some work will need to be done to make this formal, possibly via spatio-temporal horseshoe functions.

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Appendix A: Linearisations and equivariance

This Appendix contains the proofs of some auxiliary results and the persistence Theorem 3.1.

Proof of Lemma 2.3 : First we prove that for any $c \in \mathcal{C}$, the generator $\xi(Z_0^\infty(c))$ lies in the kernel of the operator $L_0(c)$. The flow of the unperturbed system commutes with the group action, hence the Poincaré map is equivariant under the group action and for all $G_\theta \in \mathcal{G}$ and Z we have $\Pi^{0,b(c)}\left(G_\theta Z\right) = G_\theta \Pi^{0,b(c)}\left(Z\right)$.

Differentiating this equation with respect to θ and setting $\theta = 0$ we obtain

$$D\Pi^{0,b(c)}(Z)\xi(Z) = \xi\left(\Pi^{0,b(c)}(Z)\right).$$

Substitution $Z = Z_0^\infty(c)$ completes the proof of (12).

Next, we prove the second statement, i.e., for any $c \in \mathcal{C}$, the vector $\frac{dZ_0^\infty(c)}{dc}$ is the generalised eigenvector corresponding to the zero eigenvalue of the operator $L_0(c)$. Recall that the relative equilibria $Z_0^\infty(c)$ are fixed points of the unperturbed Poincaré map $\Pi^{0,b(c)}$, hence $\Pi^{0,b(c)}\left(Z_0^\infty(c)\right) = Z_0^\infty(c)$. Differentiating this relation with respect to c and recalling that $b(c) = s_a'(c)$ hence $b'(c) = s_a''(c)$, gives

$$s_a''(c)\left.\frac{\partial}{\partial b}\right|_{b=b(c)} \Pi^{0,b}\left(Z_0^\infty(c)\right) + L_0(c)\frac{dZ_0^\infty(c)}{dc} = 0. \tag{A1}$$

Next use that for arbitrary b, the relative equilibria flow with the symmetry group, i.e., $\Pi^{0,b}(Z_0^\infty(c)) = G_{(b(c)-b)T}Z_0^\infty(c)$. Differentiating this relation with respect to b and evaluating at $b = b(c)$ gives

$$\left.\frac{\partial}{\partial b}\right|_{b=b(c)} \Pi^{0,b}\left(Z_0^\infty(c)\right) = -T\,\xi(Z_0^\infty(c)).$$

Combining this with (A1), we get the second relation (13).

Finally, the proof that for every $c \in \mathcal{C}$, the vector $\nabla C(Z_0^\infty(c))$ is orthogonal to the range of the operator $L_0(c)$. Since C is a constant of motion of the unperturbed Hamiltonian ODE, it follows immediately for every $c \in \mathcal{C}$ that $C(\Pi^{0,b(c)}(Z)) = C(Z)$. Taking the variation of Z in an arbitrary direction U and evaluating at $Z = Z_0^\infty(c)$ gives

$$\langle \nabla C(Z_0^\infty(c)), U\rangle = \langle \nabla C(Z_0^\infty(c)), D\Pi^{0,b(c)}(Z_0^\infty(c))U\rangle$$

hence $\langle \nabla C(Z_0^\infty(c)), L_0(c)U\rangle = 0$. □

Proof of Theorem 3.1: To find the fixed points of the Poincaré map, we want find b_ε and Z_ε such that

$$\Pi^{\varepsilon,b}(Z) - Z = 0.$$

However, for $\varepsilon = 0$ this equation does not have a unique solution, due to the equivariance under the group $\mathcal{G}$ and the invariance of the C level sets under the dynamics. In order to deal with this issue, the equation above is extended to the system

$$\begin{cases} 0 = \Pi^{\varepsilon,b}(Z) - Z + \sigma\nabla C(Z) \\ 0 = \langle \xi(Z), Z - G_\theta Z_0^\infty(c)\rangle \\ 0 = \displaystyle\int_0^T R(\Phi_x^{\varepsilon,b}(Z))\,dx \end{cases} \tag{A2}$$

with the variables/parameters b, Z, θ, c and σ. The idea behind this system is the same as the idea behind a Lyapunov-Schmidt reduction. The first equation adds a vector which is missing in the range of the unperturbed linearised operator $L_0 = D\Pi^{0,b}(Z_0^\infty) - I$. The second equation removes the invariance under the group action, which is present if $\varepsilon = 0$. Finally, the third equation is related to the selection criterion for c or θ. The role of the variables c, b, σ and θ will vary, depending on which of the three criteria the perturbation satisfies.

Case 1 Let F be equivariant and $R(Z, \tau) = 0$ for all values of Z and τ. This means that the third equation in (A2) is trivially satisfied and on each C-level set we can expect a persisting relative equilibrium. So we fix c and we will solve for (Z, σ, b) in

$$\begin{cases} 0 = \Pi^{\varepsilon,b}(Z) - Z + \sigma \nabla C(Z) \\ 0 = \langle \xi(Z), Z - Z_0^\infty(c) \rangle \\ 0 = C(Z) - c \end{cases} \tag{A3}$$

First note that for $\varepsilon = 0$ this system has a unique solution, given by $(Z_0^\infty(c), 0, b(c))$. The linearisation near this solution of this system at $\varepsilon = 0$ is

$$\text{LIN} := \begin{pmatrix} L_0(c) & \nabla C(Z_0^\infty(c)) & -T\,\xi(Z_0^\infty(c)) \\ \xi(Z_0^\infty(c))^T & 0 & 0 \\ \nabla C(Z_0^\infty(c)) & 0 & 0 \end{pmatrix}.$$

(note that Lemma 2.3 gives that $\frac{\partial}{\partial b}\Big|_{b=b(c)} \Pi^{0,b}(Z_0^\infty(c)) = -T\,\xi(Z_0^\infty(c)))$. This matrix has a trivial kernel. Indeed, assume that

$$\text{LIN}\begin{pmatrix} U \\ g \\ \beta \end{pmatrix} = 0, \quad \text{hence} \quad \begin{cases} 0 = L_0(c)\,U + g\,\nabla C(Z_0^\infty(c)) - \beta T\,\xi(Z_0^\infty(c)) \\ 0 = \langle \xi(Z_0^\infty(c)), U \rangle \\ 0 = \langle \nabla C(Z_0^\infty(c)), U \rangle \end{cases}$$

Taking the inner product of the first equation with $\nabla C(Z_0^\infty(c))$ (due to the Hypothesis **H2.1**(c), one has $\nabla C(Z_0^\infty(c)) = \mathbf{J}\xi(Z_0^\infty(c)) \neq 0$) and using the result of Lemma 2.3 that $\nabla C(Z_0^\infty(c))$ is orthogonal to the range of the operator $L_0(c)$, we obtain $g\,\|\nabla C(Z_0^\infty(c))\|^2 = 0$, and therefore $g = 0$. With Lemma 2.3, it follows that $L_0(c)\,\frac{dZ_0^\infty(c)}{dc} = b'(c)\,T\,\xi(Z_0^\infty(c))$, hence, $U - \frac{\beta}{b'(c)}\frac{dZ_0^\infty(c)}{dc} \in \ker L_0(c)$, and thus it can be written as $U - \frac{\beta}{b'(c)}\frac{dZ_0^\infty(c)}{dc} = K\xi(Z_0^\infty(c))$. Substituting this into the third equation gives that $\beta = 0$. Now substitute $U = K\xi(Z_0^\infty(c))$ into the second equation and $K\|\xi(Z_0^\infty(c))\|^2 = 0$ follows, thus $K = 0$ and therefore $U = 0$. So, it can be concluded that LIN has a trivial kernel, and therefore it is an invertible matrix.

Applying the implicit function theorem to (A3) gives that for ε small there exists a solution $(Z_\varepsilon^\infty(c), \sigma_\varepsilon(c), b_\varepsilon(c))$ of the system (A3), such that $\lim_{\varepsilon\to 0}(Z_\varepsilon^\infty(c), \sigma_\varepsilon(c), b_\varepsilon(c)) = (Z_0^\infty(c), 0, b(c))$.

To finish this part of the proof, it has to be shown that σ_ε vanishes. We have seen that $R(Z, \tau) = 0$ implies that the C level sets are invariant under the perturbed dynamics. Hence $C\left(Z_\varepsilon^\infty\right) = C\left(\Pi^{\varepsilon,b_\varepsilon}(Z_\varepsilon^\infty)\right)$. Expanding the right-hand side of this equation about Z_ε^∞, we obtain

$$C\left(Z\right) = C\left(Z\right) + \left\langle \nabla C\left(Z\right), \Pi(Z) - Z \right\rangle + \frac{1}{2}\Big\langle D^2 C(\tilde{Z})\left(\Pi(Z) - Z\right), \Pi(Z) - Z \Big\rangle,$$

where $Z = Z_\varepsilon^\infty$, $\Pi = \Pi^{\varepsilon,b_\varepsilon}$ and $|\tilde{Z} - Z_\varepsilon^\infty| \leq \mu\left|\Pi^{\varepsilon,b_\varepsilon}(Z_\varepsilon^\infty) - Z_\varepsilon^\infty\right|$, for some $0 \leq \mu \leq 1$. From (A3) it follows that $\Pi^{\varepsilon,b_\varepsilon}(Z_\varepsilon^\infty) - Z_\varepsilon^\infty = -\sigma_\varepsilon \nabla C(Z_\varepsilon^\infty)$, so we can conclude

$$\sigma_\varepsilon \|\nabla C(Z_\varepsilon^\infty)\|^2 = \frac{1}{2}\sigma_\varepsilon^2 \langle D^2 C(\tilde{Z}) \nabla C(Z_\varepsilon^\infty), \nabla C(Z_\varepsilon^\infty) \rangle \leq \frac{K}{2}\sigma_\varepsilon^2, \tag{A4}$$

for some constant K. Therefore either $\sigma_\varepsilon = 0$, or $\sigma_\varepsilon \geq \frac{2\|\nabla C(Z_\varepsilon^\infty)\|^2}{K}$. In the limit $\varepsilon \to 0$ the right-hand side of this inequality is bounded away from zero which contradicts $\lim_{\varepsilon\to 0}\sigma_\varepsilon = 0$. Thus $\sigma_\varepsilon = 0$, for all ε small.

By varying c and using the equivariance, the existence of a two-dimensional manifold of fixed points given by $G_\theta Z_\varepsilon^\infty(c)$ follows immediately for any $|\varepsilon| < \varepsilon_0$.

Case 2 Let F be equivariant and assume that there is some $\hat{c} \in \mathcal{C}$ such that $\mathcal{R}(Z_0^\infty(\hat{c})) = 0$ and $\frac{d}{dc}|_{c=\hat{c}}\mathcal{R}(Z_0^\infty(c)) \neq 0$. We will solve the system (A2) with $\theta = 0$ and $c = \hat{c}$ for (Z, b, σ). For convenience of notation, we introduce

$$\mathcal{R}_{\text{flow}}(Z, b, \varepsilon) = \int_0^T R(\Phi_x^{\varepsilon,b}(Z, 0))\, dx,$$

hence $\mathcal{R}_{\text{flow}}(Z_0^\infty(c), b(c), 0) = T\,\mathcal{R}(Z_0^\infty(c))$. Since F is equivariant, the perturbed flow is equivariant. Thus $\mathcal{R}_{\text{flow}}$ is invariant under the group $\mathcal{G}$ and $\langle \nabla\mathcal{R}_{\text{flow}}(Z, b, \varepsilon), \xi(Z)\rangle = 0$.

First note that for $\varepsilon = 0$ the system (A2) with $\theta = 0$ has a unique solution $(Z_0^\infty(\hat{c}), b(\hat{c}), 0)$. The linearisation at $\varepsilon = 0$ near this solution, is

$$\text{LIN} := \begin{pmatrix} L_0(\hat{c}) & \frac{\partial \Pi^{0,b(\hat{c})}}{\partial b}(Z_0^\infty(\hat{c})) & \nabla C(Z_0^\infty(\hat{c})) \\ \xi(Z_0^\infty)^T & 0 & 0 \\ \nabla\mathcal{R}_{\text{flow}}(Z_0^\infty, b(\hat{c}), 0) & \frac{\partial \mathcal{R}_{\text{flow}}}{\partial b}(Z_0^\infty, b(\hat{c}), 0) & 0 \end{pmatrix}.$$

Note that Lemma 2.3 implies that $\frac{\partial \Pi^{0,b(\hat{c})}}{\partial b}(Z_0^\infty(\hat{c})) = -(s_a''(\hat{c}))^{-1} L_0(\hat{c}) \frac{dZ_0^\infty(\hat{c})}{dc}$. To show that the matrix LIN is invertible, we proceed in a same way as in case 1, i.e., assume that there is some (U, β, g) such that

$$\text{LIN}\begin{pmatrix} U \\ \beta \\ g \end{pmatrix} = 0, \quad \text{i.e.,} \quad \begin{cases} 0 = L_0(\hat{c})\left[U - \beta(s_a''(\hat{c}))^{-1}\frac{dZ_0^\infty(\hat{c})}{dc}\right] + g\nabla C_v(p_0), \\ 0 = \langle \xi(p_0), U\rangle, \\ 0 = \langle \nabla\mathcal{R}_{\text{flow}}(Z_0^\infty(\hat{c}), b(\hat{c}), 0)\, ,\, U\rangle + \beta\, \frac{\partial \mathcal{R}_{\text{flow}}(Z_0^\infty(\hat{c}), b(\hat{c}), 0)}{\partial b}. \end{cases} \tag{A5}$$

As in case 1, by taking the inner product of the first equation with $\nabla C(Z_0^\infty(\hat{c}) \neq 0$ and using that $\nabla C(Z_0^\infty(\hat{c})$ is orthogonal to the range of $L_0(\hat{c})$, it follows that $g = 0$. Thus $U - \beta(s_a''(\hat{c}))^{-1}\frac{dZ_0^\infty(\hat{c})}{\partial c} \in \ker L_0(\hat{c})$, so it can be rewritten as $U = K\xi(Z_0^\infty(\hat{c})) + (s_a''(\hat{c}))^{-1}\frac{dZ_0^\infty(\hat{c})}{\partial c}$. Substituting this into the second equation of (A5), using the invariance of $\mathcal{R}_{\text{flow}}$ and $s_a''(\hat{c}) = b'(\hat{c})$, it follows

$$\frac{\beta}{s_a''(\hat{c})}\left[\left\langle \nabla\mathcal{R}_{\text{flow}}(Z_0^\infty(\hat{c}), b(\hat{c}), 0)\, ,\, \frac{dZ_0^\infty(\hat{c})}{dc}\right\rangle + b'(\hat{c})\,\frac{\partial \mathcal{R}_{\text{flow}}(Z_0^\infty(\hat{c}), b(\hat{c}), 0)}{\partial b}\right] = 0,$$

in other words

$$\frac{\beta}{s_a''(\hat{c})}\left.\frac{d}{dc}\right|_{c=\hat{c}} \mathcal{R}_{\text{flow}}(Z_0^\infty(c), b(c), 0) = 0 \quad \text{so} \quad \frac{\beta}{s_a''(\hat{c})}\left.\frac{d}{dc}\right|_{c=\hat{c}} \mathcal{R}(Z_0^\infty(c)) = 0.$$

With the assumption on the derivative of $\mathcal{R}(Z_0^\infty(c))$, this implies that $\beta = 0$, and therefore $U = K\xi(p_0)$. Substitution of this into the third equation of (A5) gives $K\|\xi(p_0)\|^2 = 0$, and thus $K = 0$. So, it can be concluded that LIN has a trivial kernel, and therefore it is an invertible matrix.

Again, applying the implicit function theorem gives that for ε small there exists a solution $(Z_\varepsilon^\infty, b_\varepsilon, \sigma_\varepsilon)$ of the system (A2) with $\theta = 0$, such that $\lim_{\varepsilon\to 0}(Z_\varepsilon^\infty, b_\varepsilon, \sigma_\varepsilon) = (Z_0^\infty(\hat{c}), b(\hat{c}), 0)$.

In a similar way as in case 1 it can be shown that $\sigma_\varepsilon = 0$. Indeed, the system (A2) with $\theta = 0$ gives that $\mathcal{R}_{\text{flow}}(Z_\varepsilon^\infty, b_\varepsilon, \varepsilon) = 0$. Since $\frac{d}{dx}C(\pi_1\Phi_x^{b_\varepsilon,\varepsilon}(Z_\varepsilon^\infty)) = -\varepsilon R(\Phi_x^{b_\varepsilon,\varepsilon}(Z_\varepsilon))$, this implies

$$C(Z_\varepsilon^\infty) - C(\Pi^{b_\varepsilon,\varepsilon}(Z_\varepsilon^\infty)) = \mathcal{R}_{\text{flow}}(Z_\varepsilon^\infty, b_\varepsilon, \varepsilon) = 0.$$

Following the same arguments as in case 1, it follows that $\sigma_\varepsilon = 0$ for all ε.

By using the equivariance, the existence of a one-dimensional manifold of fixed points given by $G_\theta Z_\varepsilon^\infty$ follows immediately for any $|\varepsilon| < \varepsilon_0$.

Case 3 Take $c = c_0$ and $b = 0 = a$ and assume that there is some $\hat{\theta}$ such that $\mathcal{R}(G_{\hat{\theta}} Z_0^\infty(c_0)) = 0$ and $\frac{d}{d\theta}|_{\theta=\hat{\theta}} \mathcal{R}(G_\theta Z_0^\infty(c_0)) \neq 0$. We will solve the system (A2) for (Z, θ, σ) (and $b = 0$, $c = c_0$). For convenience of notation, we introduce

$$\widehat{\mathcal{R}}_{\text{flow}}(Z, \varepsilon) = \int_0^T R(\Phi_x^{\varepsilon,0}(Z, 0))\, dx,$$

hence $\widehat{\mathcal{R}}_{\text{flow}}(G_\theta Z_0^\infty(c_0), 0) = T\,\mathcal{R}(G_\theta Z_0^\infty(c_0))$.

First note that for $\varepsilon = 0$ the system (A2) with $b = 0$ has a unique solution $(G_{\hat{\theta}} Z_0^\infty(c_0), \hat{\theta}, 0)$. The linearisation at $\varepsilon = 0$ near this solution, is

$$\text{LIN} := \begin{pmatrix} (DG_{\hat{\theta}}^*(Z_0^\infty(c_0)))^{-1}\, L_0(c_0)\, (DG_{\hat{\theta}}(Z_0^\infty(c_0)))^{-1} & 0 & \nabla C(G_{\hat{\theta}} Z_0^\infty(c_0)) \\ \xi(G_{\hat{\theta}} Z_0^\infty(c_0))^T & \|\xi(G_{\hat{\theta}} Z_0^\infty(c_0))\|^2 & 0 \\ \nabla \widehat{\mathcal{R}}_{\text{flow}}(G_{\hat{\theta}} Z_0^\infty(c_0), 0) & 0 & 0 \end{pmatrix}.$$

To show that the matrix LIN is invertible, we proceed in a same way as in case 1, i.e., assume that there is some (U, α, g) such that

$$\text{LIN} \begin{pmatrix} U \\ \alpha \\ g \end{pmatrix} = 0, \quad \text{i.e.,} \quad \begin{cases} 0 = L_0(c_0)\, (DG_{\hat{\theta}}(Z_0^\infty(c_0)))^{-1} U + g \nabla C(Z_0^\infty(c_0)), \\ 0 = \langle \xi(G_{\hat{\theta}} Z_0^\infty(c_0)), U \rangle + \alpha\, \|\xi(G_{\hat{\theta}} Z_0^\infty(c_0))\|^2, \\ 0 = \langle \nabla \widehat{\mathcal{R}}_{\text{flow}}(G_{\hat{\theta}} Z_0^\infty(c_0), 0)\,,\, U \rangle \end{cases} \tag{A6}$$

As in case 1, by taking the inner product of the first equation with $\nabla C(Z_0^\infty(c_0) \neq 0$ and using that $\nabla C(Z_0^\infty(c_0)$ is orthogonal to the range of $L_0(c_0)$, it follows that $g = 0$. Thus $(DG_{\hat{\theta}}(Z_0^\infty(c_0)))^{-1} U \in \ker L_0(c_0)$, so it can be rewritten as $U = K \xi(G_{\hat{\theta}} Z_0^\infty(c_0))$. Substituting this into the third equation of (A6) gives

$$0 = K\, \langle \nabla \widehat{\mathcal{R}}_{\text{flow}}(G_{\hat{\theta}} Z_0^\infty(c_0), 0)\,,\, \xi(G_{\hat{\theta}} Z_0^\infty(c_0)) \rangle = KT\, \frac{d}{d\theta}\Big|_{\theta=\hat{\theta}} \mathcal{R}(G_\theta Z_0^\infty(c_0)).$$

With the assumption on the derivative of $\mathcal{R}(G_\theta Z_0^\infty(c_0))$, this implies that $K = 0$, thus $U = 0$. Since $U = 0$, the second equation gives immediately that $\alpha = 0$. So, it can be concluded that LIN has a trivial kernel, and therefore it is an invertible matrix.

Again, applying the implicit function theorem gives that for ε small there exists a solution $(Z_\varepsilon^\infty, \theta_\varepsilon, \sigma_\varepsilon)$ of the system (A2) with $b = 0$, such that $\lim_{\varepsilon \to 0} (Z_\varepsilon^\infty, \theta_\varepsilon, \sigma_\varepsilon) = (Z_0^\infty(c_0), \hat{\theta}, 0)$.

In a similar way as in cases 1 and 2, it can be shown that $\sigma_\varepsilon = 0$ for all ε.

□

Proof of Lemma 4.2 :

Case 1 The equivariance of the system and the invariance of the C level sets under the dynamics, imply that the Poincaré map will preserve the double eigenvalue 1. The other two eigenvalues are just the continuation of the hyperbolic eigenvalues $e^{\pm\lambda(c)T}$.

Case 2 The equivariance of the system under the action of the symmetry group $\mathcal{G}$, gives that the generator of the group is an eigenvector of the Poincaré map with eigenvalue 1:

$$[D\Pi^{\varepsilon, b_\varepsilon}(Z_\varepsilon^\infty) - I]\, \xi(Z_\varepsilon^\infty) = 0.$$

This follows immediately by differentiating the equality $\Pi^{\varepsilon,b_\varepsilon}(G_\theta Z_\varepsilon^\infty) = G_\theta Z_\varepsilon^\infty$ with respect to θ and setting $\theta = 0$.

To find the eigenvalue bifurcating out of 1, the eigenvalue equation is written as

$$[D\Pi^{\varepsilon,b_\varepsilon}(Z_\varepsilon^\infty) - I](\xi(Z_\varepsilon^\infty) + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots) = (\varepsilon\mu_1 + \varepsilon^2\mu_2 + \ldots)(\xi(Z_\varepsilon^\infty) + \varepsilon v_1 + \ldots),$$

Using that $\xi(Z_\varepsilon^\infty)$ is an eigenvector of the Poincaré map, this reduces to

$$[D\Pi^{\varepsilon,b_\varepsilon}(Z_\varepsilon^\infty) - I]\,(v_1 + \varepsilon v_2 + \ldots) = (\mu_1 + \varepsilon\mu_2 + \ldots)\,(\xi(Z_\varepsilon^\infty) + \varepsilon v_1 + \ldots).$$

The lowest order equation is

$$[D\Pi^{0,b(\hat{c})}(\widehat{Z}_0^\infty) - I]\,v_1 = \mu_1\,\xi(\widehat{Z}_0^\infty) = \frac{\mu_1}{s_a''(\hat{c})T}\,[D\Pi^{0,b(\hat{c})}(\widehat{Z}_0^\infty) - I]\,\frac{dZ_0^\infty(\hat{c})}{dc},$$

where $\widehat{Z}_0^\infty = Z_0^\infty(\hat{c})$ and the last equality follows from (13). Thus $v_1 = \frac{\mu_1}{s_a''(\hat{c})T}\frac{dZ_0^\infty}{dc}(\hat{c})$.

The next order equation gives

$$L_0(\hat{c})v_2 + D\overline{\Pi}^1(\widehat{Z}_0^\infty)v_1 + D^2\Pi^{0,b(\hat{c})}(\widehat{Z}_0^\infty)(\overline{Z}_1^\infty, v_1) = \mu_1 v_1 + \mu_2\xi(\widehat{Z}_0^\infty) + \mu_1 D\xi(\widehat{Z}_0^\infty)\overline{Z}_1^\infty,$$

where the following notation is used

$$\overline{\Pi}^1(Z) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\Pi^{\varepsilon,b_\varepsilon}(Z), \quad \text{and} \quad \overline{Z}_1^\infty = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} Z_\varepsilon^\infty.$$

Taking the inner product with $\nabla C(\widehat{Z}_0^\infty)$ and using that $R(L_0) \perp \nabla C(\widehat{Z}_0^\infty)$, $\xi(\widehat{Z}_0^\infty) \perp \nabla C(\widehat{Z}_0^\infty)$ and the expression for v_1 derived above, gives

$$\begin{aligned} 0 = \tfrac{\mu_1}{s_a''(\hat{c})T}\Big(&\Big\langle D\overline{\Pi}^1(\widehat{Z}_0^\infty)\tfrac{dZ_0^\infty}{dc}(\hat{c}) + D^2\Pi^{0,b(\hat{c})}(\widehat{Z}_0^\infty)\Big(\overline{Z}_1^\infty, \tfrac{dZ_0^\infty}{dc}(\hat{c})\Big), \nabla C(\widehat{Z}_0^\infty)\Big\rangle \\ &- s_a''(\hat{c})T\Big\langle D\xi(\widehat{Z}_0^\infty)\overline{Z}_1^\infty, \nabla C(\widehat{Z}_0^\infty)\Big\rangle - \mu_1\Big\langle \tfrac{dZ_0^\infty}{dc}(\hat{c}), \nabla C(\widehat{Z}_0^\infty)\Big\rangle\Big) \end{aligned}$$

Observe that $\Big\langle \frac{dZ_0^\infty}{dc}(\hat{c}), \nabla C(\widehat{Z}_0^\infty)\Big\rangle = \frac{dC(Z_0^\infty)}{dc}(\hat{c}) = 1$, then this implies

$$\begin{aligned} \mu_1 = &\Big\langle D\overline{\Pi}^1(\widehat{Z}_0^\infty)\tfrac{dZ_0^\infty}{dc}(\hat{c}) + D^2\Pi^{0,b(\hat{c})}(\widehat{Z}_0^\infty)\Big(\overline{Z}_1^\infty, \tfrac{dZ_0^\infty}{dc}(\hat{c})\Big), \nabla C(\widehat{Z}_0^\infty)\Big\rangle \\ &- s_a''(\hat{c})T\Big\langle D\xi(\widehat{Z}_0^\infty)\overline{Z}_1^\infty, \nabla C(\widehat{Z}_0^\infty)\Big\rangle \end{aligned}$$

To simplify this expression, we manipulate the differential equation for the flow operator $\Phi_x^{\varepsilon,b_\varepsilon}(Z,0)$. For simplicity of notation, we write $\Phi_x^\varepsilon(Z) = \pi_1\Phi_x^{\varepsilon,b_\varepsilon}(Z,0)$. For any x, Z, ε, we have

$$\mathbf{J}D_x\Phi_x^\varepsilon(Z) = \nabla H(\Phi_x^\varepsilon(Z), b_\varepsilon) + \varepsilon F(\Phi_x^\varepsilon(Z), \omega x).$$

Taking the inner product with $\xi(\Phi_x^\varepsilon(Z)) = \mathbf{J}^{-1}\nabla C(\Phi_x^\varepsilon(Z))$ gives for any x, Z, ε

$$-\frac{d}{dx}C(\Phi_x^\varepsilon(Z)) = \varepsilon\langle F(\Phi_x^\varepsilon(Z), \omega x), \xi(\Phi_x^\varepsilon(Z))\rangle. \tag{A7}$$

For arbitrary c, take $Z = Z_0^\infty(c) + \varepsilon\overline{Z}_1^\infty$ and expand in ε about $\varepsilon = 0$, using the notation $\overline{\Phi}_x^1(Z) =$

$\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\Phi^\varepsilon_x(Z)$:

$$-\tfrac{d}{dx}\Big(C(\Phi^0_x(Z^\infty_0(c)))+\varepsilon\Big\langle\nabla C(\Phi^0_x(Z^\infty_0(c))),\overline{\Phi}^1_x(\Phi^0_x(Z^\infty_0(c)))+D\Phi^0_x(\Phi^0_x(Z^\infty_0(c)))\overline{Z}^\infty_1\Big\rangle+\ldots\Big)$$
$$=\varepsilon\langle F(\Phi^0_x(Z^\infty_0(c)),\omega x),\xi(\Phi^0_x(Z^\infty_0(c)))\rangle+\ldots.$$

Using the first order equation, recall that $\Phi^0_x(Z)$ is the flow for the vector field with Hamiltonian $H_0(Z,b(\hat{c}))$, thus $\Phi^0_x(Z^\infty_0(c))=G_{(b(c)-b(\hat{c}))x}Z^\infty_0(c)$ and using equivariance under the group action, we get

$$-\tfrac{d}{dx}\Big(\Big\langle\nabla C(Z^\infty_0(c)),\overline{\Phi}^1_x(Z^\infty_0(c))+D\Phi^0_x(Z^\infty_0(c))DG_{(b(c)-b(\hat{c}))x}(Z^\infty_0(c))^{-1}\overline{Z}^\infty_1\Big\rangle\Big)$$
$$=\langle F(Z^\infty_0(c),\omega x),\xi(Z^\infty_0(c))\rangle$$

Integrating this for x from 0 to T and using that for any Z, $\Phi^\varepsilon_0(Z)=Z$ (hence $D\Phi^0_0(Z)=I$ and $\overline{\Phi}^1_0(Z)=0$), gives

$$\Big\langle\nabla C(Z^\infty_0(c)),\overline{Z}^\infty_1-\overline{\Phi}^1_T(Z^\infty_0(c))-D\Phi^0_T(Z^\infty_0(c))DG_{(b(c)-b(\hat{c}))T}(Z^\infty_0(c))^{-1}\overline{Z}^\infty_1\Big\rangle$$
$$=\int_0^T\langle F(Z^\infty_0(c),\omega x),\xi(Z^\infty_0(c))\rangle\,dx=T\mathcal{R}(Z^\infty_0(c)).$$

Finally, differentiating this with respect to c and evaluating at $c=\hat{c}$, gives

$$\Big\langle D^2C(\widehat{Z}^\infty_0)\tfrac{dZ^\infty_0}{dc}(\hat{c}),\overline{Z}^\infty_1-\overline{\Phi}^1_T(\widehat{Z}^\infty_0)-D\Phi^0_T(\widehat{Z}^\infty_0)\overline{Z}^\infty_1\Big\rangle$$
$$-\Big\langle\nabla C(\widehat{Z}^\infty_0),D\overline{\Phi}^1_T(\widehat{Z}^\infty_0)\tfrac{dZ^\infty_0}{dc}(\hat{c})+D^2\Phi^0_T(\widehat{Z}^\infty_0)(\tfrac{dZ^\infty_0}{dc}(\hat{c}),\overline{Z}^\infty_1)-Ts''_a(\hat{c})D\xi(\widehat{Z}^\infty_0)\overline{Z}^\infty_1\Big\rangle$$
$$=T\ \tfrac{d}{dc}\big|_{c=\hat{c}}\mathcal{R}(Z^\infty_0(c)).$$

Observe first that

$$\overline{Z}^\infty_1-\overline{\Phi}^1_T(\widehat{Z}^\infty_0)-D\Phi^0_T(\widehat{Z}^\infty_0)\overline{Z}^\infty_1=\frac{d}{d\varepsilon}(Z^\infty_\varepsilon-\Phi^\varepsilon_T(Z^\infty_\varepsilon))=0,$$

Furthermore, the Poincaré map $\Pi^{\varepsilon,b_\varepsilon}=\Phi^\varepsilon_T$, and Z^∞_ε is a fixed point of this map, so we can conclude

$$\Big\langle\nabla C(\widehat{Z}^\infty_0),D\overline{\Pi}^1(\widehat{Z}^\infty_0)\tfrac{dZ^\infty_0}{dc}(\hat{c})+D^2\Pi^{0,b(\hat{c})}(\widehat{Z}^\infty_0)(\tfrac{dZ^\infty_0}{dc}(\hat{c}),\overline{Z}^\infty_1)-Ts''_a(\hat{c})D\xi(\widehat{Z}^\infty_0)\overline{Z}^\infty_1\Big\rangle$$
$$=-T\ \tfrac{d}{dc}\big|_{c=\hat{c}}\mathcal{R}(Z^\infty_0(c)).$$

Hence $\mu_1=-T\ \frac{d}{dc}\big|_{c=\hat{c}}\mathcal{R}(Z^\infty_0(c))$.

The final two eigenvalues are just the continuation of the hyperbolic eigenvalues $e^{\pm\lambda(c)T}$.

Case 3 In general, the eigenvalue 1 will not persist in this case as the symmetry is broken. This eigenvalue is degenerate with algebraic multiplicity 2, so it can be expected that two eigenvalues will bifurcate out of 1. So we should use and expansion in $\sqrt{\varepsilon}$ instead of ε. Therefore, to find the eigenvalues bifurcating out of 1, the eigenvalue equation is written as

$$[D\Pi^{\varepsilon,0}(Z^\infty_\varepsilon)-I](\xi(G_{\hat{\theta}}Z^\infty_0)+\sqrt{\varepsilon}v_1+\varepsilon v_2+\ldots)=(\sqrt{\varepsilon}\mu_1+\varepsilon\mu_2+\ldots)(\xi(G_{\hat{\theta}}Z^\infty_0)+\sqrt{\varepsilon}v_1+\ldots),$$

where $Z^\infty_0=Z^\infty_0(c_0)$. Using that $\xi(G_{\hat{\theta}}Z^\infty_0)$ is an eigenvector of the unperturbed Poincaré map, the lowest order equation is

$$[D\Pi^{0,0}(G_{\hat\theta}Z_0^\infty)-I]\,v_1=\mu_1\,\xi(G_{\hat\theta}Z_0^\infty)=\frac{\mu_1}{s_0''(c)T}\,[D\Pi^{0,0}(G_{\hat\theta}Z_0^\infty)-I]\,\frac{dG_{\hat\theta}Z_0^\infty(c_0)}{dc},$$

Thus $v_1=\frac{\mu_1}{s_0''(c_0)T}\,\frac{dG_{\hat\theta}Z_0^\infty(c_0)}{dc}$.

The next order equation gives

$$L_0(c_0,\hat\theta)v_2+D\overline{\Pi}^1(G_{\hat\theta}Z_0^\infty)\xi(G_{\hat\theta}Z_0^\infty)+D^2\Pi^{0,0}(G_{\hat\theta}Z_0^\infty)(\overline{Z}_1^\infty,\xi(G_{\hat\theta}Z_0^\infty))=\mu_1v_1+\mu_2\xi(G_{\hat\theta}Z_0^\infty),$$

where the following notation is used

$$L_0(c_0,\hat\theta)=D\Pi^{0,0}(G_{\hat\theta}Z_0^\infty)-I,\quad \overline{\Pi}^1(Z)=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\Pi^{\varepsilon,0}(Z),\quad\text{and}\quad \overline{Z}_1^\infty=\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}Z_\varepsilon^\infty(c_0).$$

Taking the inner product with $\nabla C(G_{\hat\theta}Z_0^\infty)$ and using that $R(L_0(c_0,\hat\theta))\perp\nabla C(G_{\hat\theta}Z_0^\infty)$, $\xi(G_{\hat\theta}Z_0^\infty)\perp\nabla C(G_{\hat\theta}Z_0^\infty)$ and the expression for v_1 gives

$$\frac{\mu_1^2}{s_0''(c_0)T}=\left\langle D\overline{\Pi}^1(G_{\hat\theta}Z_0^\infty)\xi(G_{\hat\theta}Z_0^\infty)+D^2\Pi^{0,0}(G_{\hat\theta}Z_0^\infty)\left(\overline{Z}_1^\infty,\xi(G_{\hat\theta}Z_0^\infty)\right),\nabla C(G_{\hat\theta}Z_0^\infty)\right\rangle$$

To simplify this expression, put (for arbitrary θ) $Z=G_\theta Z_\varepsilon^\infty$ into (A7) and take the derivative with respect to ε and evaluate at $\varepsilon=0$:

$$\begin{aligned}&\tfrac{d}{dx}\left[\left\langle\nabla C(\Phi_x^0(G_\theta\widehat{Z}_0^\infty)),\overline{\Phi}_x^1(G_\theta\widehat{Z}_0^\infty)+D\Phi_x^0(G_\theta\widehat{Z}_0^\infty)DG_\theta(\widehat{Z}_0^\infty)\overline{Z}_1^\infty\right\rangle\right]\\&\quad=\langle F(G_\theta\widehat{Z}_0^\infty,\omega x),\xi(G_\theta\widehat{Z}_0^\infty)\rangle,\end{aligned}$$

where $\widehat{Z}_0^\infty=G_{\hat\theta}Z_0^\infty$. Integration from 0 to T gives

$$\begin{aligned}&\left\langle\nabla C(G_\theta\widehat{Z}_0^\infty),\overline{\Phi}_T^1(G_\theta\widehat{Z}_0^\infty)+D\Phi_T^0(G_\theta\widehat{Z}_0^\infty)DG_\theta(\widehat{Z}_0^\infty)\overline{Z}_1^\infty-DG_\theta(\widehat{Z}_0^\infty)\overline{Z}_1^\infty\right\rangle\\&\quad=\int_0^T\langle F(G_\theta\widehat{Z}_0^\infty,\omega x),\xi(G_\theta\widehat{Z}_0^\infty)\rangle\,dx.\end{aligned}$$

Finally taking the derivative with respect to θ and evaluating at $\theta=0$ shows

$$\begin{aligned}&\left\langle D^2C(G_{\hat\theta}Z_0^\infty)\xi(G_{\hat\theta}Z_0^\infty),\overline{\Phi}_T^1(G_{\hat\theta}Z_0^\infty)+[D\Phi_T^0(G_\theta Z_0^\infty)-I]\overline{Z}_1^\infty\right\rangle\\&+\left\langle\nabla C(G_{\hat\theta}Z_0^\infty),D\overline{\Phi}_T^1(G_{\hat\theta}Z_0^\infty)\xi(G_{\hat\theta}Z_0^\infty)+D^2\Phi_T^0(G_{\hat\theta}Z_0^\infty)(\overline{Z}_1^\infty,\xi(Z_0^\infty))\right\rangle\\&-\left\langle\nabla C(G_{\hat\theta}Z_0^\infty),L_0(c_0,\hat\theta)D\xi(G_{\hat\theta}Z_0^\infty)\overline{Z}_1^\infty\right\rangle=\tfrac{d}{d\theta}\big|_{\theta=\hat\theta}\int_0^T\langle F(G_\theta Z_0^\infty,\omega x),\xi(G_\theta Z_0^\infty)\rangle\,dx.\end{aligned}$$

As shown in case 2, the first term is zero and using that $\nabla C(G_{\hat\theta}Z_0^\infty)$ is perpendicular to the range of $L_0(c_0,\hat\theta)$, we get

$$\begin{aligned}&\left\langle\nabla C(G_{\hat\theta}Z_0^\infty),D\overline{\Phi}_T^1(G_{\hat\theta}Z_0^\infty)\xi(G_{\hat\theta}Z_0^\infty)+D^2\Phi_T^0(G_{\hat\theta}Z_0^\infty)(\overline{Z}_1^\infty,\xi(Z_0^\infty))\right\rangle\\&\quad=\tfrac{d}{d\theta}\big|_{\theta=\hat\theta}\int_0^T\langle F(G_\theta Z_0^\infty,\omega x),\xi(G_\theta Z_0^\infty)\rangle\,dx=T\,\frac{d}{d\theta}\bigg|_{\theta=\hat\theta}\mathcal{R}(G_\theta Z_0^\infty).\end{aligned}$$

and this implies that μ_1^2 satisfies the expression as given in the lemma.

The final two eigenvalues are just the continuation of the hyperbolic eigenvalues $e^{\pm\lambda(c)T}$.

$\square$