

# Perturbations of embedded eigenvalues for the planar bilaplacian

Gianne Derks  
Department of Mathematics  
University of Surrey  
Guildford, GU2 7XH, UK

Sara Maad Sasane\*  
Department of Mathematics  
Stockholm University  
106 91 Stockholm, Sweden

Björn Sandstede†  
Division of Applied Mathematics  
Brown University  
Providence, RI 02912, USA

August 23, 2010

## Abstract

Operators on unbounded domains may acquire eigenvalues that are embedded in the essential spectrum. Determining the fate of these embedded eigenvalues under small perturbations of the underlying operator is a challenging task, and the persistence properties of such eigenvalues is linked intimately to the multiplicity of the essential spectrum. In this paper, we consider the planar bilaplacian with potential and show that the set of potentials for which an embedded eigenvalue persists is locally an infinite-dimensional manifold with infinite codimension in an appropriate space of potentials.

## 1 Introduction

Determining the dependence of the spectrum of operators on perturbations is an important issue that is of relevance in many applications. Of course, much is known in this direction: the persistence of point eigenvalues and the behavior of the essential spectrum under small bounded perturbations, for instance, have been analyzed comprehensively, and we refer to [10] for many results along these lines. Here, we consider differential operators that are posed on unbounded domains and are interested in the interaction between eigenvalues, with proper eigenfunctions in the underlying domain of the operator, and the essential spectrum. More precisely, we study the fate of eigenvalues that are embedded in the essential spectrum under small perturbations of the operator. Typically, such eigenvalues will disappear under generic perturbations of the potential, and it is therefore of interest to determine the class of perturbations for which an embedded eigenvalue persists. For the bilaplacian on cylindrical domains, we showed in our previous work [7] that the set of perturbations for which an embedded eigenvalue persists is an infinite-dimensional manifold of finite codimension. Furthermore, we showed that the codimension of this set is given by the multiplicity of the essential spectrum, defined as the number of independent continuum eigenfunctions or, more rigorously, via the spectral resolution of the Fourier transform of the bilaplacian (see e.g. [2, Definition 2 in §85]). In this paper, we continue the investigation that we began in [7] and consider the bilaplacian posed on the plane: the challenge is that the essential spectrum of the planar bilaplacian has infinite multiplicity. Thus, we may expect that the set of potentials for which an embedded

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\*Sara Maad Sasane was partially supported by the European Union under the Marie Curie Fellowship MEIF-CT-2005-024191 and by the Swedish Research Council.

†Björn Sandstede was partially supported by a Royal Society–Wolfson Research Merit Award and by the NSF under grant DMS-0907904.

eigenvalue persists is an infinite-dimensional manifold of infinite codimension, and this is indeed what we shall prove for an appropriate class of potentials.

Before stating our results, we briefly outline why embedded eigenvalues are of interest. Our first motivation comes from quantum mechanics: the eigenfunctions associated with eigenvalues of an energy operator correspond to bound states that can be attained by the physical system modelled by the energy operator. If such an eigenvalue is embedded in the essential spectrum, then its fate under perturbations of the potential determines whether the associated bound states persists or not (see [9, 15] for examples). A second example comes from inverse scattering theory, where eigenvalues correspond to coherent soliton structures of the underlying integrable system, while the essential spectrum describes radiative scattering behavior. Thus, bifurcations of solitons are reflected by the disappearance or persistence of embedded eigenvalues [12, 13]. Finally, embedded eigenvalues provide a common mechanism for the destabilization of travelling waves in near-integrable Hamiltonian partial differential equations, and we refer to [16] for further background information and pointers to the literature.

As mentioned above, we focus in this paper on the persistence of embedded eigenvalues for the planar bilaplacian. Our primary reason for considering the bilaplacian is that this operator is complex enough to exhibit the underlying difficulties, while not adding technical complications that have nothing to do with the issue we are interested in. In other words, the planar bilaplacian provides a useful paradigm for the issues that we expect to encounter for other more complicated differential operators. Note also that the applications we mentioned above all involve selfadjoint operators, a feature shared by the bilaplacian.

We now describe the precise setting that we consider. Let  $r_0 > 0$ , and assume that  $\theta \in C_0^\infty(B_{r_0}(0); \mathbb{R})$  is a radially symmetric potential. Hence, we use polar coordinates  $(r, \varphi)$ , write  $\theta = \theta(r)$ , and consider the multiplication operator on  $L^2(\mathbb{R}^2)$  (also denoted by  $\theta$ ) defined by

$$[\theta u](r, \varphi) := \theta(r)u(r, \varphi).$$

We define  $\mathcal{L} := \Delta^2 + \theta$  on  $L^2(\mathbb{R}^2)$ , where  $\Delta^2$  is the bilaplace operator which is densely defined on  $L^2(\mathbb{R}^2)$  with domain  $H^4(\mathbb{R}^2)$ . It is known that the spectrum of  $\Delta^2$  is  $\sigma(\Delta^2) = [0, \infty)$ . Since  $\theta$  has compact support, the essential spectra of  $\mathcal{L}$  and  $\Delta^2$  coincide, and so  $\sigma_c(\mathcal{L}) = [0, \infty)$ . We assume that  $\theta$  is chosen so that  $\mathcal{L}$  has a simple positive eigenvalue  $\lambda_0$ :

(A1)  $\mathcal{L}$  has an eigenvalue  $\lambda_0 > 0$  of multiplicity 1.

We are mainly interested in the case where  $\lambda_0$  is an embedded eigenvalue, i.e. when  $\lambda_0 \geq 0$ , since when  $\lambda_0$  is isolated from the rest of the spectrum, the persistence of eigenvalues is well known, [10, pp. 213–215]. We also exclude the case  $\lambda_0 = 0$  which lies on the boundary between spectrum and resolvent set. We denote by  $u_*(r, \varphi)$  the eigenfunction associated with the embedded eigenvalue  $\lambda_0$ . Since  $\theta$  is radially symmetric and the laplacian  $\Delta$  is invariant under rotations of the underlying cartesian coordinates, we see that the functions  $u_*(r, \varphi + \varphi_0)$  are, for each fixed  $\varphi_0$ , also eigenfunctions of  $\mathcal{L}$  belonging to the eigenvalue  $\lambda_0$ . The simplicity of  $\lambda_0$  required in assumption (A1) therefore implies that  $u_*$  is a radial function, and we henceforth write  $u_* = u_*(r)$ . It is clear by existence and uniqueness of solutions of ODEs that  $u_*(r)$  cannot vanish for all  $r \geq r_1$ , and so we assume that

(A2)  $r_1 > r_0$  is such that  $u_*(r_1) \neq 0$ .

Lemma 1 below shows that our hypotheses can be satisfied. We now perturb the potential  $\theta$  by potentials  $\rho$  in the weighted  $L^2$ -space  $\mathcal{R} := L^2([0, r_1], H^{1/2}(S^1), r dr)$  of functions that map the interval  $[0, r_1]$  into  $H^{1/2}(S^1)$ , where the interval  $[0, r_1]$  is the domain of the radial variable  $r$ , while  $H^{1/2}(S^1)$  describes the dependence on the angular variable  $\varphi$ . Our main result is as follows.

**Theorem 1.** *Let  $0 < r_0 \leq r_1$ ,  $\theta \in C_0^\infty(B_{r_0}(0); \mathbb{R})$  be radially symmetric, and assume that (A1) and (A2) hold. Then there exists  $\delta > 0$  and a neighbourhood  $\mathcal{O}$  of 0 in  $\mathcal{R} = L^2([0, r_1], H^{1/2}(S^1), r dr)$  such that the set*

$$\mathcal{R}_{emb} := \{\rho \in \mathcal{O}; \mathcal{L} + \rho \text{ has an embedded eigenvalue in } (\lambda_0 - \delta, \lambda_0 + \delta)\}$$

*is a smooth manifold in  $\mathcal{R}$  of infinite dimension and codimension.*

Before commenting on the ideas behind the proof of Theorem 1, we illustrate that our hypotheses can be met.

**Lemma 1.** *There exists a smooth radial potential  $\theta(r)$  with compact support such that  $\mathcal{L} = \Delta^2 + \theta$  satisfies Hypothesis (A1).*

*Proof.* Let  $K_0(r)$  denote the modified Bessel function of the second kind and define a smooth, strictly positive function  $u_0(r)$  via

$$u_0(r) = \begin{cases} 1 & 0 \leq r \leq 1 \\ K_0(r) & 2 \leq r \end{cases}$$

together with a smooth interpolation in the intermediate region  $r \in [1, 2]$ . Note that  $u_0$  decays to exponentially as  $r \rightarrow \infty$  and can be chosen so that  $u_0(r) > 0$  for all  $r$ . Thus, the radial potential

$$\theta := \frac{1}{u_0}(-\Delta^2 + 1)u_0 = \frac{1}{u_0}(\Delta + 1)(-\Delta + 1)u_0 = \begin{cases} 1 & 0 \leq r \leq 1 \\ 0 & 2 \leq r \end{cases}$$

is well-defined, smooth, and has support in  $[0, 2]$  since  $K_0(r)$  satisfies  $(-\Delta + 1)K_0 = 0$ . Furthermore, we have

$$\mathcal{L}u_0 = (\Delta^2 + \theta)u_0 = \Delta^2u_0 + \frac{1}{u_0}[(-\Delta^2 + 1)u_0]u_0 = u_0,$$

and  $u_0$  is a positive radial eigenfunction belonging to the embedded eigenvalue  $\lambda_0 = 1$  of  $\mathcal{L}$ .

It remains to show that  $\lambda_0 = 1$  is simple. Using the radial symmetry of  $\theta$ , the results presented in the rest of this paper imply that it suffices to show that the equation

$$\begin{aligned} \left[ \partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \right]^2 u &= (1 - \theta)u \quad \text{for } r \in (0, 2) \\ u_r(0) = u_{rrr}(0) &= 0, \quad u(r) = K_k(r) \text{ for } r \geq 2 \end{aligned} \quad (1)$$

does not have a solution  $u(r)$  for each integer  $k \geq 1$ . We will now outline why (1) will not have solutions for  $k \geq 1$  provided  $\theta$  is modified appropriately but omit the straightforward details. Using variations of parameters, it can be shown that (1) cannot have solutions for  $k \gg 1$ . If it does have solutions for some or all of the remaining finitely many integers  $k \geq 1$ , then we can modify the potential  $\theta$  to remove these solutions while retaining the eigenfunction for  $k = 0$ . Indeed, any solution of (1) for  $k \geq 1$  is of the explicit form  $u(r) = r^\ell$  for some integer  $\ell = \ell(k) \geq 1$  since  $\theta(r) - 1 = 0$  for  $0 \leq r \leq 1$ . Replacing  $u_0$  in the above construction of  $\theta$  by  $u_0 + \epsilon v_0$  for bounded functions  $v_0$  with support in  $(\frac{1}{2}, 1)$  and using the necessary expressions (64) derived in section 7 for the persistence of eigenvalues, it is then not difficult to see that any nonzero choice of  $v_0 \geq 0$  removes the solutions of (1) for  $k \geq 1$  for  $0 \ll \epsilon \ll 1$ .  $\square$

The idea for proving Theorem 1 is to characterize embedded eigenvalues as roots of a regular function, since such a characterization would allow us to use the implicit function theorem. As it appears difficult to find a functional-analytic characterization of embedded eigenvalues, we pursue here a dynamical-systems formulation similar to that used in our precursor work [7] for the bilaplacian on cylinders. The eigenvalue problem can be written as a system of differential equations in the radial evolution variable  $r$  posed on an appropriate function space  $X$  of functions that are defined in the angular variable  $\varphi$ . The issue is that this system is ill-posed in the sense that, for given initial data, solutions may not exist. Using a similar approach as in Scheel [18], we will show, however, that this dynamical system has an exponential dichotomy: there are two infinite-dimensional subspaces  $X^{cu}$  and  $X^s$  of  $X$  at  $r = r_1$  so that solutions with initial data in  $X^{cu}$  exist and stay bounded for  $r \leq r_1$ , while solutions with data in  $X^s$  exist and decay as  $r \rightarrow \infty$ . The intersection of these spaces corresponds to eigenfunctions of the underlying operator, and our goal is therefore to characterize those perturbations for which this intersection is nontrivial. We show that there are infinitely many conditions that characterize such intersections and prove that we can solve them using an implicit function theorem. A key issue is the space for the perturbation  $\rho$ . For the conditions of the implicit function theorem to be satisfied, the space for  $\rho$  needs to be  $L^2([0, r_1]; H^{1/2}(S^1), r dr)$ , a space with very low regularity. This low regularity forces us to work with different

function spaces for  $r \leq r_1$  (where  $\rho$  has its support) and for  $r \geq r_1$  (where we have an explicit formulation of the solutions of the system in terms of Bessel functions), and so we need to take extra care when matching the solutions at  $r = r_1$ .

The rest of this paper is organized as follows. In section 2, we introduce the spatial-dynamics formulations of the eigenvalue problem. In sections 3 and 4, we prove the existence of exponential dichotomies for the bilaplacian and for the operator  $\mathcal{L}$ , respectively, near the core  $r = 0$ . We then construct dichotomies for  $\mathcal{L}$  in the far field for  $r \gg 1$  in section 5 and discuss similar properties for the adjoint spatial dynamical system in section 6. These results are then used in section 7 where we match the solutions from the core and the far field by using Lyapunov–Schmidt reduction and prove Theorem 1. The paper is concluded with suggestions for extensions and some open problems.

## 2 Spatial-dynamics formulation

If  $\lambda$  is an eigenvalue of  $\mathcal{L} + \rho$ , then there exists  $u \in H^4(\mathbb{R}^2)$  such that

$$\Delta^2 u + (\theta + \rho)u = \lambda u. \quad (2)$$

Let  $r_3 > r_2 > \max(1, r_1)$ . We introduce a new radial variable

$$s(r) = \begin{cases} \log r & \text{if } r \leq r_2, \\ r & \text{if } r \geq r_3, \end{cases} \quad (3)$$

and for  $r \in (r_2, r_3)$ , we define  $s$  such that  $s \in C^\infty(\mathbb{R}^+; \mathbb{R})$  is strictly increasing. Note that this implies that there exist constants  $c$  and  $C$  such that  $0 < c < C$  and  $c \leq s'(r) \leq C$  for every  $r_2 \leq r \leq r_3$ . We define  $\tilde{\theta}$  and  $\tilde{\rho}$  by  $\tilde{\theta}(s(r)) = \theta(r)$ , etc. Since  $s$  is an increasing function, it is invertible, and we denote the inverse function by  $r(s)$ . Let  $s_j := s(r_j)$ ,  $j = 1, \dots, 3$ . Under the coordinate transformation (3), the space  $\mathcal{R}$  transforms into the space  $\tilde{\mathcal{R}}$  given by

$$\tilde{\mathcal{R}} := L^2((-\infty, s_1]; H^{1/2}(S^1), e^{2s} ds),$$

that is, the weighted  $L^2$  space with values in  $H^{1/2}(S^1)$  and weight  $e^{2s}$ .

Setting  $v = \Delta u$ , equation (2) is equivalent to the system

$$\begin{aligned} \Delta u &= v, \\ \Delta v &= (\lambda - \tilde{\theta} - \tilde{\rho})u, \end{aligned} \quad (4)$$

where in the variables  $s$  and  $\varphi$ , the Laplacian is given by

$$\Delta = \frac{1}{r'(s)^2} \left[ \frac{\partial^2}{\partial s^2} + \left( \frac{r'(s)}{r(s)} - \frac{r''(s)}{r'(s)} \right) \frac{\partial}{\partial s} + \left( \frac{r'(s)}{r(s)} \right)^2 \frac{\partial^2}{\partial \varphi^2} \right].$$

Rewriting this intermediate system as a first order system, with  $u_1 = u$ ,  $u_2 = u'$ ,  $u_3 = v$  and  $u_4 = v'$ , where  $'$  denotes differentiation with respect to  $s$ , we obtain a system of the form

$$U'(s) = A(s; \lambda, \tilde{\rho})U, \quad (5)$$

where  $A(s; \lambda, \tilde{\rho})$  is given by

$$A(s; \lambda, \tilde{\rho}) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{r'(s)^2}{r(s)^2} \partial^2 & \frac{r''(s)}{r'(s)} - \frac{r'(s)}{r(s)} & r'(s)^2 & 0 \\ 0 & 0 & 0 & 1 \\ (\lambda - \tilde{\theta} - \tilde{\rho})r'(s)^2 & 0 & -\frac{r'(s)^2}{r(s)^2} \partial^2 & \frac{r''(s)}{r'(s)} - \frac{r'(s)}{r(s)} \end{pmatrix} \quad (6)$$

where  $\partial$  denotes differentiation with respect to  $\varphi$ , i.e.,  $\partial = \frac{\partial}{\partial \varphi}$ . The expression for  $A(s; \lambda, \tilde{\rho})$  simplifies significantly for  $s < s_2$  or  $s > s_3$ ; see sections 3 and 5.

The perturbation  $\tilde{\rho} \in \tilde{\mathcal{R}}$  is, in general, not continuous or even bounded, so we need to study weak solutions of (5). Let

$$\begin{aligned} X &= H^2(S^1) \times H^1(S^1) \times H^1(S^1) \times L^2(S^1); \\ Y &= H^3(S^1) \times H^2(S^1) \times H^2(S^1) \times H^1(S^1). \end{aligned}$$

**Definition 1.** Let  $J$  be an interval of  $\mathbb{R}$ . A function  $U : J \rightarrow X$  is a weak solution of (5) in  $J$  if

1.  $U \in L^2_{loc}(J; Y) \cap H^1_{loc}(J; X)$ ,
2. for every  $V \in C_0^\infty(J; X)$  we have

$$-\int_J U(s)V'(s) ds = \int_J A(s; \lambda, \tilde{\rho})U(s)V(s) ds.$$

**Lemma 2.** Let  $\lambda \in \mathbb{R}$ . The eigenvalue equation (2) has a solution  $u \in H^4_{loc}(\mathbb{R}^2)$  if and only if (5) has a weak solution  $U \in H^1_{loc}(\mathbb{R}; X) \cap L^2_{loc}(\mathbb{R}; Y) \cap L^\infty(\mathbb{R}_-; X)$ .

*Proof.* Suppose that  $u \in H^4_{loc}(\mathbb{R}^2)$  is a solution of (2), and let  $U := (u, u', \Delta u, (\Delta u)')^T$ , where  $'$  denotes differentiation with respect to  $s$ .

We first consider  $u$  as a function of  $r$ , and let  $B_R(0)$  be a ball centered at 0, with  $R$  any positive radius. Then

$$\begin{aligned} u &\in H^1((0, R); H^3(S^1), r dr) \cap L^2((0, R); H^4(S^1), r dr) \\ &\subset H^1((0, R); H^2(S^1), r dr) \cap L^2((0, R); H^3(S^1), r dr), \\ u' &= r' \frac{du}{dr} \in H^1((0, R); H^2(S^1), r dr) \cap L^2((0, R); H^3(S^1), r dr) \\ &\subset H^1((0, R); H^1(S^1), r dr) \cap L^2((0, R); H^2(S^1), r dr), \\ \Delta u &\in H^1((0, R); H^1(S^1), r dr) \cap L^2((0, R); H^2(S^1), r dr), \\ (\Delta u)' &= r' \frac{d(\Delta u)}{dr} \in H^1((0, R); L^2(S^1), r dr) \cap L^2((0, R); H^1(S^1), r dr), \end{aligned}$$

where  $r' = dr/ds = r$  for  $r < r_2$ . By the Sobolev embedding theorem,  $U \in C([0, R]; X)$ , and so  $U(s(r))$  has a limit as  $r \rightarrow 0+$ , or equivalently, as  $s \rightarrow -\infty$ . Hence, viewing  $U$  as a function of  $s$ ,  $U \in L^\infty(\mathbb{R}_-; X)$ . We also see that  $U \in H^1_{loc}(\mathbb{R}; X) \cap L^2_{loc}(\mathbb{R}; Y)$ . It is clear from the construction that  $U$  is a weak solution of (5).

Conversely, let  $U \in H^1_{loc}(\mathbb{R}; X) \cap L^2_{loc}(\mathbb{R}; Y) \cap L^\infty(\mathbb{R}_-; X)$  be a weak solution of (5), and let  $u = U_1$ . Viewing  $u$  as a function of  $r$  rather than of  $s$ , it is clear that  $u \in H^1_{loc}(\mathbb{R}_+; H^2(S^1)) \subset C((0, \infty); C(S^1))$ , and so by (4),

$$\Delta^2 u = (\lambda - \theta - \rho)u \in L^2_{loc}(\mathbb{R}_+; H^{1/2}(S^1), r dr) \subset L^2_{loc}(\mathbb{R}_+; L^2(S^1), r dr) = L^2_{loc}(\mathbb{R}^2 \setminus \{0\}).$$

Since we also have

$$\begin{aligned} u &\in L^\infty([0, r_2]; H^2(S^1)) \subset L^\infty([0, r_2]; C(S^1)), \\ \lambda - \theta - \rho &\in L^2([0, r_2]; H^{1/2}(S^1), r dr) \subset L^2([0, r_2]; L^2(S^1), r dr), \end{aligned}$$

we see that  $\Delta^2 u = (\lambda - \theta - \rho)u \in L^2([0, r_2]; L^2(S^1), r dr) = L^2(B_{r_2}(0))$ . We have proved that  $u \in H^4(B_{r_2}(0)) \cap H^4_{loc}(\mathbb{R}^2 \setminus \{0\}) = H^4_{loc}(\mathbb{R}^2)$ .

Since it is also clear that  $u$  solves (2), the proof is complete.  $\square$

Note that a weak solution satisfies  $U \in C(\mathbb{R}; X)$  (see e.g. [8, p. 286]), and so the following definition for an exponential dichotomy makes sense (see also [6] for the standard definition for ODEs and [14] for an extension to PDEs):

**Definition 2.** Let  $J$  be an unbounded subinterval of  $\mathbb{R}$ . We say that equation (5) has an exponential dichotomy in  $X$  on  $J$  if there exists a family of projections  $P(s)$  for  $s \in J$  such that for any  $s \in J$ ,  $P(s) \in \mathcal{L}(X)$ ,  $P(s)^2 = P(s)$  and  $P(\cdot)U \in C(J; X)$  for every  $U \in X$ , and there exist constants  $K > 0$  and  $\kappa^s < \kappa^u$  with the following properties:

(i) For each  $t \in J$  and  $U \in X$  there exists a unique weak solution  $\Phi^s(s, t)U$  of (5) defined for  $s \geq t$ ,  $s, t \in J$  such that  $\Phi^s(t, t)U = P(t)U$  and

$$\|\Phi^s(s, t)U\|_X \leq Ke^{\kappa^s(s-t)}\|U\|_X$$

for all  $s \geq t$ ,  $s, t \in J$ .

(ii) For each  $t \in J$  and  $U \in X$  there exists a unique weak solution  $\Phi^u(s, t)U$  of (5) defined for  $s \leq t$ ,  $s, t \in J$  such that  $\Phi^u(t, t)U = (I - P(t))U$  and

$$\|\Phi^u(s, t)U\|_X \leq Ke^{\kappa^u(s-t)}\|U\|_X$$

for all  $s \leq t$ ,  $s, t \in J$ .

(iii) The solutions  $\Phi^s(s, t)U$  and  $\Phi^u(s, t)U$  satisfy

$$\begin{aligned} \Phi^s(s, t)U &\in \text{Ran } P(s) && \text{for every } s \geq t, s, t \in J, \\ \Phi^u(s, t)U &\in \text{ker } P(s) && \text{for every } s \leq t, s, t \in J. \end{aligned}$$

We also need the definition of time-dependent exponential dichotomy, which will be used for  $J = [s_1, \infty)$  and with  $\mathcal{X}^s := H^1 \times L^2 \times H^1 \times L^2$  with the  $s$ -dependent norm

$$\|U\|_{\mathcal{X}^s}^2 := \frac{1}{s^2} \|u_1\|_{H^1(S^1)}^2 + \|u_1\|_{L^2(S^1)}^2 + \|u_2\|_{L^2(S^1)}^2 + \frac{1}{s^2} \|u_3\|_{H^1(S^1)}^2 + \|u_3\|_{L^2(S^1)}^2 + \|u_4\|_{L^2(S^1)}^2,$$

where  $u_j$  are the components of  $U$ ,  $j = 1, \dots, 4$ .

**Definition 3.** Let  $J$  be an unbounded subinterval of  $\mathbb{R}$ . We say that equation (5) has a time-dependent exponential dichotomy in  $\mathcal{X}^s$  on  $J$  if there exists a family of projections  $P(s)$  for  $s \in J$  such that for any  $s \in J$ ,  $P(s) \in \mathcal{L}(\mathcal{X}^s)$ ,  $P(s)^2 = P(s)$  and  $P(\cdot)U \in C(J; \mathcal{X})$  for every  $U \in \mathcal{X}$ , and there exist constants  $K > 0$  and  $\kappa^s < \kappa^u$  with the following properties:

(i) For each  $t \in J$  and  $U \in \mathcal{X}^t$  there exists a unique solution  $\Phi^s(s, t)U$  of (5) defined for  $s \geq t$ ,  $s, t \in J$  such that  $\Phi^s(t, t)U = P(t)U$  and

$$\|\Phi^s(s, t)U\|_{\mathcal{X}^s} \leq Ke^{\kappa^s(s-t)}\|U\|_{\mathcal{X}^t}$$

for all  $s \geq t$ ,  $s, t \in J$ .

(ii) For each  $t \in J$  and  $U \in \mathcal{X}^t$  there exists a unique solution  $\Phi^u(s, t)U$  of (5) defined for  $s \leq t$ ,  $s, t \in J$  such that  $\Phi^u(t, t)U = (I - P(t))U$  and

$$\|\Phi^u(s, t)U\|_{\mathcal{X}^s} \leq Ke^{\kappa^u(s-t)}\|U\|_{\mathcal{X}^t}$$

for all  $s \leq t$ ,  $s, t \in J$ .

(iii) The solutions  $\Phi^s(s, t)U$  and  $\Phi^u(s, t)U$  satisfy

$$\begin{aligned} \Phi^s(s, t)U &\in \text{Ran } P(s) && \text{for every } s \geq t, s, t \in J, \\ \Phi^u(s, t)U &\in \text{ker } P(s) && \text{for every } s \leq t, s, t \in J. \end{aligned}$$

In the following sections, we will consider the intervals

$$J_- := (-\infty, s_1] \quad \text{and} \quad J_+ := [s_1, \infty),$$

and show that the system (5) has an exponential dichotomy on  $J_-$  and a time-dependent exponential dichotomy on  $J_+$ .

### 3 Dichotomies for the system at $-\infty$

For  $s \leq s_1$ ,  $r(s) = e^s$ , and hence the system (5) is given by

$$\begin{cases} u'_1 = u_2, \\ u'_2 = -\partial^2 u_1 + e^{2s} u_3, \\ u'_3 = u_4, \\ u'_4 = (\lambda - \tilde{\theta}(s) - \tilde{\rho}(s, \cdot)) e^{2s} u_1 - \partial^2 u_3. \end{cases} \quad (7)$$

In the limit as  $s \rightarrow -\infty$  we have the system

$$\begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\partial^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\partial^2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

or

$$U' = A_- U. \quad (8)$$

We expand  $U = (u_1, u_2, u_3, u_4)^T$  as a Fourier series in the  $\varphi$  variable, and denote the  $k$ th Fourier coefficient by  $\widehat{U}_k(s)$ . For  $j \in \mathbb{R}$ , we define the weighted  $l^2$  spaces  $l_j^2$  with norm defined by

$$\|\{a_k\}_{k \in \mathbb{Z}}\|_{l_j^2}^2 := \sum_{k \in \mathbb{Z}} (1 + k^2)^j |a_k|^2$$

The function space induced by  $X$  is

$$\widehat{X} := l_2^2 \times l_1^2 \times l_1^2 \times l^2. \quad (9)$$

The system (8) decouples in the Fourier space and for  $k \in \mathbb{Z}$  we have

$$\widehat{U}'_k(s) = \widehat{A}_-(k) \widehat{U}_k(s), \quad (10)$$

where

$$\widehat{A}_-(k) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ k^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & k^2 & 0 \end{pmatrix}.$$

The eigenvalues of  $\widehat{A}_-(k)$  are  $\pm|k|$ , and for  $k \neq 0$  both eigenvalues have geometric multiplicity 2. The eigenvectors for  $k \neq 0$  are  $(\pm 1/k^2, 1/|k|, 0, 0)^T$  and  $(0, 0, \pm 1/|k|, 1)$  (we normalize the eigenvectors so that their  $\widehat{X}$  norm is approximately constant and bounded away from 0 as  $k \rightarrow \infty$ ). Let

$$M_k := \begin{pmatrix} -1/k^2 & 0 & 1/k^2 & 0 \\ 1/|k| & 0 & 1/|k| & 0 \\ 0 & -1/|k| & 0 & 1/|k| \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$D_k := \begin{pmatrix} -|k| & 0 & 0 & 0 \\ 0 & -|k| & 0 & 0 \\ 0 & 0 & |k| & 0 \\ 0 & 0 & 0 & |k| \end{pmatrix},$$

so that  $\widehat{A}_-(k) = M_k D_k M_k^{-1}$  for  $k \neq 0$ . Note that

$$M_k^{-1} = \frac{1}{2} \begin{pmatrix} -k^2 & |k| & 0 & 0 \\ 0 & 0 & -|k| & 1 \\ k^2 & |k| & 0 & 0 \\ 0 & 0 & |k| & 1 \end{pmatrix},$$

For  $k = 0$ , the eigenvalue 0 has algebraic multiplicity 4 and geometric multiplicity 2, so  $\widehat{A}_-(0)$  is not diagonalizable. Note however that  $\widehat{A}_-(0)$  is already in Jordan normal form. We define  $M_0 = I$  and  $D_0 = \widehat{A}_-(0)$ .

**Lemma 3.** *The operator  $A_- : X \rightarrow X$  is a closed densely defined operator with spectrum  $\sigma(A_-) = \mathbb{Z}$ .*

*Proof.* Recall that  $X = H^2(S^1) \times H^1(S^1) \times H^1(S^1) \times L^2(S^1)$ , and  $Y = H^3(S^1) \times H^2(S^1) \times H^2(S^1) \times H^1(S^1)$ . It is easy to check that the domain of  $A_-$  is  $Y$ , which is dense in  $X$ . To see that  $A_-$  is closed, let  $U_j \in Y$  be such that  $U_j \rightarrow U$  in  $X$  and  $A_-U_j \rightarrow f$  in  $X$ . We write  $U_j = (u_{1,j}, u_{2,j}, u_{3,j}, u_{4,j})^T$  etc. By the definition of  $A_-$  we have

$$\begin{aligned} u_{1,j} &\rightarrow u_1 && \text{in } H^2, \\ u_{2,j} &\rightarrow u_2 && \text{in } H^1, \\ u_{3,j} &\rightarrow u_3 && \text{in } H^1, \\ u_{4,j} &\rightarrow u_4 && \text{in } L^2, \end{aligned}$$

while

$$\begin{aligned} u_{2,j} &\rightarrow f_1 && \text{in } H^2, \\ -\partial^2 u_{1,j} &\rightarrow f_2 && \text{in } H^1, \\ u_{4,j} &\rightarrow f_3 && \text{in } H^1, \\ -\partial^2 u_{3,j} &\rightarrow f_4 && \text{in } L^2. \end{aligned}$$

It follows that  $u_2 = f_1 \in H^2$ , and that  $u_{1,j}$  converges in  $H^3$ . Since  $u_{1,j} \rightarrow u_1$  in  $H^2 \supset H^3$ , and since limits (in  $H^2$ ) are unique if they exist, we also have  $u_{1,j} \rightarrow u_1$  in  $H^3$ , and so  $-\partial^2 u_1 = f_2$ . It follows in exactly the same way that  $u_4 = f_3 \in H^1$ , that  $u_3 \in H^2$  and that  $-\partial^2 u_3 = f_4$ . This shows that  $U \in Y$  and  $A_-U = F$ , and so  $A_- : X \rightarrow X$  is closed.

The operator  $A_- : X \rightarrow X$  induces an operator  $\widehat{A}_- : \widehat{X} \rightarrow \widehat{X}$  defined by

$$(\widehat{A}_-\widehat{U})_k := \widehat{A}_-(k)\widehat{U}_k.$$

Then  $\widehat{A}_-$  is a densely defined operator on  $\widehat{X}$  with domain  $\widehat{Y} := l_3^2 \times l_2^2 \times l_2^2 \times l_1^2$ .

It is clear that  $(A_- - \mu I) : X \rightarrow X$  has a bounded inverse if and only if  $(\widehat{A}_- - \mu I) : \widehat{X} \rightarrow \widehat{X}$  has a bounded inverse. It is also clear that  $k \in \sigma(\widehat{A}_-)$  for  $k \in \mathbb{Z}$ . To prove that there are no other points in the spectrum of  $A_-$ , let  $\mu \in \mathbb{C} \setminus \mathbb{Z}$ .

Define  $\widehat{M} : l^2 \times l^2 \times l^2 \times l^2 \rightarrow l_2^2 \times l_1^2 \times l_1^2 \times l^2$  by

$$(\widehat{M}\widehat{U})_k = M_k\widehat{U}_k,$$

and note that  $\widehat{M}$  is a linear homeomorphism between these spaces. Define also the unbounded operator  $\widehat{D}$  on  $l^2 \times l^2 \times l^2 \times l^2$  by

$$(\widehat{D}\widehat{U})_k = D_k\widehat{U}_k.$$

Note that  $\widehat{D}$  is a closed densely defined operator with domain  $l_1^2 \times l_1^2 \times l_1^2 \times l_1^2$ , and that  $\sigma(\widehat{D}) = \mathbb{Z}$ .

If  $\mu \in \mathbb{C} \setminus \mathbb{Z}$ , then

$$(\widehat{A}_- - \mu I)^{-1} = \widehat{M}(\widehat{D} - \mu I)^{-1}\widehat{M}^{-1}$$

It is now easy to see that  $(\widehat{A}_- - \mu I)^{-1} : \widehat{X} \rightarrow \widehat{X}$  is bounded, and consequently also  $(A_- - \mu I)^{-1} : X \rightarrow X$ .  $\square$

Having established that the spectrum of  $A_-$  consists exactly of its eigenvalues, we define the (generalized) spectral projections  $P^s, P^c, P^u$  in  $X$ , corresponding to the negative, the zero and the positive eigenvalues of  $A_-$ , respectively. Let  $X^s = P^s X$ , etc. so that  $X = X^s \oplus X^c \oplus X^u$ , where  $X^s$  and  $X^u$  are infinite-dimensional whereas  $X^c$  is four-dimensional. We also define corresponding spectral projections  $P_k^s, P_k^u$  of  $\widehat{A}_-(k)$ , in the spaces  $X_k$ ,  $k \in \mathbb{Z} \setminus \{0\}$  and note that if  $U = \sum_{k \in \mathbb{Z}} \widehat{U}_k e^{ik} \in X$ , then

$$\begin{aligned} P^s U &= \sum_{k \in \mathbb{Z} \setminus \{0\}} P_k^s \widehat{U}_k e^{ik}, \\ P^u U &= \sum_{k \in \mathbb{Z} \setminus \{0\}} P_k^u \widehat{U}_k e^{ik}, \\ P^c U &= \widehat{U}_0. \end{aligned}$$



**Lemma 4.** *The operator  $A_-$  possesses an exponential dichotomy in  $X$  on  $J_- = (-\infty, s_1]$  with constant  $K$  and rates  $\kappa^s = 0$  and  $\kappa^u = 1$ , and another exponential dichotomy in  $X$  on  $J_-$  with constant  $K$  and rates  $\kappa^s = -1$  and  $\kappa^u = 0$ .*

*Proof.* Let  $\eta \in (0, 1)$  be arbitrary. We apply Lemma 2.1 of [14] for the operators  $A_- - \eta I$  and  $A_- + \eta I$ , and obtain exponential dichotomies with constant  $K$  and rates  $\kappa^s = -\eta$  and  $\kappa^u = 1 - \eta$ , and  $\kappa^s = -1 + \eta$  and  $\kappa^u = \eta$ , respectively. The existence of exponential dichotomies for  $A_-$  with rates  $\kappa^s = -1$  and  $\kappa^u = 0$ , and  $\kappa^s = 0$  and  $\kappa^u = 1$ , respectively then follows by using the transformation  $V = e^{\pm\eta \cdot} U$ .

We only consider the operator  $A_- - \eta I$ , since the proof for  $A_- + \eta I$  is similar. The result follows from Lemma 2.1 of [14] if we can verify condition (H1) of [14] for the operator  $A_- - \eta I$ , namely

(H1) Suppose that there exists a constant  $C > 0$  such that

$$\|(A_- - \eta I - i\mu I)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{1 + |\mu|}$$

for every  $\mu \in \mathbb{R}$ .

As in the proof of Lemma 3, it suffices to prove that there exists a constant  $\tilde{C}$  such that

$$\left\| \left( \widehat{D} - \eta I - i\mu I \right)^{-1} \widehat{U} \right\|_{l^2 \times l^2 \times l^2 \times l^2} \leq \frac{\tilde{C}}{1 + |\mu|} \|\widehat{U}\|_{l^2 \times l^2 \times l^2 \times l^2}$$

for every  $\widehat{U} \in l^2 \times l^2 \times l^2 \times l^2$ . Note that for  $k \neq 0$  we have

$$\left| (D_k - \eta I - i\mu I)^{-1} \widehat{U}_k \right|^2 = \frac{|\widehat{U}_k|^2}{(k - \eta)^2 + \mu^2} \leq \frac{2}{\min(\eta^2, (1 + \eta)^2)(1 + |\mu|)^2} |\widehat{U}_k|^2,$$

and it is not difficult to see that a similar estimate holds for  $|(D_0 - \eta I - i\mu I)\widehat{U}_0|^2$ . Hence

$$\begin{aligned} \left\| \left( \widehat{D} - \eta I - i\mu I \right)^{-1} \widehat{U} \right\|_{l^2 \times l^2 \times l^2 \times l^2}^2 &= \sum_{k \in \mathbb{Z}} \left| (D_k - \eta I - i\mu I)^{-1} \widehat{U}_k \right|^2 \\ &\leq \frac{\tilde{C}^2}{(1 + |\mu|)^2} \sum_{k \in \mathbb{Z}} |\widehat{U}_k|^2. \end{aligned}$$

□

## 4 Dichotomies near the core

The system (7) can be abbreviated and written as

$$U' = (A_- + B(s; \lambda, \tilde{\rho}))U, \tag{11}$$

where

$$B(s; \lambda, \tilde{\rho}) := e^{2s} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda - \tilde{\theta}(s) - \tilde{\rho}(s, \cdot) & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

We will show that the system (11) has an exponential dichotomy on the interval  $J_- = (-\infty, s_1]$ .

To show this, we would like to apply Theorem 1 of [14]. This is not possible, however, since  $\tilde{\rho} \in \tilde{\mathcal{R}}$  is not smooth enough in  $s$ . We are interested in  $\tilde{\rho}$  small and consider therefore first  $\tilde{\rho} = 0$ , and show that the  $\lambda$ -perturbed system

$$U' = (A_- + B(s; \lambda, 0))U \tag{13}$$

possesses an exponential dichotomy in  $X$  on  $J_-$ . Then we will use the implicit function theorem to show that also the system (11) possesses an exponential dichotomy. Note that from its definition, it follows immediately that  $B(s; \lambda, 0) \in \mathcal{L}(X)$ .

As  $\tilde{\theta}$  does not depend on  $\varphi$ , the system (13) decouples in Fourier space, just as the limiting system (8). Using the same notation in the Fourier spaces as before, we get for  $k \in \mathbb{Z}$

$$\widehat{U}'_k(s) = \left[ \widehat{A}_-(k) + B(s; \lambda, 0) \right] \widehat{U}_k(s). \quad (14)$$

As  $\widehat{A}_-(k) = M_k D_k M_k^{-1}$ , we rescale both  $\widehat{U}_k$  and  $s$  to get estimates which are uniform in  $k$ . For  $k \neq 0$ , define  $\tau = |k|(s - s_1)$  and  $V_k(\tau) = M_k^{-1} \widehat{U}_k(\tau/|k| + s_1)$ . Then (14) becomes

$$\frac{d}{d\tau} V_k = \left[ D_1 + \frac{1}{|k|} M_k^{-1} B(\tau/|k|; \lambda, 0) M_k \right] V_k. \quad (15)$$

A short calculation shows that

$$|M_k^{-1} B(\tau/|k|; \lambda, 0) M_k| \leq \frac{e^{2\tau/|k|}}{2} \sup_{s \leq s_1} \{1, |\lambda - \tilde{\theta}(s)|/k^2\}.$$

Hence there exists a constant  $C$  such that for all  $k \neq 0$ ,  $|M_k^{-1} B(\tau/|k|; \lambda, 0) M_k| \leq 2C e^{2\tau/|k|}$  and

$$\int_{-\infty}^0 \frac{1}{|k|} |M_k^{-1} B(\tau/|k|; \lambda, 0) M_k| d\tau \leq \int_{-\infty}^0 \frac{2C e^{2\tau/|k|}}{|k|} d\tau = C. \quad (16)$$

By the proof of the roughness theorem for ordinary dichotomies (see [6] for details), the system (15) has an exponential dichotomy which we denote by  $\Psi_k^{u/s}(\tau, \sigma)$ , with constants  $K$ ,  $\kappa^u = 1$ ,  $\kappa^s = -1$ . We choose the dichotomy in such a way that  $\text{Ran } \Psi_-^s(s_1, s_1; \lambda, 0) \subset \text{span}\{e_1, e_2\}$ , where  $e_j$ ,  $j = 1, \dots, 4$  are the standard basis vectors of  $\mathbb{C}^4$  (again see [6]). This implies that the stable and unstable solutions satisfy

$$\begin{aligned} |\Psi_k^s(\sigma, \tau) V_k| &\leq K e^{-(\sigma-\tau)} |V_k|, & \tau \leq \sigma \leq 0; \\ |\Psi_k^u(\sigma, \tau) V_k| &\leq K e^{(\sigma-\tau)} |V_k|, & \sigma \leq \tau \leq 0. \end{aligned}$$

The norm in  $X$  induces a norm on the Fourier space  $X_k$  with

$$\|\widehat{U}_k\|_{X_k}^2 := \|\widehat{U}_k e^{ik\cdot}\|_X^2 = (k^2 + 1)^2 ([\widehat{U}_k]_1)^2 + (k^2 + 1) ([\widehat{U}_k]_2)^2 + (k^2 + 1) ([\widehat{U}_k]_3)^2 + ([\widehat{U}_k]_4)^2.$$

As seen in the proof of Lemma 3,  $M_k$  is a linear homeomorphism between  $\mathbb{C}^4$  and  $X_k$ . Thus if we denote the exponential dichotomy of the unscaled system (14) by  $\Phi_k^{u/s}$ , then  $\Phi_k^{u/s}(s, t) = M_k \Psi_k^{u/s}(|k|(s - s_1), |k|(t - s_1)) M_k^{-1}$ , and they satisfy for  $\widehat{U}_k \in X_k$

$$\begin{aligned} \|\Phi_k^s(s, t) \widehat{U}_k\|_{X_k} &\leq K e^{-|k|(s-t)} \|\widehat{U}_k\|_{X_k} \leq K e^{-(s-t)} \|\widehat{U}_k\|_{X_k}, & t \leq s \leq s_1; \\ \|\Phi_k^u(s, t) \widehat{U}_k\|_{X_k} &\leq K e^{|k|(s-t)} \|\widehat{U}_k\|_{X_k} \leq K e^{(s-t)} \|\widehat{U}_k\|_{X_k}, & s \leq t \leq s_1. \end{aligned} \quad (17)$$

for some constant  $K$ , which is independent of  $k$ .

For the central space, corresponding to  $k = 0$ , the scaling  $V_0(s) = e^{\pm \epsilon s} \widehat{U}_0(s)$  and the integrability of  $B(s; \lambda, 0)$  shows that, for any  $\epsilon > 0$

$$\begin{aligned} |\Phi_0(s, t) \widehat{U}_0| &\leq K e^{\epsilon(s-t)} |\widehat{U}_0|, & t \leq s \leq s_1; \\ |\Phi_0(s, t) \widehat{U}_0| &\leq K e^{-\epsilon(s-t)} |\widehat{U}_0|, & s \leq t \leq s_1. \end{aligned} \quad (18)$$

Thus for the full solutions, we can define the stable and center-unstable solutions

$$\begin{aligned} \Phi_-^s(s, t; \lambda, 0) U &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \Phi_k^s(s, t) \widehat{U}_k e^{ik\cdot}, & s \leq t \leq s_1, \\ \Phi_-^{cu}(s, t; \lambda, 0) U &= \Phi_0(s, t) \widehat{U}_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \Phi_k^u(s, t) \widehat{U}_k e^{ik\cdot}, & t \leq s \leq s_1, \end{aligned}$$

and the unstable and center-stable solutions

$$\begin{aligned}\Phi_-^{cs}(s, t; \lambda, 0)U &= \Phi_0(s, t)\widehat{U}_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \Phi_k^s(s, t)\widehat{U}_k e^{ik \cdot}, \quad s \leq t \leq s_1, \\ \Phi_-^u(s, t; \lambda, 0)U &= \sum_{k \in \mathbb{Z} \setminus \{0\}} \Phi_k^u(s, t)\widehat{U}_k e^{ik \cdot}, \quad t \leq s \leq s_1.\end{aligned}$$

These solutions are related to dichotomies for (13) in  $X$  on  $J_-$ .

**Lemma 5.** *Let  $-1 = \kappa^s < \kappa^{cu} < 0 < \kappa^{cs} < \kappa^u = 1$  and  $\lambda \in \mathbb{R}$ . Then the system (13) has an exponential dichotomy in  $X$  on  $J_-$  with constant  $K$  and rates  $\kappa^{cu}$  and  $\kappa^s$ , and another with constant  $K$  and rates  $\kappa^u$  and  $\kappa^{cs}$ . The dichotomies can be chosen such that  $\text{Ran } \Phi_-^s(s_1, s_1; \lambda, 0) = P^s$  and  $\text{Ran } \Phi_-^{cs}(s_1, s_1; \lambda, 0) = P^{cs}$ . Moreover, for any  $t \in (-\infty, s_1]$  and  $U_0 \in X$ , the solutions  $\Phi_-^{cu}(\cdot, t; \lambda, 0)U_0$  and  $\Phi_-^s(\cdot, t; \lambda, 0)U_0$  belong to  $C^\infty((-\infty, t); X)$  and  $C^\infty((t, s_1); X)$ , respectively. Similarly, the solutions  $\Phi_-^u(\cdot, t; \lambda, 0)U_0$  and  $\Phi_-^{cs}(\cdot, t; \lambda, 0)U_0$  belong to  $C^\infty((-\infty, t); X)$  and  $C^\infty((t, s_1); X)$ , respectively. All solutions also depend smoothly on the parameter  $\lambda$ .*

*Proof.* The scaling  $e^{\pm \eta s}U$  for  $0 < \eta < 1$  and the dichotomy estimates in (17) and (18) immediately prove the first part of the Lemma. The dichotomies satisfy  $\text{Ran } \Phi_-^s(s_1, s_1; \lambda, 0) = P^s$  and  $\text{Ran } \Phi_-^{cs}(s_1, s_1; \lambda, 0) = P^c + P^s$  since we have chosen the  $\Psi_k^{u/s}$  above to satisfy  $\text{Ran } \Psi_-^s(s_1, s_1; \lambda, 0) \subset \text{span}\{e_1, e_2\}$  (cf. the definition of  $D_k$ ).

The smoothness with respect to  $s$  follows since  $\theta$  is smooth in  $s$  and smoothness in  $\lambda$  can be proved using an implicit function theorem argument. First observe that for any  $\lambda, \tilde{\lambda}$  close to each other, the solutions  $\Phi^{cu}$  and  $\Phi^s$  satisfy the integral equations

$$\begin{aligned}0 &= -\Phi_-^{cu}(s, t; \tilde{\lambda}, 0) + \Phi_-^{cu}(s, t; \lambda, 0) + (\tilde{\lambda} - \lambda) \left[ \int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0) e^{2\tau} B_0 \Phi_-^{cu}(\tau, t; \tilde{\lambda}, 0) d\tau \right. \\ &\quad \left. - \int_s^t \Phi_-^{cu}(s, \tau; \lambda, 0) e^{2\tau} B_0 \Phi_-^{cu}(\tau, t; \tilde{\lambda}, 0) d\tau + \int_t^{s_1} \Phi_-^{cu}(s, \tau; \lambda, 0) e^{2\tau} B_0 \Phi_-^s(\tau, t; \tilde{\lambda}, 0) d\tau \right], \\ 0 &= -\Phi_-^s(s, t; \tilde{\lambda}, 0) + \Phi_-^s(s, t; \lambda, 0) - (\tilde{\lambda} - \lambda) \left[ \int_{-\infty}^t \Phi_-^s(s, \tau; \lambda, 0) e^{2\tau} B_0 \Phi_-^{cu}(\tau, t; \tilde{\lambda}, 0) d\tau \right. \\ &\quad \left. - \int_t^s \Phi_-^s(s, \tau; \lambda, 0) e^{2\tau} B_0 \Phi_-^s(\tau, t; \tilde{\lambda}, 0) d\tau + \int_s^{s_1} \Phi_-^{cu}(s, \tau; \lambda, 0) e^{2\tau} B_0 \Phi_-^s(\tau, t; \tilde{\lambda}, 0) d\tau \right],\end{aligned}$$

for  $s \leq t \leq s_1$  and  $t \leq s \leq s_1$ , respectively, where  $B_0$  is the matrix

$$B_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (19)$$

Define the function spaces

$$\begin{aligned}X^s &:= \{\Phi^s; \Phi^s(s, t) \in \mathcal{L}(X) \text{ is defined and continuous for } t \leq s \leq s_1 \\ &\quad \text{with } \|\Phi^s\|_s := \sup_{t \leq s \leq s_1} e^{-\kappa^s(s-t)} \|\Phi^s(s, t)\|_{\mathcal{L}(X)}\}; \\ X^{cu} &:= \{\Phi^{cu}; \Phi^{cu}(s, t) \in \mathcal{L}(X) \text{ is defined and continuous for } s \leq t \leq s_1 \\ &\quad \text{with } \|\Phi^{cu}\|_{cu} := \sup_{s \leq t \leq s_1} e^{-\kappa^{cu}(s-t)} \|\Phi^{cu}(s, t)\|_{\mathcal{L}(X)}\}.\end{aligned}$$

For  $\lambda$  fixed, the integral equations can be written as  $F(\Phi^{cu}, \Phi^s; \tilde{\lambda}) = 0$ , where  $F : X^{cu} \times X^s \times \mathbb{R} \rightarrow X^{cu} \times X^s$ . The estimates of the exponential dichotomies immediately give that  $F$  is indeed a mapping between those spaces, for example,

$$\left\| \int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0) e^{2\tau} B_0 \Phi_-^{cu}(\tau, t; \tilde{\lambda}, 0) d\tau \right\|_{\mathcal{L}(X)} \leq \int_{-\infty}^s K^2 e^{\kappa^s(s-\tau)} e^{2\tau} e^{\kappa^{cu}(\tau-t)} d\tau = \frac{K^2 e^{2s_1} e^{\kappa^{cu}(s-t)}}{2 + \kappa^{cu} - \kappa^s}.$$

The other integrals can be estimated in a similar way. Since  $D_{(\Phi^{cu}, \Phi^s)} F(\Phi^{cu}(s, t; \lambda, 0), \Phi^s(s, t; \lambda, 0); \lambda) = I$ , the implicit function theorem can be applied and the smoothness with respect to  $\lambda$  follows immediately.  $\square$

**Remark 1.** The  $\varphi$ -independence of  $\tilde{\theta}$  is not essential in Lemma 5. The lemma can be proved for smooth  $\varphi$ -dependent functions  $\theta$  by using Theorem 1 of [14] and verifying the conditions (H1), (H2), (H3) and (H5) of that paper.

Next, we prove four technical lemmas needed in the proof of the existence of exponential dichotomies for the full system (11) with  $\tilde{\rho} \in \tilde{\mathcal{R}}$ . We work in exponentially weighted spaces, and for an unbounded interval  $J \subset J_-$  and  $\eta \in \mathbb{R}$ , we let  $C_\eta(J; X)$  be the space defined by

$$C_\eta(J; X) := \{U \in C(J; X); \|U\|_{C_\eta} := \sup_{s \in J} e^{\eta s} \|U(s)\|_X < \infty\}.$$

Hence  $C_0(J, X)$  is the space of continuous functions with an  $X$ -norm that is uniformly bounded in  $J$ .

**Lemma 6.** Let  $J \subset J_-$  and pick  $u \in C_0(J; H^2(S^1))$  and  $\rho \in L^2(J; H^{1/2}(S^1))$ , then  $\rho u \in L^2(J; H^{1/2}(S^1))$ .

*Proof.* We need to prove that for  $s$  fixed,

$$\|\rho(s)u(s)\|_{H^{1/2}(S^1)} \leq C \|u(s)\|_{H^2(S^1)} \|\rho(s)\|_{H^{1/2}(S^1)}. \quad (20)$$

Indeed, if this is proved, the claim follows, since

$$\begin{aligned} \|\rho u\|_{L^2(J; H^{1/2}(S^1))}^2 &= \int_J \|\rho(s)u(s)\|_{H^{1/2}(S^1)}^2 ds \\ &\leq C^2 \int_J \|\rho(s)\|_{H^{1/2}(S^1)}^2 \|u(s)\|_{H^2(S^1)}^2 ds \\ &\leq C^2 \sup_{s \in J} \|u(s)\|_{H^2(S^1)}^2 \int_J \|\rho(s)\|_{H^{1/2}(S^1)}^2 ds \\ &= C^2 \|u\|_{C_0(J; H^2(S^1))}^2 \|\rho\|_{L^2(J; H^{1/2}(S^1))}^2. \end{aligned}$$

To prove (20), let  $u \in H^2(S^1)$  and  $\rho \in H^{1/2}(S^1)$  (we suppress the variable  $s$  for simplicity of notation). Let  $\hat{\rho}_k$  and  $\hat{u}_k$  be the Fourier coefficients of  $\rho$  and  $u$ , respectively. We have

$$\begin{aligned} \|u\|_{H^2}^2 &= \sum_{k \in \mathbb{Z}} \hat{u}_k^2 (1 + k^2)^2, \\ \|\rho\|_{H^{1/2}}^2 &= \sum_{k \in \mathbb{Z}} \hat{\rho}_k^2 (1 + k^2)^{1/2}. \end{aligned}$$

Then  $(\widehat{u\rho})_k = \sum_{j \in \mathbb{Z}} \hat{u}_j \hat{\rho}_{k-j}$ , and so  $\|u\rho\|_{H^{1/2}}^2 = \sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \hat{u}_j \hat{\rho}_{k-j} \right)^2 (1 + k^2)^{1/2}$ . Let  $v$  and  $\sigma$  be the functions with Fourier coefficients  $\hat{u}_k (1 + k^2)^{1/4}$  and  $\hat{\rho}_k (1 + k^2)^{1/4}$ , respectively. Note that  $v \in H^{3/2}(S^1)$  and  $\sigma \in L^2(S^1)$ .

Now observe that

$$1 + k^2 = 1 + ((k - j) + j)^2 \leq 2(1 + j^2) + 2(1 + (k - j)^2),$$

and hence

$$(1 + k^2)^{1/4} \leq 2^{1/4} ((1 + j^2)^{1/4} + (1 + (k - j)^2)^{1/4})$$

for any  $j \in \mathbb{Z}$ . Thus

$$\begin{aligned} \|u\rho\|_{H^{1/2}}^2 &\leq \sqrt{2} \sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \hat{u}_j \hat{\rho}_{k-j} (1 + j^2)^{1/4} + \sum_{j \in \mathbb{Z}} \hat{u}_j \hat{\rho}_{k-j} (1 + (k - j)^2)^{1/4} \right)^2 \\ &\leq 2\sqrt{2} \sum_{k \in \mathbb{Z}} \left( \left( \sum_{j \in \mathbb{Z}} \hat{u}_j \hat{\rho}_{k-j} (1 + j^2)^{1/4} \right)^2 + \left( \sum_{j \in \mathbb{Z}} \hat{u}_j \hat{\rho}_{k-j} (1 + (k - j)^2)^{1/4} \right)^2 \right) \\ &= 2\sqrt{2} (\|v\rho\|_{L^2}^2 + \|u\sigma\|_{L^2}^2) \\ &\leq 2\sqrt{2} \left( \sup_{\varphi \in S^1} |v(\varphi)|^2 \|\rho\|_{L^2}^2 + \sup_{\varphi \in S^1} |u(\varphi)|^2 \|\sigma\|_{L^2}^2 \right) \\ &\leq C^2 (\|v\|_{H^{3/2}}^2 \|\rho\|_{L^2}^2 + \|u\|_{H^2}^2 \|\sigma\|_{L^2}^2) \\ &\leq C^2 (\|u\|_{H^2}^2 \|\rho\|_{H^{1/2}}^2) \end{aligned}$$

for some constant  $C > 0$ . This completes the proof.  $\square$

For each  $\tilde{\rho} \in \tilde{\mathcal{R}}$  and  $s \in J_-$ , let

$$\delta B(s; \tilde{\rho}) := B(s; \lambda, \tilde{\rho}) - B(s; \lambda, 0) = -e^{2s} \tilde{\rho}(s) B_0,$$

where  $B_0$  has been defined in (19). Note that for any  $s \in J_-$ ,

$$\begin{aligned} \|\delta B(s; \tilde{\rho})\|_{\mathcal{L}(X)} &\leq \sup_{\substack{u_1 \in H^2(S^1) \\ \|u_1\|_{H^2}=1}} \|e^{2s} \tilde{\rho}(s) u_1\|_{L^2(S^1)} \leq C \sup_{\substack{u_1 \in H^2(S^1) \\ \|u_1\|_{H^2}=1}} \sup_{\varphi \in S^1} |u_1(\varphi)| \|e^{2s} \tilde{\rho}(s)\|_{L^2(S^1)} \\ &\leq C e^{2s} \|\tilde{\rho}(s)\|_{L^2(S^1)}, \end{aligned} \quad (21)$$

where we use the notation  $C$  for the different constants occurring. It follows that

$$\begin{aligned} \int_{-\infty}^{s_1} \|\delta B(s; \tilde{\rho})\|_{\mathcal{L}(X)}^2 ds &\leq C^2 \int_{-\infty}^{s_1} e^{2s} \|\tilde{\rho}(s)\|_{L^2(S^1)}^2 e^{2s} ds \leq C^2 e^{2s_1} \int_{-\infty}^{s_1} \|\tilde{\rho}(s)\|_{L^2(S^1)}^2 e^{2s} ds \\ &\leq C^2 e^{2s_1} \|\tilde{\rho}\|_{\tilde{\mathcal{R}}}^2. \end{aligned} \quad (22)$$

**Lemma 7.** For  $\eta \in (-1, \kappa^{cu})$ , where  $\kappa^{cu}$  is as in Lemma 5, pick  $U_-^{cu} \in C_\eta(J_-; X)$ , and  $\tilde{\rho} \in \tilde{\mathcal{R}}$ . Let  $s \in J_-$ . Then the integral

$$I := \int_{-\infty}^s A_- e^{A_- P^s (s-\tau)} P^s \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau$$

belongs to  $X$ .

*Proof.* Let  $H(\tau) := e^{(\eta-1)\tau} \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau)$ . By the definition of  $\delta B(\tau, \tilde{\rho})$ ,

$$H(\tau) = (0, 0, 0, -e^{(\eta+1)\tau} \tilde{\rho}(\tau) u(\tau))^T =: (0, 0, 0, h(\tau))^T,$$

where  $u(\tau)$  is the first component of  $U_-^{cu}(\tau)$ . Then  $e^\eta u \in C_0(J_-; H^2(S^1))$  and  $e \tilde{\rho} \in L^2(J_-; H^{1/2}(S^1))$ , and so by Lemma 6,  $h \in L^2(J_-; H^{1/2}(S^1))$ . For  $k \in \mathbb{Z}$ , let  $H_k(\tau)$  and  $h_k(\tau)$  be the Fourier coefficients of  $H(\tau)$  and  $h(\tau)$ , respectively. Let  $\tilde{P}_k^s := M_k^{-1} P_k^s M_k$ . To show that  $I$  exists in  $X$ , it suffices to show that  $\{I_k\}_{k \in \mathbb{Z}} \in \hat{X}$  (see (9)), where

$$\begin{aligned} I_k &:= \int_{-\infty}^s e^{(1-\eta)\tau} M_k D_k e^{D_k \tilde{P}_k^s (s-\tau)} M_k^{-1} P^s H_k(\tau) d\tau \\ &= \frac{1}{2} \int_{-\infty}^s e^{(1-\eta)\tau} e^{-|k|(s-\tau)} h_k(\tau) d\tau (0, 0, 1, -|k|)^T. \end{aligned}$$

We therefore need to prove that

$$\left\{ |k| \int_{-\infty}^s e^{(1-\eta)\tau} e^{-|k|(s-\tau)} h_k(\tau) d\tau \right\}_{k \in \mathbb{Z}} \in l^2.$$

Using that  $\eta < 0$  and  $h_k \in L^2(J_-)$  (as  $h \in L^2(J_-; H^{1/2}(S^1))$ ), we note that

$$\begin{aligned} |k| \left| \int_{-\infty}^s e^{(1-\eta)\tau} e^{-|k|(s-\tau)} h_k(\tau) d\tau \right| &\leq |k| e^{-|k|s} \left( \int_{-\infty}^s e^{2(1-\eta+|k|)\tau} d\tau \right)^{1/2} \left( \int_{-\infty}^s h_k(\tau)^2 d\tau \right)^{1/2} \\ &= |k| \frac{1}{\sqrt{2(1-\eta+|k|)}} e^{(1-\eta)s} \|h_k\|_{L^2((-\infty, s])} \\ &\leq e^{(1-\eta)s_1} (1+|k|^2)^{1/4} \|h_k\|_{L^2(J_-)}. \end{aligned}$$

Since  $\{(1+|k|^2)^{1/4} \|h_k\|_{L^2(J_-)}\}_{k \in \mathbb{Z}} \in l^2$ , the proof is complete.  $\square$

**Lemma 8.** For  $-1 < \eta < \kappa^{cu}$ , where  $\kappa^{cu} < 0$  is as in Lemma 5, pick  $U_-^{cu} \in C_\eta(J_-; X)$  and  $\tilde{\rho} \in \tilde{\mathcal{R}}$ . Then the integrals

$$\int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau \quad \text{and} \quad \int_{-\infty}^s B(s; \lambda, 0) \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau$$

exist in  $X$  for each  $s \in J_-$ .

*Proof.* We use (21) and compute

$$\begin{aligned}
& \left\| \int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau \right\|_X \\
& \leq C \int_{-\infty}^s \|\Phi_-^s(s, \tau; \lambda, 0)\|_{\mathcal{L}(X)} e^{2\tau} \|\tilde{\rho}(\tau)\|_{L^2(S^1)} \|U_-^{cu}(\tau)\|_X d\tau \\
& \leq C \int_{-\infty}^s e^{2\tau} e^{\kappa^s(s-\tau)} \|\tilde{\rho}(\tau)\|_{L^2(S^1)} \|U_-^{cu}(\tau)\|_X d\tau \\
& \leq C \|U_-^{cu}\|_{C_\eta(J_-; X)} e^{\kappa^s s} \left( \int_{-\infty}^s e^{2(1-\kappa^s-\eta)\tau} d\tau \right)^{1/2} \left( \int_{-\infty}^s e^{2\tau} \|\tilde{\rho}(\tau)\|_{L^2(S^1)}^2 d\tau \right)^{1/2} \\
& \leq C \|U_-^{cu}\|_{C_\eta(J_-; X)} \frac{1}{\sqrt{2(1-\kappa^s-\eta)}} e^{(1-\eta)s} \|\tilde{\rho}\|_{\tilde{\mathcal{R}}}.
\end{aligned}$$

Using that  $B(s; \lambda, 0) \in \mathcal{L}(X)$ , it follows that both integrals converge in  $X$ .  $\square$

**Lemma 9.** Let  $-1 < \eta < \kappa^{cu}$ , where  $\kappa^{cu} < 0$  is as in Lemma 5. Let  $U_-^{cu} \in C_\eta(J_-; X)$  and  $\tilde{\rho} \in \tilde{\mathcal{R}}$ . For every  $s \in J_-$ , the integral

$$\int_{-\infty}^s A_- \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau \quad (23)$$

exists in  $X$ .

*Proof.* By (3.1) of [14], for  $\tau \leq s \leq s_1$ ,

$$\begin{aligned}
\Phi_-^s(s, \tau; \lambda, 0) &= e^{A_- P^s(s-\tau)} P^s - \int_{-\infty}^{\tau} e^{A_- P^s(s-\xi)} P^s B(\xi; \lambda, 0) \Phi_-^{cu}(\xi, \tau; \lambda, 0) d\xi \\
&+ \int_{\tau}^s e^{A_- P^s(s-\xi)} P^s B(\xi; \lambda, 0) \Phi_-^s(\xi, \tau; \lambda, 0) d\xi \\
&- \int_s^{s_1} e^{A_- P^{cu}(s-\xi)} P^{cu} B(\xi; \lambda, 0) \Phi_-^s(\xi, \tau; \lambda, 0) d\xi.
\end{aligned}$$

Note that we used here that  $\text{Ran } \Phi_-^s(s_1, s_1; \lambda, 0)$  has been chosen so that it coincides with  $\text{Ran } P^s$ .

Substituting this into (23), we have four integrals to estimate, the first of which was dealt with in Lemma 7. The other three integrals are

$$\begin{aligned}
I_1 &:= \int_{-\infty}^s A_- \int_{-\infty}^{\tau} e^{A_- P^s(s-\xi)} P^s B(\xi; \lambda, 0) \Phi_-^{cu}(\xi, \tau; \lambda, 0) d\xi \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau, \\
I_2 &:= \int_{-\infty}^s A_- \int_{\tau}^s e^{A_- P^s(s-\xi)} P^s B(\xi; \lambda, 0) \Phi_-^s(\xi, \tau; \lambda, 0) d\xi \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau, \\
I_3 &:= \int_{-\infty}^s A_- \int_s^{s_1} e^{A_- P^{cu}(s-\xi)} P^{cu} B(\xi; \lambda, 0) \Phi_-^s(\xi, \tau; \lambda, 0) d\xi \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) d\tau.
\end{aligned}$$

We carry out the calculations for  $I_1$ , since the others are similar. Let  $(\phi_{jl}(\xi, \tau))$ ,  $j, l = 1, \dots, 4$ , be the entries of the matrix corresponding to  $\Phi_-^{cu}(\xi, \tau; \lambda, 0)$ , and as in the proof of Lemma 7, let  $h(\tau) = -e^{(\eta+1)\tau} \tilde{\rho}(\tau) u(\tau)$ . Recall that  $h \in L^2(J_-; H^{1/2}(S^1))$ . A short calculation shows that

$$B(\xi; \lambda, 0) \Phi_-^{cu}(\xi, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) U_-^{cu}(\tau) e^{2\xi+(1-\eta)\tau} \begin{pmatrix} 0 \\ \phi_{34}(\xi, \tau) h(\tau) \\ 0 \\ (\lambda - \tilde{\theta}(\tau)) \phi_{14}(\xi, \tau) h(\tau) \end{pmatrix}.$$

Note that  $\phi_{34}(\xi, \tau)$  and  $\phi_{14}(\xi, \tau)$  map  $L^2(S^1)$  boundedly into  $H^1(S^1)$  and  $H^2(S^1)$ , respectively, and that by Lemma 5 for  $\xi \leq \tau \leq s_1$

$$\begin{aligned}
\|\phi_{34}(\xi, \tau)\|_{\mathcal{L}(L^2; H^1)} &\leq K e^{\kappa^{cu}(\xi-\tau)}, \\
\|\phi_{14}(\xi, \tau)\|_{\mathcal{L}(L^2; H^2)} &\leq K e^{\kappa^{cu}(\xi-\tau)}.
\end{aligned}$$

Introducing the notation  $f(\xi, \tau) := e^{-\kappa^{cu}(\xi-\tau)}\phi_{34}(\xi, \tau)h(\tau)$ , and  $g(\xi, \tau) := e^{-\kappa^{cu}(\xi-\tau)}(\lambda - \tilde{\theta}(\tau))\phi_{14}(\xi, \tau)h(\tau)$ , we note that  $\max(\|f(\xi, \tau)\|_{H^1}, \|g(\xi, \tau)\|_{H^2}) \leq K\|h(\tau)\|_{L^2}$ , for  $\xi < \tau < s_1$ . The Fourier coefficients of  $f(\xi, \tau)$  and  $g(\xi, \tau)$  are denoted by  $\widehat{f}_k(\xi, \tau)$  and  $\widehat{g}_k(\xi, \tau)$ , respectively.

To prove that  $I_1 \in X$ , it suffices to prove that  $\{J_k\}_{k \in \mathbb{Z}} \in \widehat{X}$ , where

$$\begin{aligned} J_k &:= \int_{-\infty}^s M_k D_k \int_{-\infty}^{\tau} e^{2\xi+(1-\eta)\tau} e^{\kappa^{cu}(\xi-\tau)} e^{D_k \tilde{P}_k^s(s-\xi)} M_k^{-1} P^s \begin{pmatrix} 0 \\ \widehat{f}_k(\xi, \tau) \\ 0 \\ \widehat{g}_k(\xi, \tau) \end{pmatrix} d\xi d\tau \\ &= \frac{1}{2} \int_{-\infty}^s \int_{-\infty}^{\tau} e^{2\xi+(1-\eta)\tau} e^{\kappa^{cu}(\xi-\tau)} e^{-|k|(s-\xi)} \begin{pmatrix} \widehat{f}_k(\xi, \tau) \\ -|k|\widehat{f}_k(\xi, \tau) \\ \widehat{g}_k(\xi, \tau) \\ -|k|\widehat{g}_k(\xi, \tau) \end{pmatrix} d\xi d\tau, \end{aligned} \tag{24}$$

where  $\tilde{P}_k^s = M_k^{-1} P_k^s M_k$  as before. The first component of  $J_k$  can be written

$$\begin{aligned} &\frac{1}{2} e^{-|k|s} \int_{-\infty}^s e^{(1-\kappa^{cu}-\eta)\tau} \int_{-\infty}^{\tau} e^{(1+\kappa^{cu}+|k|)\xi} e^{\xi} \widehat{f}_k(\xi, \tau) d\xi d\tau \\ &\leq \frac{1}{2} e^{-|k|s} \int_{-\infty}^s \frac{1}{\sqrt{2(1+\kappa^{cu}+|k|)}} e^{(2+|k|-\eta)\tau} \left( \int_{-\infty}^{\tau} e^{2\xi} \widehat{f}_k(\xi, \tau)^2 d\xi \right)^{1/2} d\tau \\ &\leq \frac{1}{4} \frac{e^{(2-\eta)s}}{\sqrt{(1+\kappa^{cu}+|k|)(2+|k|-\eta)}} \left( \int_{-\infty}^s \int_{-\infty}^{\tau} e^{2\xi} \widehat{f}_k(\xi, \tau)^2 d\xi d\tau \right)^{1/2}. \end{aligned}$$

The square of the  $l_2^2$  norm of the first component of  $\{J_k\}_{k \in \mathbb{Z}}$  can then be estimated by

$$\begin{aligned} &\frac{e^{2(2-\eta)s}}{16} \sum_{k \in \mathbb{Z}} \frac{1+k^2}{(1+\kappa^{cu}+|k|)(2+|k|-\eta)} \int_{-\infty}^s \int_{-\infty}^{\tau} (1+k^2) e^{2\xi} \widehat{f}_k(\xi, \tau)^2 d\xi d\tau \\ &\leq \frac{e^{2(2-\eta)s}}{16(1+\kappa^{cu})} \sum_{k \in \mathbb{Z}} \int_{-\infty}^s \int_{-\infty}^{\tau} e^{2\xi} \widehat{h}_k(\tau)^2 d\xi d\tau \\ &= \frac{e^{2(2-\eta)s}}{32(1+\kappa^{cu})} \sum_{k \in \mathbb{Z}} \int_{-\infty}^s e^{2\tau} \widehat{h}_k(\tau)^2 d\tau \\ &\leq \frac{e^{2(3-\eta)s_1}}{32(1+\kappa^{cu})} \|h\|_{L^2(J_-; L^2(S^1))}^2. \end{aligned}$$

The other three components of  $\{J_k\}_{k \in \mathbb{Z}}$  are estimated in a completely similar way, and by adding these estimates we see that  $I_1 \in X$ .  $\square$

We are now ready to prove the existence of exponential dichotomies for the full system.

**Theorem 2.** *Let  $-1 < \kappa^s < \kappa^{cu} < 0$  and  $0 < \kappa^{cs} < \kappa^u < 1$ . Then there exists a neighbourhood  $\mathcal{U}$  of 0 in  $\tilde{\mathcal{R}}$  such that for any  $\tilde{\rho} \in \mathcal{U}$  and any  $\lambda \in \mathbb{R}$ , the system (5) has an exponential dichotomy on  $J_-$  with constants  $K$  and rates  $\kappa^{cu}$ ,  $\kappa^s$ , and another with constants  $K$  and rates  $\kappa^u$ ,  $\kappa^{cs}$ . Moreover, the projections and evolution operators depend smoothly on  $\lambda \in \mathbb{R}$  and  $\tilde{\rho} \in \mathcal{U}$ . The dichotomies are denoted by  $\Phi_-^s(s, t; \lambda, \tilde{\rho})$ ,  $\Phi_-^{cu}(s, t; \lambda, \tilde{\rho})$ , and  $\Phi_-^{cs}(s, t; \lambda, \tilde{\rho})$ ,  $\Phi_-^u(s, t; \lambda, \tilde{\rho})$ , respectively. The associated projections will be denoted by  $P_-^s(s; \lambda, \tilde{\rho}) := \Phi_-^s(s, s; \lambda, \tilde{\rho})$ ,  $P_-^{cu}(s; \lambda, \tilde{\rho})$ ,  $P_-^{cs}(s; \lambda, \tilde{\rho})$ , and  $P_-^u(s; \lambda, \tilde{\rho})$ , respectively.*

*Proof.* We will show that there exists a neighbourhood of 0 in  $\tilde{\mathcal{R}}$  such that if  $\tilde{\rho}$  belongs to this neighbourhood then there exist exponential dichotomies for the system (11) with this  $\tilde{\rho}$ . Let  $U_0 \in X$  and  $t \in J_-$  be fixed but arbitrary. We will use the implicit function theorem to solve the system of integral equations for the pair of

functions  $(U_-^{cu}, U_-^s)$  as functions of the parameters  $\lambda \in \mathbb{R}$  and  $\tilde{\rho} \in \tilde{\mathcal{R}}$  near 0

$$\begin{aligned}
0 &= \Phi_-^{cu}(s, t; \lambda, 0)U_0 - U_-^{cu}(s) + \int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0)\delta B(\tau; \tilde{\rho})U_-^{cu}(\tau) d\tau \\
&\quad - \int_s^t \Phi_-^{cu}(s, \tau; \lambda, 0)\delta B(\tau; \tilde{\rho})U_-^{cu}(\tau) d\tau \\
&\quad + \int_t^{s_1} \Phi_-^{cu}(s, \tau; \lambda, 0)\delta B(\tau; \tilde{\rho})U_-^s(\tau) d\tau, \quad \text{for } s \leq t \leq s_1, \\
0 &= \Phi_-^s(s, t; \lambda, 0)U_0 - U_-^s(s) - \int_{-\infty}^t \Phi_-^s(s, \tau; \lambda, 0)\delta B(\tau; \tilde{\rho})U_-^{cu}(\tau) d\tau \\
&\quad + \int_t^s \Phi_-^s(s, \tau; \lambda, 0)\delta B(\tau; \tilde{\rho})U_-^s(\tau) d\tau \\
&\quad - \int_s^{s_1} \Phi_-^{cu}(s, \tau; \lambda, 0)\delta B(\tau; \tilde{\rho}, 0)U_-^s(\tau) d\tau, \quad \text{for } t \leq s \leq s_1.
\end{aligned} \tag{25}$$

By Lemma 5, the dichotomies  $\Phi_-^{cu}(s, t; \lambda, 0)U_0$  and  $\Phi_-^s(s, t; \lambda, 0)U_0$  exist and have constants  $K$ ,  $\tilde{\kappa}^s = -1$  and  $\tilde{\kappa}^{cu} \in (-1, 0)$ .

Let  $\eta \in (\kappa^s, \kappa^{cu})$  and rewrite equation (25) as  $F(U_-^{cu}, U_-^s; \lambda, \tilde{\rho}) = 0$ , where  $F : C_\eta((-\infty, t]; X) \times C_\eta([t, s_1]; X) \times \mathbb{R} \times \tilde{\mathcal{R}} \rightarrow C_\eta((-\infty, t]; X) \times C_\eta([t, s_1]; X)$  is the right hand side of (25).

We first verify that  $F$  is indeed a map between the above spaces. We do the estimates for the first integral in the first equation of (25). The other estimates are similar. Lemma 5 gives that for any  $s \in (-\infty, t]$  and  $U_-^{cu} \in C_\eta((-\infty, t]; X)$ :

$$\begin{aligned}
&e^{\eta s} \left\| \int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0)\delta B(\tau; \tilde{\rho})U_-^{cu}(\tau) d\tau \right\|_X \\
&\leq K \sup_{\tau \in (-\infty, s]} (e^{\eta \tau} \|U_-^{cu}(\tau)\|_X) \int_{-\infty}^s e^{(\kappa^s - \eta)(s - \tau)} \|\delta B(\tau; \tilde{\rho})\|_{\mathcal{L}(X)} d\tau \\
&\leq K \|U_-^{cu}\|_{C_\eta((-\infty, t], X)} \int_{-\infty}^s \left( e^{2(\kappa^s - \eta)(s - \tau)} + \|\delta B(\tau; \tilde{\rho})\|_{\mathcal{L}(X)}^2 \right) d\tau \\
&\leq K \|U_-^{cu}\|_{C_\eta((-\infty, t], X)} \left( \frac{1}{2(\eta - \kappa^s)} + e^{2s_1} \|\tilde{\rho}\|_{\tilde{\mathcal{R}}}^2 \right),
\end{aligned} \tag{26}$$

where we have used (22). After taking the supremum over all  $s \in (-\infty, s_1]$  we see that the function defined by the first integral in (25) belongs to  $C_\eta(J_-, X)$ . Using similar estimates for the other integrals, we can conclude that  $F$  is indeed a map between the spaces as stated.

That  $F$  is smooth with respect to  $\lambda$  and  $\tilde{\rho}$  follows since the evolution operators  $\Phi_-^{cu}(\cdot, t; \lambda, 0)U_0$  and  $\Phi_-^s(\cdot, t; \lambda, 0)U_0$  are smooth in  $\lambda$  by Lemma 5 (using that the  $H^1$  norm is weaker than the  $C^1$  norm on bounded intervals), and since  $\delta B$  depends smoothly on  $\tilde{\rho}$  (indeed,  $\delta B$  is a bounded linear mapping with respect to  $\tilde{\rho}$ ). Note that

$$F(\Phi_-^{cu}(\cdot, t; \lambda, 0)U_0, \Phi_-^s(\cdot, t; \lambda, 0)U_0; \lambda, 0) = 0.$$

The Fréchet derivative of  $F$  with respect to its two first variables evaluated at  $(\Phi_-^{cu}(\cdot, t; \lambda, 0)U_0, \Phi_-^s(\cdot, t; \lambda, 0)U_0; \lambda, 0)$  is  $-I$  on  $C_\eta((-\infty, t]; X) \times C_\eta([t, s_1]; X)$ . In particular, this derivative is a linear homeomorphism on this space, and so the implicit function theorem is applicable, and we obtain solutions  $\Phi_-^{cu}(\cdot, t; \lambda, \tilde{\rho})U_0 := U_-^{cu}$  and  $\Phi_-^s(\cdot, t; \lambda, \tilde{\rho})U_0 := U_-^s$  of the integral equation (25), which exist in a neighbourhood of  $(\lambda_0, 0)$  in  $\mathbb{R} \times \tilde{\mathcal{R}}$ . Smoothness of these solutions with respect to parameters also follows from a corollary of the implicit function theorem (see e.g. [4, p.115]).

Next, we need to verify that  $\Phi_-^{cu}(\cdot, t; \lambda_0, \tilde{\rho})U_0$  and  $\Phi_-^s(\cdot, t; \lambda_0, \tilde{\rho})U_0$  are weak solutions of (11), and that they satisfy the conditions of Definition 2. We first check that  $\Phi_-^{cu}(\cdot, t; \lambda, \tilde{\rho})U_0$  is a weak solution on the interval  $(-\infty, t]$ . By Lemma 5,  $\Phi_-^{cu}(\cdot, t; \lambda, 0)U_0$  is a  $C^\infty$  solution of

$$U' = (A_- + B(s; \lambda, 0))U$$



on  $(-\infty, t]$ , and hence it is also a weak solution of this equation. Next we deal with the integral terms. For the first integral we use the abbreviation

$$g(s) := \int_{-\infty}^s f(s, \tau) d\tau, \quad \text{with } f(s, \tau) = \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) \Phi_-^{cu}(\tau, t; \lambda_0, \tilde{\rho}) U_0.$$

Thus  $f$  is  $C^\infty$  in the first variable and  $L^1$  in the second. From its definition, it follows immediately that  $g$  is continuous. We will see that  $g$  is weakly differentiable and that

$$g'(s) = f(s, s) + \int_{-\infty}^s \frac{\partial f}{\partial s}(s, \tau) d\tau. \quad (27)$$

In order to prove this, we need to check that the integral on the right hand side of (27) exists, and that the equality (27) holds. The integral in the right hand side of (27) is

$$\int_{-\infty}^s \frac{\partial f}{\partial s}(s, \tau) d\tau = \int_{-\infty}^s (A_- + B(s; \lambda, 0)) \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) \Phi_-^{cu}(\tau, t; \lambda, \tilde{\rho}) U_0 d\tau,$$

and it exists in  $X$  by Lemma 8 and Lemma 9.

Next, we calculate the distributional derivative of  $g$  and let  $V \in C_0^\infty((-\infty, t]; X)$  be a test function. Then by Fubini's Theorem and integration by parts

$$\begin{aligned} \int_{-\infty}^t g'(s) V(s) ds &= - \int_{-\infty}^t g(s) V'(s) ds \\ &= - \int_{-\infty}^t \int_{-\infty}^s f(s, \tau) d\tau V'(s) ds \\ &= - \int_{-\infty}^t \int_{\tau}^t f(s, \tau) V'(s) ds d\tau \\ &= \int_{-\infty}^t \left( f(\tau, \tau) V(\tau) + \int_{\tau}^t \frac{\partial f}{\partial s}(s, \tau) V(s) ds \right) d\tau \\ &= \int_{-\infty}^t \left( f(s, s) + \int_{-\infty}^s \frac{\partial f}{\partial s}(s, \tau) d\tau \right) V(s) ds, \end{aligned}$$

and we see that the weak derivative of  $g$  is indeed given by (27). Hence

$$\begin{aligned} \frac{d}{ds} \int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) \Phi_-^{cu}(\tau, t; \lambda, \tilde{\rho}) U_0 d\tau \\ = (I - P_-^{cu}(s; \lambda, 0)) \delta B(s; \tilde{\rho}) \Phi_-^{cu}(s, t; \lambda, \tilde{\rho}) U_0 \\ + \int_{-\infty}^s (A_- + B(s; \lambda, 0)) \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) \Phi_-^{cu}(\tau, t; \lambda, \tilde{\rho}) U_0 d\tau. \end{aligned} \quad (28)$$

We have already noticed that  $g(s) = \int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) \Phi_-^{cu}(\tau, t; \lambda, \tilde{\rho}) U_0 d\tau$  is continuous, and so it belongs to  $L_{loc}^2((-\infty, t]; X)$ . The right hand side of (28) also belongs to  $L_{loc}^2((-\infty, t]; X)$  since the first term belongs to  $L_{loc}^2((-\infty, t]; X)$  and the second term is continuous on  $(-\infty, t]$ . This shows that

$$\int_{-\infty}^s \Phi_-^s(s, \tau; \lambda, 0) \delta B(\tau; \tilde{\rho}) \Phi_-^{cu}(\tau, t; \lambda, \tilde{\rho}) U_0 d\tau$$

belongs to  $H_{loc}^1((-\infty, t]; X)$ .

Similar calculations for the other integral terms of the first equation of (25) show that these are also weakly differentiable on  $(-\infty, t]$  and belong to  $H_{loc}^1((-\infty, t], X)$ . After adding the terms up, we conclude that

$$\frac{d}{ds} \Phi_-^{cu}(s, t; \lambda, \tilde{\rho}) U_0 = (A_- + B(s; \lambda, \tilde{\rho})) \Phi_-^{cu}(s, t; \lambda, \tilde{\rho}) U_0,$$

i.e.  $\Phi_-^{cu}(\cdot, t; \lambda, \tilde{\rho}) U_0$  is a weak solution of (11).

Similar calculations for the terms of the second equation of (25) show that  $\Phi_-^s(\cdot, t; \lambda, \tilde{\rho}) U_0$  is a weak solution of (11) on the interval  $[t, s_1]$ .

Finally we check that the conditions of Definition 2 are satisfied. A similar computation as in (26) also shows that the estimates in (i) and (ii) of Definition 2 are satisfied for  $\Phi_-^{cu}(s, t; \lambda, \rho)$  and  $\Phi_-^s(s, t; \lambda, \tilde{\rho})$  for any  $\kappa^{cu}$  and  $\kappa^s$  such that  $-1 = \tilde{\kappa}^s < \kappa^s < \kappa^{cu} < \tilde{\kappa}^{cu} < 0$ . Since  $\tilde{\kappa}^{cu}$  can be taken arbitrarily close to 1, the same is true also for  $\kappa^{cu}$ . Note that (iii) of Definition 2 is satisfied with  $P_-^{cu}(s; \lambda, \tilde{\rho}) := \Phi_-^{cu}(s, s; \lambda, \tilde{\rho})$  and  $P_-^s(s; \lambda, \tilde{\rho}) := \Phi_-^s(s, s; \lambda, \tilde{\rho})$ .  $\square$

To finish this section, we derive some more details about the solutions of (5) in the case when  $\lambda = \lambda_0$  and  $\tilde{\rho} = 0$ . We are particularly interested in the solutions on  $J_-$ , and we study the exact growth/decay rate of solutions as  $s \rightarrow -\infty$ . As we have seen before, the space  $X$  decouples into a direct sum of 4-dimensional pairwise orthogonal Fourier subspaces  $X_k$ , and that since  $\theta$  is radially symmetric, the subspaces  $X_k$  are invariant both under the flow of (5) with  $\tilde{\rho} = 0$  and under the flow of the asymptotic system (8).

**Lemma 10.** *Let  $e_j$ ,  $j = 1, \dots, 4$  be the standard basis of  $\mathbb{C}^4$  and consider the four-dimensional invariant central space corresponding to  $k = 0$  of the unperturbed equation obtained when  $\tilde{\rho} = 0$  and  $\lambda = \lambda_0$  in (5). Then there exist two unique solutions  $U_{0,j}(s)$  with  $j = 1, 3$  such that*

$$\lim_{s \rightarrow -\infty} U_{0,j}(s) = e_j, \quad j = 1, 3.$$

We may also pick two solutions  $U_{0,j}$  with  $j = 2, 4$  which grow algebraically as  $s \rightarrow -\infty$  and satisfy

$$\lim_{s \rightarrow -\infty} \frac{1}{s} U_{0,j}(s) = e_{j-1}, \quad j = 2, 4.$$

The solutions  $U_{0,j}$ ,  $j = 1, \dots, 4$ , are linearly independent.

*Proof.* It is straightforward to check the assertions of the lemma, using [5, Chapter 3.8].  $\square$

In Section 6 we will specify the solutions  $U_{0,2}$  and  $U_{0,4}$  using the adjoint system.

**Lemma 11.** *For every  $k \in \mathbb{Z} \setminus \{0\}$ , there exist solutions  $U_{k,j}$  of (5) with  $\tilde{\rho} = 0$  and  $\lambda = \lambda_0$  such that (together with the solutions specified in Lemma 10 for  $k = 0$ ) we have*

$$\text{span}\{U_{k,j}(s_1); k \in \mathbb{Z}, j = 1, \dots, 4\} = X,$$

and for  $s \rightarrow -\infty$ ,

$$\begin{aligned} e^{|k|(s-s_1)} U_{k,1}(s) &\rightarrow (-1/k^2, 1/|k|, 0, 0)^T e^{ik}, \\ e^{|k|(s-s_1)} U_{k,2}(s) &\rightarrow (0, 0, -1/|k|, 1)^T e^{ik}, \\ e^{-|k|(s-s_1)} U_{k,3}(s) &\rightarrow (1/k^2, 1/|k|, 0, 0)^T e^{ik}, \\ e^{-|k|(s-s_1)} U_{k,4}(s) &\rightarrow (0, 0, 1/|k|, 1)^T e^{ik}. \end{aligned}$$

*Proof.* As seen in the beginning of this section, the system (5) with  $\tilde{\rho} = 0$  and  $\lambda = \lambda_0$  leaves the subspaces  $X_k$  invariant. The estimates on the matrix  $B(s; \lambda_0, 0)$  in (16) now show that there are solutions of (14) (and hence of (5)) which converge to the solutions of the system at infinity, see e.g. [5, Chapter 3.8]. In Section 3 we have seen that (8) has two solutions in  $X_k$  with decay rate  $e^{|k|s}$  and two with growth rate  $e^{-|k|s}$  for  $s \rightarrow -\infty$ . A comparison with the eigenvectors of  $\widehat{A}_-(k)$  in Section 3, we obtain solutions  $U_{k,j}$ , with  $k \in \mathbb{Z} \setminus \{0\}$  and  $j = 1, \dots, 4$  with the desired properties.  $\square$

Next, we perturb the solutions  $U_{0,j}(s)$  with  $j = 1, 3$  described in Lemma 10 to solutions of (5) for all sufficiently small potentials  $\tilde{\rho}$ . First, we will show that the four-dimensional central subspace corresponding to  $k = 0$  persists in (5) as the intersection of the ranges of  $\Phi_-^{cs}(s_1, s_1; \lambda, \tilde{\rho})$  and  $\Phi_-^{cu}(s_1, s_1; \lambda, \tilde{\rho})$ . Note that the difference between the operators  $A(s; \lambda, \tilde{\rho})$  and  $A(s; \lambda_0, 0)$  in (5) is

$$A(s; \lambda, \tilde{\rho}) - A(s; \lambda_0, 0) = r'(s)^2 (\lambda - \lambda_0 - \tilde{\rho}) B_0 e^{2s} (\lambda - \lambda_0 - \tilde{\rho}) B_0,$$

as  $r'(s)^2 = e^{2s}$  for  $s \leq s_1$  (see (19) for the definition of  $B_0$ ).

The function  $e^\tau \rho(\tau)$  belongs to  $L^2(J_-, H^{1/2})$ . By Lemma 5,  $e^\tau \rho(\tau) u_1(\tau)$  also belongs to this space. Thus  $\|e^\tau \rho(\tau) B_0 U\|_X \in L^2(J_-)$  and  $\|e^{2\tau} \rho(\tau) B_0 U\|_X$  is the product of an  $L^2$  function and the exponentially decaying function  $e^\tau$ .

This allows us to use the Gap Lemma as in [16, §4.3 and (4.12)] and [3, Proof of Lemma 4.1] to show that (5) has two linearly independent solutions  $U_{0,j}^{cb}(s; \lambda, \tilde{\rho})$  for  $j = 1, 3$  that converge to  $e_j$  as  $s \rightarrow -\infty$ , and two other solutions which grow algebraically. In fact, the results in these works show that any linear combination of the bounded solutions  $U_{0,j}^{cb}(s; \lambda, \tilde{\rho})$  with  $j = 1, 3$  can be found as a fixed point of the equation

$$\begin{aligned} U(s) &= \Phi_-^{cu}(s; \lambda_0, 0) U_0^{cb} + \int_{-\infty}^s \Phi_-^{cs}(s, \tau; \lambda_0, 0) e^{2\tau} (\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 U(\tau) d\tau \\ &\quad - \int_s^{s_1} \Phi_-^u(s, \tau; \lambda_0, 0) e^{2\tau} (\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 U(\tau) d\tau, \end{aligned} \quad (29)$$

where  $U_0^{cb}$  belongs to the unperturbed bounded central subspace spanned by  $U_{0,j}(s_1)$  for  $j = 1, 3$ . We denote the fixed point by  $U_-^{cb}(s; \lambda, \tilde{\rho}, U_0^{cb})$  and write

$$P_-^{cb}(s_1; \lambda, \tilde{\rho}) U_0^{cb} := U_-^{cb}(s_1; \lambda, \tilde{\rho}, U_0^{cb}) = U_0^{cb} + \int_{-\infty}^{s_1} \Phi_-^{cs}(s_1, \tau; \lambda_0, 0) e^{2\tau} (\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 U_-^{cb}(\tau; \lambda, \tilde{\rho}, U_0^{cb}) d\tau. \quad (30)$$

Similarly, we can use (25) to describe the solutions of (5) with exponential decay as  $s \rightarrow -\infty$  by

$$P_-^u(s_1; \lambda, \tilde{\rho}) = P_-^u(s_1; \lambda_0, 0) + \int_{-\infty}^{s_1} \Phi_-^{cs}(s_1, \tau; \lambda_0, 0) e^{2\tau} (\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 \Phi_-^u(\tau, s_1; \lambda, \tilde{\rho}) d\tau. \quad (31)$$

These results will be used later to characterize eigenfunctions of the perturbed operator.

## 5 Dichotomies for the far field

The method of Section 4 is not available for determining the dichotomies for  $s$  large. Going back to (2), we observe that  $\theta$  and  $\rho$  have support in a ball with radius  $r_1$ , and thus for  $r \geq r_1$  the eigenvalue problem (2) reduces to  $(\Delta^2 - \lambda)u = 0$ , which can be factorized:

$$(\Delta - \sqrt{\lambda})(\Delta + \sqrt{\lambda})u = (\Delta + \sqrt{\lambda})(\Delta - \sqrt{\lambda})u = 0.$$

Expanding  $u(r, \varphi)$  as a Fourier series in the angular variable  $\varphi$ , we see that the Fourier coefficients  $\hat{u}_k$  satisfy the differential equations

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} - \sqrt{\lambda} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} + \sqrt{\lambda} \right) \hat{u}_k = 0. \quad (32)$$

For  $k$  fixed, this is a fourth order linear ODE, so it has a four-dimensional space of solutions. The general solution can then be obtained as a linear combination of the solutions of

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} - \sqrt{\lambda} \right) \hat{u}_k = 0 \quad (33)$$

and the solutions of

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} + \sqrt{\lambda} \right) \hat{u}_k = 0, \quad (34)$$

so that the general solution of (32) is given by

$$\hat{u}_k(r) = C_1 I_k(\lambda^{1/4} r) + C_2 K_k(\lambda^{1/4} r) + C_3 J_k(\lambda^{1/4} r) + C_4 Y_k(\lambda^{1/4} r),$$

where  $J_k$  and  $Y_k$  are Bessel functions of the first and second kind, respectively, which satisfy equation (34), and  $I_k$  and  $K_k$  are modified Bessel functions of the first and second kind, respectively, which satisfy equation (33).

For  $r \geq r_3 = s_3$ , we have  $s = r$ . Thus, we can define the systems corresponding to equations (33) and (34) with the variable  $s$  for  $s \geq s_3$  as

$$\begin{aligned} u'_1 &= u_2, \\ u'_2 &= \left( \frac{k^2}{s^2} + \sqrt{\lambda} \right) u_1 - \frac{1}{s} u_2, \end{aligned} \quad (35)$$

and

$$\begin{aligned} u'_1 &= u_2, \\ u'_2 &= \left( \frac{k^2}{s^2} - \sqrt{\lambda} \right) u_1 - \frac{1}{s} u_2, \end{aligned} \quad (36)$$

respectively. We will consider these systems for  $s \geq r_1$  and define  $\phi_k(s, t)$  and  $\psi_k(s, t)$  to be the evolution operators corresponding to the systems (35) and (36), respectively. After deriving dichotomies for those systems, we will derive dichotomies for the original system (5).

To derive dichotomies for (35) and (36), we introduce for  $s \geq r_1$  the function spaces  $\tilde{X}^s = H^1(S^1) \times L^2(S^1)$ ,  $\tilde{Y}^s = H^2(S^1) \times H^1(S^1)$  and  $\tilde{Z}^s = H^3(S^1) \times H^2(S^1)$  with norms

$$\begin{aligned} \|\underline{u}\|_{\tilde{X}^s}^2 &:= \frac{1}{s^2} \|u_1\|_{H^1(S^1)}^2 + \|u_1\|_{L^2(S^1)}^2 + \|u_2\|_{L^2(S^1)}^2, \\ \|\underline{u}\|_{\tilde{Y}^s}^2 &:= \frac{1}{s^2} \|u_1\|_{H^2(S^1)}^2 + \|u_1\|_{H^1(S^1)}^2 + \|u_2\|_{H^1(S^1)}^2, \\ \|\underline{u}\|_{\tilde{Z}^s}^2 &:= \frac{1}{s^2} \|u_1\|_{H^3(S^1)}^2 + \|u_1\|_{H^2(S^1)}^2 + \|u_2\|_{H^2(S^1)}^2. \end{aligned}$$

We decompose the spaces  $\tilde{X}^s$ ,  $\tilde{Y}^s$ , and  $\tilde{Z}^s$  into their Fourier subspaces

$$\tilde{X}^s = \overline{\bigoplus_{k \in \mathbb{Z}} \tilde{X}_k^s}, \quad \tilde{Y}^s = \overline{\bigoplus_{k \in \mathbb{Z}} \tilde{Y}_k^s}, \quad \text{and} \quad \tilde{Z}^s = \overline{\bigoplus_{k \in \mathbb{Z}} \tilde{Z}_k^s}, \quad (37)$$

where

$$\tilde{X}_k^s = \tilde{Y}_k^s = \tilde{Z}_k^s = \{(ae^{ik\cdot}, be^{ik\cdot})^T; a, b \in \mathbb{C}\},$$

and the completion in (37) is in the respective norms of  $\tilde{X}^s$ ,  $\tilde{Y}^s$  and  $\tilde{Z}^s$ . The norms on  $\tilde{X}_k^s$ ,  $\tilde{Y}_k^s$  and  $\tilde{Z}_k^s$  are given by the restriction of the norms of  $\tilde{X}^s$ ,  $\tilde{Y}^s$ , and  $\tilde{Z}^s$  respectively, and so

$$\begin{aligned} \|(ae^{ik\cdot}, be^{ik\cdot})^T\|_{\tilde{X}_k^s}^2 &= \left(1 + \frac{k^2}{s^2}\right) |a|^2 + |b|^2; \\ \|(ae^{ik\cdot}, be^{ik\cdot})^T\|_{\tilde{Y}_k^s}^2 &= (1 + k^2) \|(ae^{ik\cdot}, be^{ik\cdot})^T\|_{\tilde{X}_k^s}^2; \\ \|(ae^{ik\cdot}, be^{ik\cdot})^T\|_{\tilde{Z}_k^s}^2 &= (1 + k^2)^2 \|(ae^{ik\cdot}, be^{ik\cdot})^T\|_{\tilde{X}_k^s}^2. \end{aligned} \quad (38)$$

For each  $\epsilon \in (0, \lambda^{1/4})$ , we now prove the existence of a time-dependent exponential dichotomy for (35) with constant  $K > 0$  and rates  $\kappa^s = -(\lambda^{1/4} - \epsilon)$  and  $\kappa^u = (\lambda^{1/4} - \epsilon)$  that are independent of  $k$ . For (36), we will show that the evolution operator always acts in the center-unstable manifold and derive that its growth can be bounded by any exponential.

**Lemma 12.** *There exists an  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$  there exists a  $K > 0$  such that for any  $k \in \mathbb{Z}$  and  $\lambda \in (\lambda_0/2, 2\lambda_0)$  there exists a time-dependent exponential dichotomy of (35) on  $J_+$  so that  $\phi_k^s(s, t; \lambda)$  and  $\phi_k^u(s, t; \lambda)$  satisfy*

$$\begin{aligned} \|\phi_k^s(s, t; \lambda)\|_{\mathcal{L}(\tilde{X}_k^t, \tilde{X}_k^s)} &= \|\phi_k^s(s, t; \lambda)\|_{\mathcal{L}(\tilde{Y}_k^t, \tilde{Y}_k^s)} \leq Ke^{-(\lambda^{1/4} - \epsilon)(s-t)} & s \geq t \geq r_1, \\ \|\phi_k^u(s, t; \lambda)\|_{\mathcal{L}(\tilde{X}_k^t, \tilde{X}_k^s)} &= \|\phi_k^u(s, t; \lambda)\|_{\mathcal{L}(\tilde{Y}_k^t, \tilde{Y}_k^s)} \leq Ke^{-(\lambda^{1/4} - \epsilon)(t-s)} & t \geq s \geq r_1. \end{aligned}$$

*Proof.* Let  $(u_1, u_2)^T$  satisfy equation (35). To get estimates which are uniform in  $k$ , we follow [18], and let

$$\tilde{u}_1(s) := \left( \sqrt{\lambda} + \frac{k^2}{s^2} \right)^{1/2} u_1(s).$$

Note that

$$\min(1, \sqrt{\lambda_0/2}) \|(u_1, u_2)^T\|_{\tilde{X}_k^s} \leq \|(\tilde{u}_1, u_2)^T\|_{\mathbb{C}^2} \leq \max(1, \sqrt{2\lambda_0}) \|(u_1, u_2)^T\|_{\tilde{X}_k^s},$$

and that the constants above are independent of  $k$  and  $\lambda$ . This shows that, when using the new variables  $\tilde{u}_1$  and  $u_2$ , we can use the standard norm in  $\mathbb{C}^2$ .

Next, we rewrite the system (35) in the new variables  $\tilde{u}_1, u_2$ :

$$\begin{aligned} \tilde{u}_1' &= \left(\sqrt{\lambda} + \frac{k^2}{s^2}\right)^{1/2} u_2 - \frac{k^2}{s^3} \left(\sqrt{\lambda} + \frac{k^2}{s^2}\right)^{-1} \tilde{u}_1, \\ u_2' &= -\frac{1}{s} u_2 + \left(\sqrt{\lambda} + \frac{k^2}{s^2}\right)^{1/2} \tilde{u}_1, \end{aligned}$$

Now, we change the independent variable by making the substitution  $d\tau/ds = (\sqrt{\lambda} + k^2/s^2)^{1/2}$ . We write  $s(\tau)$  to describe the dependence of  $s$  on  $\tau$ . We then obtain (where  $'$  now denotes differentiation with respect to  $\tau$ )

$$\begin{aligned} \tilde{u}_1' &= -\frac{k^2}{s(\tau)^3} \left(\sqrt{\lambda} + \frac{k^2}{s(\tau)^2}\right)^{-3/2} \tilde{u}_1 + u_2, \\ u_2' &= \tilde{u}_1 - \frac{1}{s(\tau)} \left(\sqrt{\lambda} + \frac{k^2}{s(\tau)^2}\right)^{-1/2} u_2. \end{aligned} \tag{39}$$

Noting that  $s(\tau) \rightarrow \infty$  as  $\tau \rightarrow \infty$  we find that the limiting system at  $+\infty$  is

$$\begin{aligned} \tilde{u}_1' &= u_2, \\ u_2' &= \tilde{u}_1, \end{aligned} \tag{40}$$

which is independent of  $k$ . The matrix associated with this system has eigenvalues  $\pm 1$ . Hence equation (40) possesses exponential dichotomies with  $\kappa^u = -\kappa^s = 1$ . To get estimates for the perturbed system (39), we will use the estimates

$$\begin{aligned} \left| \frac{k^2}{s(\tau)^3} \left(\sqrt{\lambda} + \frac{k^2}{s(\tau)^2}\right)^{-3/2} \right| &= \frac{1}{s(\tau)} \frac{1}{\sqrt{\sqrt{\lambda} + k^2/s(\tau)^2}} \frac{k^2/s(\tau)^2}{\sqrt{\lambda} + k^2/s(\tau)^2} \leq \frac{1}{\lambda^{1/4}s(\tau)}, \\ \left| \frac{1}{s(\tau)} \left(\sqrt{\lambda} + \frac{k^2}{s(\tau)^2}\right)^{-1/2} \right| &\leq \frac{1}{\lambda^{1/4}s(\tau)}. \end{aligned}$$

This estimate is uniform in  $\lambda$  in a neighbourhood of  $\lambda_0$ . The roughness theorem for exponential dichotomies [6, Chapter 4] now guarantees the existence of an exponential dichotomy also for the system (39), and we denote the corresponding evolution operators by  $\tilde{\phi}_k^s(\sigma, \tau)$  and  $\tilde{\phi}_k^u(\sigma, \tau)$ . For each positive  $\tilde{\epsilon}$  sufficiently small, there exists a  $K \geq 0$  such that

$$\begin{aligned} \|\tilde{\phi}_k^s(\sigma, \tau)\|_{\mathcal{L}(\mathbb{C}^2)} &\leq K e^{-(1-\tilde{\epsilon})(\sigma-\tau)} & \sigma &\geq \tau, \\ \|\tilde{\phi}_k^u(\sigma, \tau)\|_{\mathcal{L}(\mathbb{C}^2)} &\leq K e^{-(1-\tilde{\epsilon})(\tau-\sigma)} & \sigma &\leq \tau. \end{aligned}$$

Moreover,  $K$  does not depend on  $\lambda$  in a neighbourhood of  $\lambda_0$  or on  $k \in \mathbb{Z}$ .

It remains to translate this result back to the  $s$  variable. We write  $s = s(\sigma)$  and  $t = s(\tau)$ . Note that  $ds/d\tau \leq \lambda^{1/4}$ , and so by the chain rule we have for  $s > t$

$$\begin{aligned} \|\phi_k^s(s, t)\|_{\mathcal{L}(\tilde{X}_k^t, \tilde{X}_k^s)} &\leq C \|\tilde{\phi}_k^s(\sigma, \tau)\|_{\mathcal{L}(\mathbb{C}^2)} \leq K e^{-(1-\tilde{\epsilon})(\sigma-\tau)} \\ &\leq K e^{-(1-\tilde{\epsilon})\lambda^{1/4}(s-t)} \leq K e^{-(\lambda^{1/4}-\epsilon)(s-t)}, \end{aligned}$$

where we have put  $\epsilon = \tilde{\epsilon}\lambda^{1/4}$ . A similar calculation proves that for  $t > s$

$$\|\phi_k^u(s, t)\|_{\mathcal{L}(\tilde{X}_k^t, \tilde{X}_k^s)} \leq K e^{-(\lambda^{1/4}-\epsilon)(t-s)}.$$

The estimates for  $\tilde{Y}_k^s$  also follow from these estimates, since it is only a matter of multiplying both sides of the inequalities by a factor  $(1 + k^2)$ .  $\square$

**Lemma 13.** *Let  $\epsilon > 0$  be given. Then there exists a  $K > 0$  such that for any  $k \in \mathbb{Z}$  and  $\lambda \in (\lambda_0/2, 2\lambda_0)$  we have*

$$\|\psi_k(s, t; \lambda)\|_{\mathcal{L}(\tilde{X}_k^t, \tilde{X}_k^s)} = \|\psi_k(s, t; \lambda)\|_{\mathcal{L}(\tilde{Y}_k^t, \tilde{Y}_k^s)} \leq K e^{\epsilon(t-s)} \quad \text{for } t \geq s \geq r_1.$$

*Proof.* First we note a scaling invariance in (36). If  $(\bar{u}_1(s, t), \bar{u}_2(s, t))$  is a solution of (36) with  $\lambda = 1$ , then  $(u_1(s, t), u_2(s, t)) = (\bar{u}_1(\lambda_1^{1/4}s, \lambda_1^{1/4}t), \lambda_1^{1/4}\bar{u}_2(\lambda_1^{1/4}s, \lambda_1^{1/4}t))$  is a solution of (36) with  $\lambda = \lambda_1$ . So it is sufficient to prove the estimate in the Lemma in case  $\lambda = 1$ . Using the explicit expressions for the solution in terms of Bessel function, it follows that, for  $\lambda = 1$ ,  $\psi_k(s, t)$  is given by

$$\begin{aligned} \psi_k(s, t) &= \begin{pmatrix} J_k(s) & Y_k(s) \\ J'_k(s) & Y'_k(s) \end{pmatrix} \begin{pmatrix} J_k(t) & Y_k(t) \\ J'_k(t) & Y'_k(t) \end{pmatrix}^{-1} \\ &= \frac{\pi t}{2} \begin{pmatrix} J_k(s) & Y_k(s) \\ J'_k(s) & Y'_k(s) \end{pmatrix} \begin{pmatrix} Y'_k(t) & -Y_k(t) \\ -J'_k(t) & J_k(t) \end{pmatrix} \\ &= \frac{\pi t}{2} \begin{pmatrix} a_k(s, t) & b_k(s, t) \\ c_k(s, t) & d_k(s, t) \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} a_k(s, t) &:= (J_k(s)Y'_k(t) - Y_k(s)J'_k(t)), \\ b_k(s, t) &:= -J_k(s)Y_k(t) + Y_k(s)J_k(t), \\ c_k(s, t) &:= (J'_k(s)Y'_k(t) - Y'_k(s)J'_k(t)), \\ d_k(s, t) &:= (-J'_k(s)Y_k(t) + Y'_k(s)J_k(t)), \end{aligned}$$

and we have used that the Wronskian of  $J_k(t)$  and  $Y_k(t)$  is  $\frac{2}{\pi t}$  [1, (9.1.16)]. Writing  $\underline{u} = (u_1, u_2)^T \in \tilde{X}_k^t$ , we have

$$\|\psi_k(s, t)\underline{u}\|_{\tilde{X}_k^s}^2 = \frac{\pi^2 t^2}{4} \left( \left(1 + \frac{k^2}{s^2}\right) (a_k(s, t)u_1 + b_k(s, t)u_2)^2 + (c_k(s, t)u_1 + d_k(s, t)u_2)^2 \right),$$

and so we need to show that there exists a  $K > 0$  such that for every  $u_1, u_2 \in \mathbb{R}$ ,  $k \in \mathbb{Z}$  and  $t \geq s \geq r_1$ ,

$$t^2 \left( \left(1 + \frac{k^2}{s^2}\right) (a_k(s, t)u_1 + b_k(s, t)u_2)^2 + (c_k(s, t)u_1 + d_k(s, t)u_2)^2 \right) \leq K^2 e^{2\epsilon(t-s)} \left( \left(1 + \frac{k^2}{t^2}\right) u_1^2 + u_2^2 \right).$$

By choosing  $u_1$  and  $u_2$  appropriately, we note that this inequality holds if and only if the following two inequalities hold for some  $K > 0$  and all  $k \in \mathbb{Z}$  and  $t > s > r_1$ :

$$\begin{aligned} \left( \left(1 + \frac{k^2}{s^2}\right) a_k(s, t)^2 + c_k(s, t)^2 \right) t^2 &\leq K^2 e^{2\epsilon(t-s)} \left(1 + \frac{k^2}{t^2}\right), \\ \left( \left(1 + \frac{k^2}{s^2}\right) b_k(s, t)^2 + d_k(s, t)^2 \right) t^2 &\leq K^2 e^{2\epsilon(t-s)}. \end{aligned}$$

To simplify further, we note that the above two inequalities hold if there exists a constant  $K > 0$  such that for  $t \geq s \geq r_1$ ,

$$\begin{aligned} \sqrt{1 + \frac{k^2}{s^2}} |a_k(s, t)| t &\leq K e^{\epsilon(t-s)} \sqrt{1 + \frac{k^2}{t^2}}, \\ \sqrt{1 + \frac{k^2}{s^2}} |b_k(s, t)| t &\leq K e^{\epsilon(t-s)}, \\ |c_k(s, t)| t &\leq K e^{\epsilon(t-s)} \sqrt{1 + \frac{k^2}{t^2}}, \\ |d_k(s, t)| t &\leq K e^{\epsilon(t-s)}. \end{aligned} \tag{41}$$

First we will prove the fourth inequality of (41). Let

$$f_k(s, t) := e^{-\epsilon(t-s)} t d_k(s, t).$$

We need to show that  $f_k(s, t)$  is uniformly bounded for  $k \in \mathbb{Z}$  and  $t \geq s \geq r_1$ . Since  $J_{-k} = (-1)^k J_k$  and  $Y_{-k} = (-1)^k Y_k$ , it is sufficient to consider  $k \in \mathbb{N}$ .

We start with showing that  $f_k$  is bounded for  $k \in \mathbb{N}$  fixed. In the slightly smaller sector  $t \geq (1 + \delta)s \geq (1 + \delta)r_1$  (where  $\delta > 0$  is arbitrary), we have  $|f_k(s, t)| \rightarrow 0$  as  $s^2 + t^2 \rightarrow \infty$ , or equivalently, as  $t \rightarrow \infty$ . Indeed,  $Y_k(s)$ ,  $J_k(s)$ ,  $Y'_k(s)$  and  $J'_k(s)$  are bounded by a constant  $C_k$  for  $s \geq r_1$  [1, (9.2.1)], and so

$$|f_k(s, t)| \leq C_k^2 e^{-\epsilon(t-s)} t \leq C_k^2 t e^{-\epsilon \delta t / (1+\delta)} \rightarrow 0 \quad (42)$$

as  $t \rightarrow \infty$ . Furthermore,  $\sqrt{s}Y_k(s)$ ,  $\sqrt{s}J_k(s)$ ,  $\sqrt{s}Y'_k(s)$  and  $\sqrt{s}J'_k(s)$  are bounded by a constant  $D_k$  for  $s \geq r_1$  [1, (9.2.1)], and so for  $r_1 \leq s \leq t \leq (1 + \delta)s$  we have

$$|f_k(s, t)| \leq e^{-\epsilon(t-s)} \sqrt{(1 + \delta)st} d_k(s, t) \leq \sqrt{1 + \delta} D_k^2.$$

Altogether this implies that  $f_k(s, t)$  is bounded in the whole sector  $t \geq s \geq r_1$  by a constant, possibly depending on  $k$ .

To show that in fact  $f_k(s, t)$  is bounded by a  $k$ -independent constant, we consider  $s \geq r_1$  as being fixed for the moment. First we note that in (42), we proved that for fixed  $s$  and  $k$ ,  $f_k(s, t) \rightarrow 0$  as  $t \rightarrow \infty$ . Thus for  $s \geq r_1$  fixed, the function  $f_k(s, t)$  attains its maximum in an interior point  $t > s$  or at the boundary  $t = s$ . We use a method by L. Landau [11] to analyze the behaviour of  $f_k(s, \cdot)$  at its critical points. At a critical point, we have  $\partial f_k / \partial t(s, t) = 0$ . By equation (11) of [11], at the points where  $\partial f_k / \partial t = 0$  we have

$$\frac{\partial}{\partial k} f_k(s, t)^2 = 2t \frac{f_k(s, t)^2}{e^{-2\epsilon(t-s)} t^2} \frac{\partial}{\partial t} \left( e^{-2\epsilon(t-s)} t^2 A_k(t) \right),$$

where  $A_k(t) = \int_0^\infty K_0(2t \cosh \tau) e^{-2k\tau} d\tau$ , and  $K_0$  is a modified Bessel function of the second kind, satisfying

$$K_0(x) = \int_0^\infty e^{-x \cosh \tau} d\tau.$$

In particular (since  $2t f_k(s, t)^2 / e^{2\epsilon(t-s)} t^2 > 0$ ),  $f_k(s, t)^2$  is decreasing in  $k$  at a point where  $\partial f_k / \partial t = 0$  if and only if  $e^{-2\epsilon(t-s)} t^2 A_k(t)$  is decreasing in  $t$ . Note that  $A_k(t)$  is monotonically decreasing for  $t > 0$ , and since  $e^{-2\epsilon(t-s)} t^2$  is monotonically decreasing for  $t > 1/\epsilon$ , we conclude that  $|f_k(s, \cdot)|$  is monotonically decreasing in  $k$  at its critical points for  $t > 1/\epsilon$ .

Note that in the case when  $r_1 > 1/\epsilon$ , we have proved that if the maximum of  $f_k(s, \cdot)$  occurs for  $t > s \geq r_1$ , then the maximum is decreasing in  $k$ , and hence stays bounded as  $k$  increases. At the boundary  $t = s \geq r_1$ , we have  $f_k(s, s) = 2/\pi$ , which is independent of  $k$ . As each function  $f_k(s, t)$  is bounded, in particular  $f_1(s, t)$  is bounded, it follows that  $f_k(s, t)$  is bounded in the whole sector  $t \geq s \geq r_1$  by a  $k$ -independent constant for all  $k \in \mathbb{Z}$ . This shows that  $f_k$  is uniformly bounded in the case when  $r_1 > 1/\epsilon$ .

When  $r_1 < 1/\epsilon$ , we also need to estimate  $f_k(s, t)$  in the triangle  $1/\epsilon > t > s \geq r_1$ . Here we use the estimate  $e^{-\epsilon(t-s)} t \leq 1/\epsilon$ . It follows that  $|f_k(s, t)| \leq |g_k(s, t)|$ , where

$$g_k(s, t) := \frac{1}{\epsilon} (Y'_k(s) J_k(t) - J'_k(s) Y_k(t)).$$

Applying Section 3 of [11] we conclude that  $g_k(s, t)^2$  is decreasing in  $k$  at the points where  $\partial g_k(s, t) / \partial t = 0$ . Furthermore,  $g_k(s, t) \rightarrow 0$ , for  $t \rightarrow \infty$  and  $g_k(s, s) = \frac{2}{\pi \epsilon s} \leq \frac{2}{\pi \epsilon r_1}$  for  $s \geq r_1$ . The proof of the fourth inequality of (41) is complete.

Next, we prove the second inequality of (41). By [1, (9.1.27)] we have

$$\begin{aligned} \left(1 + \frac{k}{s}\right) Y_k(s) &= Y_k(s) + \frac{1}{2} (Y_{k-1}(s) + Y_{k+1}(s)), \\ \left(1 + \frac{k}{s}\right) J_k(s) &= J_k(s) + \frac{1}{2} (J_{k-1}(s) + J_{k+1}(s)). \end{aligned}$$

Note that

$$\left(1 + \frac{k^2}{s^2}\right)^{1/2} \leq 1 + \frac{k}{s} \leq \sqrt{2} \left(1 + \frac{k^2}{s^2}\right)^{1/2},$$

and so the second inequality of (41) is equivalent to

$$\left| t \left( (Y_k(s) + \frac{1}{2}Y_{k-1}(s) + \frac{1}{2}Y_{k+1}(s))J_k(t) - (J_k(s) + \frac{1}{2}J_{k-1}(s) + \frac{1}{2}J_{k+1}(s))Y_k(t) \right) \right| \leq Ke^{\epsilon(t-s)}.$$

To prove the inequality, we use the same method as above, with

$$f_k(s, t) := te^{-\epsilon(t-s)} \left( (Y_k(s) + \frac{1}{2}Y_{k-1}(s) + \frac{1}{2}Y_{k+1}(s))J_k(t) - (J_k(s) + \frac{1}{2}J_{k-1}(s) + \frac{1}{2}J_{k+1}(s))Y_k(t) \right)$$

and

$$g_k(s, t) := te^{-s} \frac{e^{-1}}{\epsilon} \left( (Y_k(s) + \frac{1}{2}Y_{k-1}(s) + \frac{1}{2}Y_{k+1}(s))J_k(t) - (J_k(s) + \frac{1}{2}J_{k-1}(s) + \frac{1}{2}J_{k+1}(s))Y_k(t) \right).$$

Then  $|f_k(s, t)| \rightarrow 0$  as  $s^2 + t^2 \rightarrow \infty$  in the sector  $t \geq s \geq r_1$  and as above,  $f_k(s, t)^2$  is decreasing in  $k$  at the points where  $\partial f_k / \partial t = 0$  if  $t > 1/\epsilon$ . In the triangle  $1/\epsilon \geq t \geq s \geq r_1$ ,  $|f_k(s, t)| \leq |g_k(s, t)|$  and  $g_k$  is decreasing in  $k$  at the points where  $\partial g_k / \partial t = 0$ . Finally, at the boundary points where  $s = t \geq r_1$ , we have  $f_k(s, s) = g_k(s, s) = 0$ . We conclude that the second inequality is valid in the whole sector  $t \geq s \geq r_1$ .

For the third inequality of (41), we use the last identity of [1, (9.1.27)] which shows that the inequality is equivalent to the two inequalities

$$\begin{aligned} \frac{k}{t} |J'_k(s)Y_k(t) - Y'_k(s)J_k(t)| &\leq Ke^{\epsilon(t-s)} \left( 1 + \frac{k^2}{t^2} \right)^{1/2}, \\ |J'_k(s)Y_{k+1}(t) - Y'_k(s)J_{k+1}(t)| &\leq Ke^{\epsilon(t-s)} \left( 1 + \frac{k^2}{t^2} \right)^{1/2}. \end{aligned}$$

The first of these inequalities follows from the fourth inequality of (41), and the second can be handled as in the proof of the second and fourth inequalities after noting that at the boundary where  $t = s \geq r_1$  we have

$$|J'_k(s)Y_{k+1}(s) - Y'_k(s)J_{k+1}(s)| = \frac{k}{s} |J'_k(s)Y_k(s) - Y'_k(s)J_k(s)| = \frac{2k}{\pi s^2},$$

thus  $f_k(s, s) = \frac{2}{\pi\sqrt{1+s^2/k^2}} \leq \frac{2}{\pi}$ . We omit the details.

It remains to prove the first inequality of (41). This can be handled as the second inequality by using

$$J_k(s)Y'_k(t) - Y_k(s)J'_k(t) = \frac{k}{t} (J_k(s)Y_k(t) - Y_k(s)J_k(t)) + (Y_k(s)J_{k+1}(t) - J_k(s)Y_{k+1}(t))$$

and splitting the inequality up into the two inequalities

$$\begin{aligned} \left( 1 + \frac{k^2}{s^2} \right)^{1/2} \frac{k}{t} |J_k(s)Y_k(t) - Y_k(s)J_k(t)| &\leq Ke^{\epsilon(t-s)} \left( 1 + \frac{k^2}{t^2} \right)^{1/2}, \\ |Y_k(s)J_{k+1}(t) - J_k(s)Y_{k+1}(t)| \left( 1 + \frac{k^2}{s^2} \right)^{1/2} &\leq Ke^{\epsilon(t-s)} \left( 1 + \frac{k^2}{t^2} \right)^{1/2}. \end{aligned}$$

The first inequality follows directly from the second inequality of (41), and the second inequality is proved in the same way as the second inequality of (41), except that at the boundary where  $t = s$  we have

$$|Y_k(s)J_{k+1}(s) - J_k(s)Y_{k+1}(s)| = 2/\pi s.$$

The details are omitted.

It is also clear that the estimates above are uniform in  $\lambda$  for  $\lambda \in (\lambda_0/2, 2\lambda_0)$ .

The estimates for  $\tilde{Y}_k^s$  follow by the same estimates, since it is just a matter of multiplying each side of the inequalities by the factor  $(1 + k^2)$ .  $\square$

We are ready to prove that there exist time-dependent exponential dichotomies on  $J_+$  for the full system (5). First we define the spaces  $\mathcal{X}^s := \tilde{X}^s \times \tilde{X}^s$  and  $\mathcal{Y}^s := \tilde{Y}^s \times \tilde{Y}^s$ . Note that  $\mathcal{Y}^s \subset X \subset \mathcal{X}^s$ . As before, we can decompose those spaces into  $\mathcal{X}^s = \overline{\bigoplus_{k \in \mathbb{Z}} \mathcal{X}_k^s}$  and  $\mathcal{Y}^s = \overline{\bigoplus_{k \in \mathbb{Z}} \mathcal{Y}_k^s}$  with  $\mathcal{X}_k^s = \tilde{X}_k^s \times \tilde{X}_k^s$  and  $\mathcal{Y}_k^s = \tilde{Y}_k^s \times \tilde{Y}_k^s$ .



For  $s > s_3 = r_3$ , we have that  $r = s$  and hence, for  $s \in [s_3, \infty)$ , the system (5) reduces to

$$U' = A(s; \lambda)U \quad \text{with} \quad A(s; \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{s^2}\partial^2 & -\frac{1}{s} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & -\frac{1}{s^2}\partial^2 & -\frac{1}{s} \end{pmatrix}. \quad (43)$$

We consider the system (43) for  $s \in [r_1, \infty)$  and record that (43) and (5) coincide on the smaller interval  $[s_3, \infty)$ . The exponential dichotomy for (5) on the whole interval  $J_+$  will follow from the fact that the systems (5) and (43) are linked by the smooth transformation  $r(s)$  of the independent variable on the compact interval  $[s_1, s_3]$ . Thus, since we are using  $r = s$  in (43), we see that if  $\tilde{U}(s)$  is a solution of (43) for  $s \in [r_1, \infty)$ , then  $U(s) = \text{diag}(1, r'(s), 1, r'(s))\tilde{U}(r(s))$  is a solution of (5) for all  $s \in [s_1, \infty)$ . Recall that there are constants  $0 < c < C$  such that  $c \leq r'(s) \leq C$  for all  $s \in \mathbb{R}$ , thus dichotomy results for  $\tilde{U}$  will immediately give similar dichotomy results for  $U$ .

It is easy to check (similarly to Lemma 3) that  $A(s; \lambda) : \mathcal{X}^s \rightarrow \mathcal{X}^s$  is closed and densely defined with domain  $\mathcal{Y}^s$  and that  $A(s; \lambda) : \mathcal{Y}^s \rightarrow \mathcal{Y}^s$  is closed and densely defined with domain  $\tilde{\mathcal{Z}}^s \times \tilde{\mathcal{Z}}^s$ .

The Fourier coefficients of  $U = (u_1, u_2, u_3, u_4)^T$  satisfy the system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= \frac{k^2}{s^2}u_1 - \frac{1}{s}u_2 + u_3, \\ u_3' &= u_4, \\ u_4' &= \lambda u_1 + \frac{k^2}{s^2}u_3 - \frac{1}{s}u_4, \end{aligned} \quad (44)$$

of ODEs, where we omit the subscript  $k$ . We denote by  $\Phi_k(s, t)$  the evolution operator corresponding to the system (44) and consider this evolution operator in either  $\mathcal{X}^s$  or  $\mathcal{Y}^s$ .

Now we can use the earlier dichotomy results to show the existence of a uniform ( $s$ -dependent) exponential dichotomy for the system (44) and hence for (43).

**Lemma 14.** *There exists an  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$  there exists a  $K > 0$  such that for any  $\lambda \in (\lambda_0/2, 2\lambda_0)$  there exists an  $s$ -dependent exponential dichotomy of (43) on  $J_+$  such that the evolution operators  $\Phi_+^s(s, t; \lambda)$  and  $\Phi_+^{cu}(s, t; \lambda)$  solve (43) and*

$$\begin{aligned} \|\Phi_+^s(s, t; \lambda)\|_{\mathcal{L}(\mathcal{X}^t, \mathcal{X}^s)} = \|\Phi_+^s(s, t; \lambda)\|_{\mathcal{L}(\mathcal{Y}^t, \mathcal{Y}^s)} &\leq Ke^{-(\lambda^{1/4} - \epsilon)(s-t)}, & s \geq t \geq r_1, \\ \|\Phi_+^{cu}(s, t; \lambda)\|_{\mathcal{L}(\mathcal{X}^t, \mathcal{X}^s)} = \|\Phi_+^{cu}(s, t; \lambda)\|_{\mathcal{L}(\mathcal{Y}^t, \mathcal{Y}^s)} &\leq Ke^{\epsilon(t-s)}, & t \geq s \geq r_1. \end{aligned} \quad (45)$$

The dichotomy is smooth in  $\lambda$  for  $\lambda$  near  $\lambda_0$ .

*Proof.* Since  $\mathcal{X}^s = \overline{\oplus \mathcal{X}_k^s}$  and each  $\mathcal{X}_k^t$  is mapped into  $\mathcal{X}_k^s$  under the flow of (43), we write for  $U \in \mathcal{X}^t$

$$\begin{aligned} \Phi_+^s(s, t)U &:= \sum_{k \in \mathbb{Z}} \Phi_k^s(s, t) \hat{U}_k e^{ik\cdot}, & s \geq t \geq r_1, \\ \Phi_+^{cu}(s, t)U &:= \sum_{k \in \mathbb{Z}} \Phi_k^{cu}(s, t) \hat{U}_k e^{ik\cdot}, & t \geq s \geq r_1. \end{aligned} \quad (46)$$

Moreover, for each  $k \in \mathbb{Z}$  the evolution operator  $\Phi_k(s, t)$  associated with (44) can be expressed in terms of  $\phi_k(s, t)$  and  $\psi_k(s, t)$ . Indeed, it can be seen that

$$\Phi_k(s, t) = \frac{1}{2} \begin{pmatrix} \phi_k(s, t) + \psi_k(s, t) & \frac{1}{\sqrt{\lambda}}(\phi_k(s, t) - \psi_k(s, t)) \\ \sqrt{\lambda}(\phi_k(s, t) - \psi_k(s, t)) & \phi_k(s, t) + \psi_k(s, t) \end{pmatrix}.$$

Similarly,

$$\Phi_k^s(s, t) = \frac{1}{2} \begin{pmatrix} \phi_k^s(s, t) & \frac{1}{\sqrt{\lambda}}\phi_k^s(s, t) \\ \sqrt{\lambda}\phi_k^s(s, t) & \phi_k^s(s, t) \end{pmatrix}$$

and

$$\Phi_k^{cu}(s, t) = \frac{1}{2} \begin{pmatrix} \phi_k^u(s, t) + \psi_k(s, t) & \frac{1}{\sqrt{\lambda}}(\phi_k^u(s, t) - \psi_k(s, t)) \\ \sqrt{\lambda}(\phi_k^u(s, t) - \psi_k(s, t)) & \phi_k^u(s, t) + \psi_k(s, t) \end{pmatrix}.$$

Introducing the temporary notation  $\widehat{U}_k = (\underline{u}, \underline{v})^T \in \mathcal{X}_k^t$ , where  $\underline{u} = (u_1, u_2)^T$  and  $\underline{v} = (u_3, u_4)^T$ , we have

$$\|\Phi_k^s(s, t)\widehat{U}_k\|_{\mathcal{X}_k^s}^2 = \frac{1}{4}(1 + \lambda) \left\| \phi_k^s(s, t)\underline{u} + \frac{1}{\sqrt{\lambda}}\phi_k^s(s, t)\underline{v} \right\|_{\widetilde{\mathcal{X}}_k^s}^2. \quad (47)$$

Since we will take the supremum over all  $\widehat{U}_k \in \mathcal{X}_k^t$  such that  $\|\widehat{U}_k\|_{\mathcal{X}_k^t} = 1$ , we may without loss of generality assume that  $\underline{v} = \sqrt{\lambda}\underline{u}$ , since all other choices will result in a smaller value of the right hand side of (47). For any such  $\underline{u}$  and  $\underline{v} \in \widetilde{\mathcal{X}}_k^t$ , the condition that  $\|\widehat{U}_k\|_{\mathcal{X}_k^t}^2 = 1$  implies that  $\|\underline{u}\|_{\widetilde{\mathcal{X}}_k^t}^2(1 + \lambda) = 1$ . We therefore have

$$\begin{aligned} \sup_{\|\widehat{U}_k\|_{\mathcal{X}_k^t}^2=1} \|\Phi_k^s(s, t)\widehat{U}_k\|_{\mathcal{X}_k^s}^2 &= \sup_{\|\underline{u}\|_{\widetilde{\mathcal{X}}_k^t}^2=1/(1+\lambda)} \frac{1 + \lambda}{2} \|\phi_k^s(s, t)\underline{u}\|_{\widetilde{\mathcal{X}}_k^s}^2 \\ &= \frac{1}{2} \sup_{\|\underline{u}\|_{\widetilde{\mathcal{X}}_k^t}^2=1} \|\phi_k^s(s, t)\underline{u}\|_{\widetilde{\mathcal{X}}_k^s}^2, \end{aligned}$$

which shows that

$$\|\Phi_k^s(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)} = \frac{1}{\sqrt{2}} \|\phi_k^s(s, t)\|_{\mathcal{L}(\widetilde{\mathcal{X}}_k^t, \widetilde{\mathcal{X}}_k^s)}.$$

Likewise,

$$\|\Phi_k^{cu}(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)} \leq \|\Phi_k^c(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)} + \|\Phi_k^u(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)},$$

where

$$\Phi_k^c(s, t) = \begin{pmatrix} \psi_k(s, t) & -\frac{\psi_k(s, t)}{\sqrt{\lambda}} \\ -\sqrt{\lambda}\psi_k(s, t) & \psi_k(s, t) \end{pmatrix} \quad \text{and} \quad \Phi_k^u(s, t) = \Phi_k^{cu}(s, t) - \Phi_k^c(s, t).$$

As above,

$$\begin{aligned} \|\Phi_k^c(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)} &= \frac{1}{\sqrt{2}} \|\psi_k(s, t)\|_{\mathcal{L}(\widetilde{\mathcal{X}}_k^t, \widetilde{\mathcal{X}}_k^s)}^2, \\ \|\Phi_k^u(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)} &= \frac{1}{\sqrt{2}} \|\phi_k^u(s, t)\|_{\mathcal{L}(\widetilde{\mathcal{X}}_k^t, \widetilde{\mathcal{X}}_k^s)}^2. \end{aligned}$$

From Lemmas 12 and 13 it follows that

$$\begin{aligned} \|\Phi_k^s(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)} &\leq K e^{-(\lambda^{1/4}-\epsilon)(s-t)}, & s \geq t \geq r_1, \\ \|\Phi_k^{cu}(s, t)\|_{\mathcal{L}(\mathcal{X}_k^t, \mathcal{X}_k^s)} &\leq K e^{\epsilon(t-s)}, & t \geq s \geq r_1. \end{aligned} \quad (48)$$

By (46) and (48) we see that for  $s \geq t \geq r_1$

$$\begin{aligned} \|\Phi_+^s(s, t)\|_{\mathcal{L}(\mathcal{X}^t, \mathcal{X}^s)}^2 &= \sup_{\|U\|_{\mathcal{X}^t}=1} \|\Phi^s(s, t)U\|_{\mathcal{X}^s}^2 = \sup_{\|U\|_{\mathcal{X}^t}=1} \left\| \sum_{k \in \mathbb{Z}} \Phi_k^s(s, t)\widehat{U}_k e^{ik \cdot} \right\|_{\mathcal{X}^s}^2 \\ &= \sup_{\|U\|_{\mathcal{X}^t}=1} \sum_{k \in \mathbb{Z}} \|\Phi_k^s(s, t)\widehat{U}_k\|_{\mathcal{X}_k^s}^2 \leq \sup_{\|U\|_{\mathcal{X}^t}=1} \sum_{k \in \mathbb{Z}} \|\Phi_k^s(s, t)\|_{\mathcal{L}(\mathcal{X}^t, \mathcal{X}^s)}^2 \|\widehat{U}_k\|_{\mathcal{X}_k^t}^2 \\ &\leq \sup_{\|U\|_{\mathcal{X}^t}=1} \sum_{k \in \mathbb{Z}} K^2 e^{-2(\lambda^{1/4}-\epsilon)(s-t)} \|\widehat{U}_k\|_{\mathcal{X}_k^t}^2 = K^2 e^{-2(\lambda^{1/4}-\epsilon)(s-t)}. \end{aligned}$$

A similar calculation shows that  $\Phi_+^{cu}$  satisfies the second equation of (45). The estimates for  $\widetilde{Y}_k^s$  follow by the same estimates since it is just a matter of multiplying each side of the inequalities by a factor  $(1 + k^2)^2$ .

Finally, the smoothness in  $\lambda$  follows from the implicit function theorem [4, Corollary 3.1.11]. As  $A(s; \lambda)$  depends linearly on  $\lambda$ , we get that  $\Phi_+^{s/cu}(s, t; \lambda)$  satisfies

$$\frac{d}{ds} \Phi_+^{s/cu}(s, t; \lambda) = [A(s; \lambda_0) + (\lambda - \lambda_0) B_0] \Phi_+^{s/cu}(s, t; \lambda),$$

where  $B_0$  is defined by (19).

For  $\lambda \in (\lambda_0/2, 2\lambda_0)$  and sufficiently close to  $\lambda_0$ , the pair  $(\Phi_+^s(s, t; \lambda), \Phi_+^{cu}(s, t; \lambda))$  satisfies the fixed point equation

$$\begin{aligned}\Phi_+^s(s, t; \lambda) &= \Phi_+^s(s, t; \lambda_0) + \Phi_+^s(s, t; \lambda_0) \Phi_+^{cu}(s_1, t; \lambda) + (\lambda - \lambda_0) \left[ - \int_{r_1}^t \Phi_+^s(s, \tau; \lambda_0) B_0 \Phi_+^{cu}(\tau, t; \lambda) d\tau \right. \\ &\quad \left. + \int_t^s \Phi_+^s(s, \tau; \lambda_0) B_0 \Phi_+^s(\tau, t; \lambda) d\tau - \int_s^\infty \Phi_+^{cu}(s, \tau; \lambda_0) B_0 \Phi_+^s(\tau, t; \lambda) d\tau \right], \\ s &\geq t \geq r_1; \\ \Phi_+^{cu}(s, t; \lambda) &= \Phi_+^{cu}(s, t; \lambda_0) - \Phi_+^s(s, t; \lambda_0) \Phi_+^{cu}(s_1, t; \lambda) + (\lambda - \lambda_0) \left[ \int_{r_1}^s \Phi_+^s(s, \tau; \lambda_0) B_0 \Phi_+^{cu}(\tau, t; \lambda) d\tau \right. \\ &\quad \left. - \int_s^t \Phi_+^{cu}(s, \tau; \lambda_0) B_0 \Phi_+^{cu}(\tau, t; \lambda) d\tau + \int_t^\infty \Phi_+^{cu}(s, \tau; \lambda_0) B_0 \Phi_+^s(\tau, t; \lambda) d\tau \right], \\ t &\geq s \geq r_1.\end{aligned}$$

This fixed point equation is considered as a mapping on  $\hat{X}^s \times \hat{X}^{cu}$ , where  $\hat{X}^{s/cu}$  are defined as

$$\begin{aligned}\hat{X}^s &= \left\{ \Phi^s(s, t) \in \mathcal{L}(\mathcal{X}^t, \mathcal{X}^s); \|\Phi^s(s, t)\|_{\hat{X}^s} = \sup_{s \geq t \geq r_1} e^{((\lambda_0/2)^{1/4} - \epsilon)(s-t)} \|\Phi^s(s, t)\|_{\mathcal{L}(\mathcal{X}^t, \mathcal{X}^s)} < \infty \right\} \\ \hat{X}^{cu} &= \left\{ \Phi^{cu}(s, t) \in \mathcal{L}(\mathcal{X}^t, \mathcal{X}^s); \|\Phi^{cu}(s, t)\|_{\hat{X}^{cu}} = \sup_{t \geq s \geq r_1} e^{-2\epsilon(t-s)} \|\Phi^{cu}(s, t)\|_{\mathcal{L}(\mathcal{X}^t, \mathcal{X}^s)} < \infty \right\}.\end{aligned}$$

With the estimates derived before on  $\Phi_+^s(s, t; \lambda)$  and  $\Phi_+^{cu}(s, t; \lambda)$ , it is easy to check that the right-hand side is well-defined in this space. With the implicit function theorem, it follows immediately that the mapping is smooth for  $\lambda$  near  $\lambda_0$ .  $\square$

## 6 The unperturbed adjoint system

The dual space of  $X$  is  $X' = H^{-2} \times H^{-1} \times H^{-1} \times L^2$  (using the  $L^2$  pairing). For the space  $X'$ , we make the decomposition

$$X' = \overline{\bigoplus_{k \in \mathbb{Z}} X'_k},$$

where  $X'_k$  are 4-dimensional pairwise orthogonal subspaces  $\text{span}\{(a, b, c, d)e^{ik\cdot}; a, b, c, d \in \mathbb{C}\} \subset X'$ . For  $W \in X'_k$  and  $U \in X_k$ , we have the pairing

$$\langle W, U \rangle := \bar{w}_1 u_1 + \bar{w}_2 u_2 + \bar{w}_3 u_3 + \bar{w}_4 u_4, \quad (49)$$

where the bar denotes the complex conjugate. This means that we may use the standard inner product on  $\mathbb{C}^4$  when computing the adjoint equation.

Similarly, the dual space of  $\mathcal{X} := H^1 \times L^2 \times H^1 \times L^2$  is  $\mathcal{X}' = H^{-1} \times L^2 \times H^{-1} \times L^2$  (using the  $L^2$  pairing) and we can make the same decomposition as above, i.e.,

$$\mathcal{X}' = \overline{\bigoplus_{k \in \mathbb{Z}} \mathcal{X}'_k},$$

where  $\mathcal{X}'_k$  are the same 4-dimensional subspaces as above but are now regarded as subspaces of  $\mathcal{X}'$ . For  $W \in \mathcal{X}'_k$  and  $U \in \mathcal{X}_k$ , the pairing is again as in (49).

At the end of Section 4, we have investigated the solutions of the unperturbed linear system  $U' = A(s; \lambda_0, 0)U$  on  $J_-$ . The adjoint unperturbed system is

$$W' = -A(s; \lambda_0, 0)^* W = -(A_-^* + B(s; \lambda_0, 0)^*) W. \quad (50)$$

Just as in the case of the unperturbed linear system itself, expanding  $W$  in a Fourier series shows that the Fourier spaces  $X'_k$  are invariant under the flow of the adjoint system (50), and the Fourier coefficients satisfy the adjoint equation of (14), i.e.,

$$\widehat{W}'_k(s) = - \left[ \widehat{A}_-(k)^* + B(s; \lambda, 0)^* \right] \widehat{W}_k(s). \quad (51)$$

It is well known and straightforward to check that the pairing of a solution of a linear system with a solution of its adjoint is constant. For our systems, this means that any two solutions  $\widehat{U}_k(s)$  of (14) and  $\widehat{W}_k(s)$  of (51) satisfy

$$\frac{d}{ds} \langle \widehat{W}_k(s), \widehat{U}_k(s) \rangle = 0, \text{ and thus } \langle \widehat{W}_k(s), \widehat{U}_k(s) \rangle = \langle \widehat{W}_k(s_1), \widehat{U}_k(s_1) \rangle \text{ for any } s \in \mathbb{R}. \quad (52)$$

From [17, p. 56] it follows that if a finite-dimensional linear system has an exponential dichotomy on an interval  $J$  with constants  $K$ ,  $\kappa^s$  and  $\kappa^u$ , then the adjoint system has an exponential dichotomy on  $J$  with the dichotomy constants  $K$ ,  $-\kappa^u$  and  $-\kappa^s$ . Furthermore, if we denote evolution operators corresponding to the exponential dichotomy of the adjoint system (51) by  $\widehat{\Phi}_{k\pm}^s(s, t)$  and  $\widehat{\Phi}_k^u(s, t)$ , respectively, then  $\widehat{\Phi}_k^s(s, t) = \Phi_k^u(t, s)^*$  for  $t \leq s$  with  $s, t \in J$  and  $\widehat{\Phi}_k^u(s, t) = \Phi_k^s(t, s)^*$  for  $s \leq t$  with  $s, t \in J$ .

On  $J_-$ , the dichotomy constant  $K$  in (17) is independent of  $k$ , and so we immediately get the following estimates about the solutions of the adjoint system in the Fourier spaces  $X'_k$  with norm

$$\|\widehat{W}_k\|_{X'_k}^2 := \|\widehat{W}_k e^{ik\cdot}\|_{X'}^2 = \frac{|w_1|^2}{(k^2 + 1)^2} + \frac{|w_2|^2}{k^2 + 1} + \frac{|w_3|^2}{k^2 + 1} + |w_4|^2 :$$

**Lemma 15.** *There exists a  $K > 0$  such that for every  $k \in \mathbb{Z} \setminus \{0\}$  and  $\widehat{W}_k \in X'_k$*

$$\begin{aligned} \|\widehat{\Phi}_k^s(s, t) \widehat{W}_k\|_{X'_k} &\leq K e^{-|k|(s-t)} \|\widehat{W}_k\|_{X'_k}, & t \leq s \leq s_1, \\ \|\widehat{\Phi}_k^u(s, t) \widehat{W}_k\|_{X'_k} &\leq K e^{|k|(s-t)} \|\widehat{W}_k\|_{X'_k}, & s \leq t \leq s_1. \end{aligned}$$

Furthermore, for any solution  $\widehat{W}_k(s)$  with  $\widehat{W}_k(s_1) \in \text{Ran}(\widehat{\Phi}_{k-}^u(s_1, s_1))$ , we have

$$|w_k(s)| \leq K e^{|k|(s-s_1)} \|\widehat{W}_k(s_1)\|_{X'}$$

for all  $s \leq s_1$ , where  $w_k(s)$  denotes the fourth component of  $\widehat{W}_k(s)$ .

Similar arguments give the dichotomy of the adjoint system on  $J_+$ . From (48), it follows that the solutions of the linearised system in the Fourier spaces  $\Phi_k^s(s, t)$  and  $\Phi_k^u(s, t)$  have an exponential dichotomy with a uniform constant  $K$ . The dual space of  $\mathcal{X}^s$  is denoted by  $(\mathcal{X}^s)'$ , and for  $s$  fixed, this space is equivalent to  $H^{-1} \times L^2 \times H^{-1} \times L^2$ . The dual Fourier space is denoted by  $(\mathcal{X}_k^s)'$ , and its norm is

$$\|\widehat{W}_k\|_{(\mathcal{X}_k^s)'}^2 := \|\widehat{W}_k e^{ik\cdot}\|_{(\mathcal{X}^s)'}^2 = \frac{s^2}{k^2 + s^2} |w_1|^2 + |w_2|^2 + \frac{s^2}{k^2 + s^2} |w_3|^2 + |w_4|^2.$$

On  $J_+$ , we have the following estimates:

**Lemma 16.** *For every  $\epsilon > 0$ , there exists a  $K > 0$  such that for every  $k \in \mathbb{Z} \setminus \{0\}$  and  $\widehat{W}_k \in \mathcal{X}'_k$*

$$\begin{aligned} \|\widehat{\Phi}_k^{cs}(s, t) \widehat{W}_k\|_{(\mathcal{X}^s)'} &\leq K e^{\epsilon(s-t)} \|\widehat{W}_k\|_{(\mathcal{X}^t)'}, & s_1 \leq t \leq s, \\ \|\widehat{\Phi}_k^u(s, t) \widehat{W}_k\|_{(\mathcal{X}^s)'} &\leq K e^{(\lambda_0^{1/4} - \epsilon)(s-t)} \|\widehat{W}_k\|_{(\mathcal{X}^t)'}, & s_1 \leq s \leq t. \end{aligned}$$

Moreover, for any solution  $\widehat{W}_k(s)$  with  $\widehat{W}_k(s_1) \in \text{Ran}(\widehat{\Phi}_k^{cs}(s_1, s_1))$ , we have

$$|w_k(s)| \leq K e^{\epsilon(s-s_1)} \|\widehat{W}_k(s_1)\|_{(\mathcal{X}^{s_1})'}$$

for all  $s \geq s_1$ , where  $w_k(s)$  denotes the fourth component of  $\widehat{W}_k(s)$ .

Next we look at the adjoint system associated with  $k = 0$ . In Lemma 10, we have seen that the solutions space of the linear system at  $k = 0$  is spanned by  $U_{0,j}$ ,  $j = 1, \dots, 4$ . Now let  $Z_{0,j}(s)$  be solutions of the adjoint system (51) with  $k = 0$  such that  $\{Z_{0,l}(s_1)\}_{l=1, \dots, 4}$  is a dual basis of  $\{U_{0,j}(s_1)\}_{j=1, \dots, 4}$  (i.e.,  $\langle Z_{0,l}(s_1), U_{0,j}(s_1) \rangle = \delta_{lj}$ ). With (52), this implies  $\delta_{lj} = \langle Z_{0,l}(s), U_{0,j}(s) \rangle$  for any  $s \leq s_1$ . The unbounded solutions  $U_{0,2}$  and  $U_{0,4}$  are not unique, but they can be chosen such that  $Z_{0,2}$  and  $Z_{0,4}$  are bounded on  $J_-$ , whereas  $Z_{0,1}$  and  $Z_{0,3}$  grow algebraically as  $s \rightarrow -\infty$ . A convenient choice for  $Z_{0,2}$  and  $Z_{0,4}$  is

$$Z_{0,j}(z) := \frac{r(s)}{r'(s)} U_{0,j-1}^\perp(s), \quad j = 2, 4, \quad (53)$$

where  $U^\perp = (-u_4, u_3, -u_2, u_1)$  if  $U = (u_1, u_2, u_3, u_4)^T$ . It is easy to check that  $Z_{0,j}^\perp(s)$  are solutions of the adjoint system (50) using that  $U_{0,j-1}(s)$  are solutions of the original system (5).

As will be shown below, a similar exponential dichotomy on  $J_-$  as in Lemma 15 also holds with norms in  $\mathcal{X}'$ .

**Lemma 17.** *There exists a  $K > 0$  such that, for every  $k \in \mathbb{Z} \setminus \{0\}$  and  $\widehat{W}_k \in (\mathcal{X}_k)'$ , we have*

$$\begin{aligned} \|\widehat{\Phi}_k^s(s, t)\widehat{W}_k\|_{\mathcal{X}'} &\leq K e^{-|k|(s-t)} \|\widehat{W}_k\|_{\mathcal{X}'}, & t \leq s \leq s_1, \\ \|\widehat{\Phi}_k^u(s, t)\widehat{W}_k\|_{\mathcal{X}'} &\leq K e^{|k|(s-t)} \|\widehat{W}_k\|_{\mathcal{X}'}, & s \leq t \leq s_1. \end{aligned}$$

For any solution  $\widehat{W}_k(s)$  with  $\widehat{W}_k(s_1) \in \text{Ran}(\widehat{\Phi}_k^u(s_1, s_1))$  and with  $w_k(s)$  denoting the fourth component of  $\widehat{W}_k(s)$ , we have  $|w_k(s)| \leq K e^{|k|(s-s_1)} \|\widehat{W}_k(s_1)\|_{\mathcal{X}'}$  for all  $s \leq s_1$ .

*Proof.* First we will show that the linear system (14) has an exponential dichotomy in  $\mathcal{X}_k$ . The proof is very similar to the one in section 4 with a slightly modified matrix  $M_k$ . Define the matrix  $\widehat{M}_k$  whose columns consist of eigenvectors of  $A_-(k)$  that are scaled different to those in  $M_k$ :

$$\widehat{M}_k = \begin{pmatrix} -1/|k| & 0 & 1/|k| & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1/|k| & 0 & 1/|k| \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

As  $\widehat{M}_k$  consists of eigenvectors of  $A_-(k)$ , it follows immediately that  $A_-(k) = \widehat{M}_k D_k \widehat{M}_k^{-1}$ . It is also straightforward to verify that  $\widehat{M}_k$  is a homeomorphism between  $\mathbb{C}^4$  and  $\mathcal{X}_k$  with  $\|\widehat{M}_k\|_{\mathcal{L}(\mathbb{C}^4, \mathcal{X}_k)} \rightarrow \sqrt{2}$  as  $|k| \rightarrow \infty$ .

Using the same ideas as in the proof of (17), with  $M_k$  replaced by  $\widehat{M}_k$  and exploiting the observation that

$$|\widehat{M}_k^{-1} B(\tau/|k|; \lambda_0, 0) \widehat{M}_k| \leq \frac{e^{2\tau/|k|}}{2|k|} \sup_{s \leq s_1} \{1, |\lambda - \tilde{\theta}(s)|\},$$

we find that

$$\begin{aligned} \|\Phi_k^s(s, t)\widehat{U}_k\|_{\mathcal{X}} &\leq K e^{-|k|(s-t)} \|\widehat{U}_k\|_{\mathcal{X}}, & t \leq s \leq s_1, \\ \|\Phi_k^u(s, t)\widehat{U}_k\|_{\mathcal{X}} &\leq K e^{|k|(s-t)} \|\widehat{U}_k\|_{\mathcal{X}}, & s \leq t \leq s_1 \end{aligned} \tag{54}$$

for any  $\widehat{U}_k \in \mathcal{X}_k$  for some constant  $K$  that is independent of  $k$ . As  $\mathcal{X}_k$  is finite-dimensional, this immediately implies the estimates of the Lemma.  $\square$

Finally we will show that, for large values of  $k$ , the solutions of (51) are close to the solutions of the asymptotic system  $\widehat{W}_k'(s) = -A_-(k)^* \widehat{W}_k(s)$ . Recall that we denote the spectral projection onto the eigenspace of  $A_-(k)$  associated with the positive eigenvalue  $|k|$  by  $P_k^u$  and the complementary projection onto the eigenspace of  $A_-(k)$  associated with the negative eigenvalue  $-|k|$  by  $P_k^s$ .

**Lemma 18.** *For every  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  and a  $\delta > 0$  such that, for every  $|k| > N$ , we have*

$$\begin{aligned} \|\widehat{\Phi}_k^u(s, s_1) - e^{|k|(s-s_1)}(P_k^s)^*\|_{\mathcal{L}(X'_k)} &\leq \epsilon e^{|k|(s-s_1)}, & \text{for all } s_1 - \delta \leq s \leq s_1, \\ \|\widehat{\Phi}_k^u(s, s_1) - e^{|k|(s-s_1)}(P_k^s)^*\|_{\mathcal{L}(X'_k)} &\leq \epsilon e^{|k|(s-s_1)}, & \text{for all } s \leq s_1. \end{aligned} \tag{55}$$

Thus, for  $|k| > N$  and  $\widehat{W}_k(s_1) \in \text{Ran}(\widehat{\Phi}_k^u(s_1, s_1))$ , we have

$$\begin{aligned} \|\widehat{W}_k(s_1) - (P_k^s)^* \widehat{W}_k(s_1)\|_{X'_k} &\leq \epsilon \|\widehat{W}_k(s_1)\|_{X'_k}, \\ \|\widehat{W}_k(s_1) - (P_k^s)^* \widehat{W}_k(s_1)\|_{X_{||}'} &\leq \epsilon \|\widehat{W}_k(s_1)\|_{X_{||}'}. \end{aligned}$$

*Proof.* The evolution operator  $\widehat{\Phi}_k^u(s, s_1)$  satisfies

$$\begin{aligned} \widehat{\Phi}_k^u(s, s_1) &= e^{|k|(s-s_1)}(P_k^s)^* - \int_{-\infty}^s e^{-|k|(s-t)}(P_k^u)^* B(t; \lambda_0, 0)^* \widehat{\Phi}_k^u(t, s_1) dt \\ &\quad + \int_s^{s_1} e^{|k|(s-t)}(P_k^s)^* B(t; \lambda_0, 0)^* \widehat{\Phi}_k^u(t, s_1) dt. \end{aligned}$$

From its definition (12), we immediately see that  $\|B(t; \lambda_0, 0)^*\|_{\mathcal{L}(X'_k)} \leq Ce^{2t}$  and  $\|B(t; \lambda_0, 0)^*\|_{\mathcal{L}(X'_k)} \leq \frac{C}{\sqrt{k^2+1}} e^{2t}$  for some constant  $C$ , independent of  $k$ . Hence, with the dichotomy estimates from Lemma 15, we get that for  $s \leq s_1$  and  $k \in \mathbb{Z} \setminus \{0\}$

$$\begin{aligned} \|\widehat{\Phi}_k^u(s, s_1) - e^{|k|(s-s_1)}(P_k^s)^*\|_{\mathcal{L}(X'_k)} &\leq CK \int_{-\infty}^s e^{-|k|(s-t)} e^{2t} e^{|k|(t-s_1)} dt \\ &\quad + CK \int_s^{s_1} e^{|k|(s-t)} e^{2t} e^{|k|(t-s_1)} dt \\ &\leq \frac{CK}{2} e^{|k|(s-s_1)} \left( \frac{e^{2s_1}}{1+|k|} + e^{2s_1} - e^{2s} \right). \end{aligned}$$

It is now easy to see that we can choose  $N > 0$  large enough and  $\delta > 0$  small enough such that the first inequality of (55) is satisfied.

With the dichotomy estimates from Lemma 17, we get

$$\begin{aligned} \|\widehat{\Phi}_k^u(s, s_1) - e^{|k|(s-s_1)}(P_k^s)^*\|_{\mathcal{L}(X'_k)} &\leq \frac{C}{\sqrt{k^2+1}} K \int_{-\infty}^s e^{-|k|(s-t)} e^{2t} e^{|k|(t-s_1)} dt \\ &\quad + \frac{C}{\sqrt{k^2+1}} K \int_s^{s_1} e^{|k|(s-t)} e^{2t} e^{|k|(t-s_1)} dt \\ &\leq \frac{CK}{2\sqrt{k^2+1}} e^{|k|(s-s_1)} \left( \frac{e^{2s_1}}{(1+|k|)} + e^{2s_1} \right) \\ &\leq \frac{CKe^{2s_1}}{\sqrt{k^2+1}} e^{|k|(s-s_1)}. \end{aligned}$$

It follows that also the second inequality of (55) is valid when  $N$  is sufficiently large.  $\square$

## 7 Matching the core and far field solutions

In the next lemma we show that  $u$  is an eigenfunction that belongs to an embedded eigenvalue  $\lambda$  of  $\mathcal{L} + \rho$  if and only if  $U$  is a solution of (5) such that  $U(s_1) \in \text{Ran } P_+^s(s_1; \lambda, \tilde{\rho}) \cap (\text{Ran } P_-^u(s_1; \lambda, \tilde{\rho}) \oplus P_-^{cb}(s_1; \lambda, \tilde{\rho}))$ .

**Lemma 19.** *Let  $u$  be an  $L^2$  solution of (2). Then the corresponding solution  $U(s)$  of the system (5) is bounded in  $X$  as  $s \rightarrow -\infty$  and decays exponentially with rate  $\lambda^{1/4} - \epsilon$  as  $s \rightarrow +\infty$  in the sense that for any  $\epsilon \in (0, \lambda^{1/4})$  there exists a constant  $K > 0$  such that*

$$\|U(s)\|_{\mathcal{X}^s} \leq Ke^{-(\lambda^{1/4}-\epsilon)s} \quad (56)$$

for every  $s \geq s_1$ . Conversely, if  $U$  is a weak solution of (5) such that  $\|U(s)\|_X$  is bounded as  $s \in J_-$  and such that  $\|U(s)\|_{\mathcal{X}^s}$  decays exponentially as  $s \rightarrow +\infty$  (with any decay rate), then it corresponds to an  $H^4$  solution  $u$  of (2).

*Proof.* If  $u$  is an eigenfunction of (5), it belongs to  $H^4(\mathbb{R}^2)$ . By Lemma 2, the associated solution  $U(s)$  of (5) is bounded in  $X$  as  $s \in J_-$ . In  $J_+$ , for  $s \geq s_3$ , the system (5) reduces to (43) and the decaying solutions of this system are series formed by Bessel functions  $K_k$ ,  $J_k$  and  $Y_k$ . The decay of  $J_k$  and  $Y_k$  is asymptotic to  $1/\sqrt{s}$  as  $s \rightarrow +\infty$ , and so these solutions do not give rise to  $L^2$  solutions of (2). It now follows from Lemma 14 that  $U(s)$  decays exponentially as  $s \rightarrow +\infty$ , in the sense that (56) holds.

Assume that  $U$  is a bounded weak solution of (5) which decays exponentially as  $s \in J_+$ . We need to show that the first component of  $U$  which we denote by  $u$  belongs to  $H^4(\mathbb{R}^2)$  when regarded as a function of the two variables  $(r, \varphi)$  in radial coordinates. As  $U$  is a weak solution of (5), by Definition 1,  $U \in L_{loc}^2(J; Y) \cap H_{loc}^1(J; X)$ . Also,  $\|U\|_X$  is bounded on  $J_-$ , and hence  $U \in L_{loc}^\infty(\mathbb{R}_-; X)$ . From Lemma 2 we know that  $u \in H_{loc}^4(\mathbb{R}^2)$ , so we only need to worry about the decay properties of  $u$  as  $r \rightarrow \infty$  (i.e. as  $s \rightarrow \infty$ ). From Lemma 14 and the definition of  $\mathcal{X}^s$  it follows that for any  $0 < \epsilon < \epsilon_0$  and  $s \geq s_1$

$$\|u(s)\|_{L^2(S^1)} \leq Ke^{-(\lambda^{1/4}-\epsilon)(s-s_1)} \|U(s_1)\|_{\mathcal{X}^{s_1}}$$

and so it is clear that  $u \in L^2(\mathbb{R}^2)$ . From (2) it then follows that  $u \in H^4(\mathbb{R}^2)$ .  $\square$

Recall that  $u_*$  is the radially symmetric eigenfunction associated with the embedded eigenvalue  $\lambda_0$  when  $\tilde{\rho} = 0$ . Let  $U_*$  be the associated solution of (5) with  $\tilde{\rho} = 0$  and  $\lambda = \lambda_0$ , defined for  $s \in \mathbb{R}$ , i.e.  $U_* = (u_*, u'_*, \Delta u_*, (\Delta u_*)')^T$ . Define  $\mathcal{X} := H^1 \times L^2 \times H^1 \times L^2$  and recall that  $X = H^2 \times H^1 \times H^1 \times L^2$ . Let

$$\begin{aligned} E_+^s &:= \{U \in \mathcal{X}; P_+^s(s_1; \lambda_0, 0)U = U\}, \\ E_-^u &:= \{U \in X; P_-^u(s_1; \lambda_0, 0)U = U\}, \\ E_-^{cb} &:= \text{span}\{U_{0,1}(s_1), U_{0,3}(s_1)\} \subset X, \end{aligned}$$

where  $U_{0,j}$  are defined in Lemma 10. Roughly speaking,  $E_+^s$  and  $E_-^u$  consist of the initial values at  $s_1$  of solutions of (5) with  $\tilde{\rho} = 0$  and  $\lambda = \lambda_0$  which decay exponentially as  $s \rightarrow \infty$  and as  $s \rightarrow -\infty$ , respectively, and  $E_-^u \oplus E_-^{cb}$  consists of the bounded solutions on  $J_-$ . Note that the norm of  $\mathcal{X}$  is used for  $E_+^s$ , while the norm of  $X$  is used for  $E_-^u$  and  $E_-^{cb}$ . We have  $E_+^s \cap (E_-^u \oplus E_-^{cb}) = \text{span}\{U_*(s_1)\}$  since  $\lambda_0$  is an eigenvalue of  $\mathcal{L}$  with multiplicity 1.

Next, we introduce a new Hilbert space  $\bar{X}$  such that  $X \subset \bar{X} \subset \mathcal{X}$ , and special solutions  $V_{k,j}$ ,  $k \in \mathbb{Z}$ ,  $j = 1, \dots, 4$ , such that  $\{V_{k,j}(s_1)\}$  is a basis for  $\bar{X}$ .

We have seen that the unperturbed system (5) decouples when  $\tilde{\rho} = 0$  so that the subspaces  $X_k$  and  $\mathcal{X}_k$  are invariant under the flow of (5) with  $\tilde{\rho} = 0$ . For  $k = 0$ , we pick  $V_{0,1}(s_1) := U_*(s_1) \in E_-^{cb} \cap E_+^s$ . Note that there are no other solutions in  $X_0$  which decay exponentially as  $s \rightarrow \infty$ . We pick  $V_{0,2}(s_1) \in E_-^{cb}$  such that  $V_{0,2}(s_1) \notin E_+^s$  and  $E_-^{cb} = \text{span}\{V_{0,1}(s_1), V_{0,2}(s_1)\}$  (thus  $\text{span}\{V_{0,1}(s_1), V_{0,2}(s_1)\} = \text{span}\{U_{0,1}(s_1), U_{0,3}(s_1)\}$ ). Next, we will choose  $V_{0,3}(s_1)$  and  $V_{0,4}(s_1) \in X_0$  such that they belong to the span of  $U_{0,2}(s_1)$  and  $U_{0,4}(s_2)$ . In order to do this, we introduce a dual basis  $W_{0,j}(s_1)$  of  $V_{0,j}(s_1)$  and choose  $W_{0,3}(s_1) := U_*^\perp(s_1)$ , where  $U_*^\perp(s_1) = (-u_4(s_1), u_3(s_1), -u_2(s_1), u_1(s_1))$  and  $u_j(s_1)$  are the components of  $U_*(s_1)$ , while  $W_{0,4}(s_1)$  is any other vector so that  $\text{span}\{W_{0,3}(s_1), W_{0,4}(s_1)\} = \text{span}\{Z_{0,2}(s_1), Z_{0,4}(s_1)\}$ . The remaining vectors  $W_{0,1}(s_1)$ ,  $W_{0,2}(s_1)$ ,  $V_{0,3}(s_1)$  and  $V_{0,4}(s_1)$  are determined by the conditions that  $\{W_{0,j}(s_1); j = 1, \dots, 4\}$  and  $\{V_{0,j}(s_1); j = 1, \dots, 4\}$  are dual bases:

$$\langle W_{0,j}(s_1), V_{0,l}(s_1) \rangle = \delta_{jl}, \quad l = 1, \dots, 4.$$

We use the notation  $E_-^{ca} := \text{span}\{V_{0,3}(s_1), V_{0,4}(s_1)\}$ . We normalise the vectors such that  $\|V_{0,j}(s_1)\|_X = 1$  for  $j = 1, \dots, 4$  and note that, for  $V \in X_0$ , we have  $\|V\|_X = \|V\|_{\mathcal{X}}$ . Define  $W_{0,j}(s)$  so that it satisfies the adjoint system (50) (and hence (51) with  $k = 0$ ) and passes through  $W_{0,j}(s_1)$  for  $s = s_1$ . From (52) and the relation above, it follows immediately that  $\langle W_{0,j}(s), V_{0,l}(s) \rangle = \delta_{jl}$  for all  $s \leq s_1$ . Furthermore, from (53), we conclude that  $W_{0,3}(s)$  and  $W_{0,4}(s)$  are bounded solutions of the adjoint system on  $J_-$ .

Next we consider  $k \neq 0$ . The spaces  $X_k$  and  $\mathcal{X}_k$  are four-dimensional, and  $E_+^s \cap X_k$  and  $E_+^s \cap \mathcal{X}_k$  are one-dimensional,  $E_-^u \cap X_k$  and  $E_-^u \cap \mathcal{X}_k$  are two-dimensional, and  $E_-^u \cap E_+^s \cap X_k = \{0\} = E_-^u \cap E_+^s \cap \mathcal{X}_k$ , since the multiplicity of the eigenfunction  $U_*$  is 1. Using this, we define base vectors in  $X_k$  and  $\mathcal{X}_k$  as follows: For  $k \neq 0$  we pick  $V_{k,1}(s_1) \in E_+^s$  (and hence  $V_{k,1}(s_1) \notin E_-^u$ ). We also pick  $V_{k,2}(s_1)$  and  $V_{k,3}(s_1)$  so that they belong to  $E_-^u$  (and hence do not belong to  $E_+^s$ ). Thus  $\{V_{k,1}(s_1), V_{k,2}(s_1), V_{k,3}(s_1)\}$  span a three-dimensional subspace in the four-dimensional spaces  $X_k$  and  $\mathcal{X}_k$ . We normalize the solutions  $V_{k,j}$  such that for  $k \in \mathbb{Z} \setminus \{0\}$ :  $\|V_{k,1}(s_1)\|_{\mathcal{X}} = 1$  and  $\|V_{k,j}(s_1)\|_X = 1$  for  $j = 2, 3$ . Hence there exists a unique (up to multiplication by a unimodular constant) vector  $W_{k,4} \in \mathcal{X}'_k$  such that

$$\langle W_{k,4}, V_{k,j} \rangle = 0 \text{ for } j = 1, 2, 3 \text{ and } \|W_{k,4}(s_1)\|_{\mathcal{X}'_k} = 1.$$

Then  $\langle W_{k,4}(s_1), V \rangle = 0$  for  $V \in \text{Ran}(\Phi_{k,+}^s(s_1, s_1)) + \text{Ran}(\Phi_{k,-}^u(s_1, s_1))$  and hence  $W_{k,4}(s_1) \in \text{Ran}(\Phi_{k,+}^{cu}(s_1, s_1)^*) \cap \text{Ran}(\Phi_{k,-}^{cs}(s_1, s_1)^*) = \text{Ran}(\widehat{\Phi}_{k,+}^{cs}(s_1, s_1)) \cap \text{Ran}(\widehat{\Phi}_{k,-}^u(s_1, s_1))$ . We take the one remaining solution in  $X_k$  and  $\mathcal{X}_k$  such that  $\langle W_{k,4}(s_1), V_{k,4}(s_1) \rangle = 1$  and  $\|V_{k,4}(s_1)\|_{\mathcal{X}} = 1$ . Then  $V_{k,4}(s_1) \notin E_-^u \cup E_+^s$  as  $\langle W_{k,4}(s_1), V \rangle = 0$  for  $V \in E_-^u + E_+^s$ .

Let  $\bar{X}$  be defined by

$$\bar{X} := \left\{ U = \sum_{\substack{k \in \mathbb{Z} \\ j=1, \dots, 4}} a_{k,j} V_{k,j}(s_1) \in \mathcal{X}; \|U\|_{\bar{X}}^2 := \left\| \sum_{\substack{k \in \mathbb{Z} \\ j=1, 4}} a_{k,j} V_{k,j}(s_1) \right\|_{\mathcal{X}}^2 + \left\| \sum_{\substack{k \in \mathbb{Z} \\ j=2, 3}} a_{k,j} V_{k,j}(s_1) \right\|_{\mathcal{X}}^2 < \infty \right\}.$$

Note that  $\overline{X}$  is the direct sum of two Hilbert spaces, which are closed subspaces of  $\mathcal{X}$  and  $X$ , respectively. It follows that  $\overline{X}$  is a Hilbert space.

Note that  $E_+^s$  and  $E_-^u$  are both closed subspaces of  $\overline{X}$ : Indeed,

$$\begin{aligned} E_+^s &= \text{cl}_{\overline{X}} \text{span}\{V_{k,1}(s_1)\}_{k \in \mathbb{Z}} = \text{cl}_{\mathcal{X}} \text{span}\{V_{k,1}(s_1)\}_{k \in \mathbb{Z}} \\ E_-^u &= \text{cl}_{\overline{X}} \text{span}\{V_{k,2}(s_1), V_{k,3}(s_1)\}_{k \in \mathbb{Z} \setminus \{0\}} = \text{cl}_X \text{span}\{V_{k,2}(s_1), V_{k,3}(s_1)\}_{k \in \mathbb{Z} \setminus \{0\}}. \end{aligned}$$

It is clear that  $E_-^{cb}$  is a closed subspace of  $\overline{X}$  since it is finite-dimensional.

Define  $\iota : E_+^s \times (E_-^u \oplus E_-^{cb}) \times \mathbb{R} \times \tilde{\mathcal{R}} \rightarrow \overline{X}$  by

$$\begin{aligned} \iota(U_0^s, U_0^u + U_0^{cb}; \lambda, \tilde{\rho}) &:= P_-^u(s_1; \lambda, \tilde{\rho})U_0^u + P_-^{cb}(s_1; \lambda, \tilde{\rho})U_0^{cb} - P_+^s(s_1; \lambda, \tilde{\rho})U_0^s \\ &= P_-^u(s_1; \lambda, \tilde{\rho})U_0^u + P_-^{cb}(s_1; \lambda, \tilde{\rho})U_0^{cb} - P_+^s(s_1; \lambda, 0)U_0^s, \end{aligned} \quad (57)$$

where we recall the definition (30) and note that the last equality holds since  $\tilde{\rho}(s) = 0$  for  $s \geq s_1$ . We will see that the range of  $\iota$  is a subspace of  $\overline{X}$ , and that  $\iota$  is smooth into this space. For this we need a more explicit formula for  $\iota$ . By (30), (31) and (49) evaluated at  $s = t = s_1$ , we have (see (19) for the definition of  $B_0$ )

$$P_-^{cb}(s_1; \lambda, \tilde{\rho})U_0^{cb} = U_0^{cb} + \int_{-\infty}^{s_1} \Phi_-^{cs}(s_1, \tau; \lambda_0, 0)r'(\tau)^2(\lambda - \lambda_0 - \tilde{\rho}(\tau))B_0U_-^{cb}(\tau; \lambda, \tilde{\rho}, U_0^{cb})d\tau \quad (58)$$

$$P_-^u(s_1; \lambda, \tilde{\rho}) = P_-^u(s_1; \lambda_0, 0) + \int_{-\infty}^{s_1} \Phi_-^{cs}(s_1, \tau; \lambda_0, 0)r'(\tau)^2(\lambda - \lambda_0 - \tilde{\rho}(\tau))B_0\Phi_-^u(\tau, s_1; \lambda, \tilde{\rho})d\tau \quad (59)$$

$$P_+^s(s_1; \lambda, 0) = P_+^s(s_1; \lambda_0, 0) - \int_{s_1}^{\infty} \Phi_+^{cu}(s_1, \tau; \lambda_0, 0)r'(\tau)^2(\lambda - \lambda_0)B_0\Phi_+^s(\tau, s_1; \lambda, 0)d\tau, \quad (60)$$

where we recall that  $r(\tau) = e^\tau$  for  $\tau < s_2$ , so that  $r'(\tau)^2 = e^{2\tau}$  in this interval. Hence we may write

$$\begin{aligned} \iota(U_0^s, U_0^u + U_0^{cb}; \lambda, \tilde{\rho}) &= U_0^u + U_0^{cb} - U_0^s + \int_{s_1}^{\infty} \Phi_+^{cu}(s_1, \tau; \lambda_0, 0)r'(\tau)^2(\lambda - \lambda_0)B_0\Phi_+^s(\tau, s_1; \lambda, 0)U_0^s d\tau \\ &\quad + \int_{-\infty}^{s_1} \Phi_-^{cs}(s_1, \tau; \lambda_0, 0)e^{2\tau}(\lambda - \lambda_0 - \tilde{\rho}(\tau))B_0 [U_-^{cb}(\tau; \lambda, \tilde{\rho}, U_0^{cb}) + \Phi_-^u(\tau, s_1; \lambda, \tilde{\rho})U_0^u] d\tau. \end{aligned} \quad (61)$$

**Lemma 20.** *The map  $\iota : E_+^s \times (E_-^u \oplus E_-^{cb}) \times \mathbb{R} \times \tilde{\mathcal{R}} \rightarrow \overline{X}$  is smooth.*

*Proof.* We have seen in Theorem 2 that  $(U_0^u, \lambda, \tilde{\rho}) \mapsto P_-^u(s_1; \lambda, \tilde{\rho})U_0^u$  and  $(U_0^{cb}, \lambda, \tilde{\rho}) \mapsto P_-^{cu}(s_1; \lambda, \tilde{\rho})U_0^{cb}$  are smooth as functions from  $E_-^u \times \mathbb{R} \times \tilde{\mathcal{R}}$  to  $X \subset \overline{X}$  and from  $E_-^{cb} \times \mathbb{R} \times \tilde{\mathcal{R}}$  to  $X \subset \overline{X}$ , respectively.

Hence it suffices to prove that  $P_+^s(s_1; \lambda, 0)$  is smooth from  $E_+^s \times \mathbb{R}$  to  $\overline{X}$ . We do this by studying the terms of (60) separately. It is clear that  $U_0^s \in \overline{X}$ .

Next, we study the integral term, and note that by Lemma 14, for  $\tau \geq s_1$ ,  $\Phi_+^s(\tau, s_1; \lambda, 0) : \mathcal{X}^{s_1} \rightarrow \mathcal{X}^\tau$  with norm bounded by  $Ke^{(\lambda^{1/4} - \epsilon)(\tau - s_1)}$  and  $\Phi_+^{cu}(s_1, \tau; \lambda_0, 0) : \mathcal{Y}^\tau \rightarrow \mathcal{Y}^{s_1}$  with norm bounded by  $Ke^{\epsilon(\tau - s_1)}$ . Recall that  $\mathcal{X}^s = H^1 \times L^2 \times H^1 \times L^2$  and  $\mathcal{Y}^s = H^2 \times H^1 \times H^2 \times H^1$  with  $s$ -dependent norms. Thus  $B_0 : \mathcal{X}^\tau \rightarrow \mathcal{Y}^\tau$  is bounded with norm  $\tau$ . Using the exponential estimates of Lemma 14 and that  $\mathcal{Y}^{s_1} \subset X \subset \overline{X} \subset \mathcal{X}^{s_1}$ , there exists a constant  $C > 0$  such that for any  $\epsilon > 0$  sufficiently small

$$\begin{aligned} \left\| \int_{s_1}^{\infty} \Phi_+^{cu}(s_1, \tau; \lambda_0, 0)r'(\tau)^2B_0\Phi_+^s(\tau, s_1; \lambda, 0)U_0^s d\tau \right\|_{\overline{X}} &\leq C \left\| \int_{s_1}^{\infty} \Phi_+^{cu}(s_1, \tau; \lambda_0, 0)B_0\Phi_+^s(\tau, s_1; \lambda, 0)U_0^s d\tau \right\|_{\mathcal{Y}^{s_1}} \\ &\leq CK^2 e^{(\lambda^{1/4} - 2\epsilon)s_1} \int_{s_1}^{\infty} \tau e^{-(\lambda^{1/4} - 2\epsilon)\tau} d\tau \|U_0^s\|_{\mathcal{X}^{s_1}} \\ &\leq \frac{\tilde{C}K^2}{(\lambda^{1/4} - 2\epsilon)^2} \|U_0^s\|_{\overline{X}}, \end{aligned}$$

for some constant  $\tilde{C}$ . To show that the integral term is smooth in  $\lambda$  into  $\overline{X}$ , it suffices to prove that for  $n \geq 1$

$$\int_{s_1}^{\infty} \Phi_+^{cu}(s_1, \tau; \lambda_0, 0)r'(\tau)^2B_0 \frac{d^n}{d\lambda^n} \Phi_+^s(\tau, s_1; \lambda, 0)U_0^s d\tau$$

belongs to  $\overline{X}$ . This follows since, by Lemma 14,  $\frac{d^n}{d\lambda^n} \Phi_+^s(\tau, s_1; \lambda, 0)$  satisfies a similar exponential decay estimate as  $\Phi_+^s(\tau, s_1; \lambda, 0)$  does. It follows as above that the integral in question converges in  $\mathcal{Y}^{s_1} \subset \overline{X}$ . Smoothness in  $U_0^s$  is immediate, since  $\iota$  is bounded and linear in  $U_0^s$  into  $\overline{X}$ .  $\square$



**Lemma 21.** *The operator  $\Delta^2 + \theta + \rho$  has an embedded eigenvalue  $\lambda > 0$  if and only if there exist  $U_0^s \in E_+^s$ ,  $U_0^u \in E_-^u$  and  $U_0^{cb} \in E_-^{cb}$  such that*

$$\iota(U_0^s, U_0^u + U_0^{cb}; \lambda, \tilde{\rho}) = 0. \quad (62)$$

*Proof.* If  $\lambda$  is an eigenvalue of  $\mathcal{L} + \rho$ , then by Lemma 19, the corresponding solution of the system (5) is bounded as  $s \rightarrow -\infty$  and decays exponentially as  $s \rightarrow +\infty$ . Hence there exists a solution of (5) with initial condition

$$P_+^s(s_1; \lambda, \tilde{\rho})U_0^s = P_-^u(s_1; \lambda, \tilde{\rho})U_0^u + P_-^{cb}(s_1; \lambda, \tilde{\rho})U_0^{cb}$$

at  $s = s_1$ , i.e. (62) holds.

Conversely, suppose that (62) is satisfied for some  $(U_0^s, U_0^u + U_0^{cb}, \lambda, \tilde{\rho}) \in E_+^s \times (E_-^u \oplus E_-^{cb}) \times \mathbb{R} \times \tilde{\mathcal{R}}$ . Then there exists a solution of (5) with initial condition  $U_0^s = U_0^u + U_0^{cb}$ . By Lemma 19, this implies that  $\lambda$  is an eigenvalue of  $\mathcal{L} + \rho$ .  $\square$

**Lemma 22.** *The subspaces  $E_-^{cb} \oplus E_-^u$  and  $E_+^s$  have complements in  $\overline{X}$  denoted by  $E_-^{ca} \oplus E_-^s$  and  $E_+^{cu}$ . Moreover,  $(E_-^{ca} \oplus E_-^s) \cap E_+^{cu}$  is infinite-dimensional and has a basis with elements  $V_{0,3}(s_1), V_{0,4}(s_1) \in X_0, V_{k,4}(s_1) \in X_k, k \in \mathbb{Z} \setminus \{0\}$ .*

*Proof.* Recall that  $E_-^{ca} = \text{span}\{V_{0,3}(s_1), V_{0,4}(s_1)\}$ . Let

$$\begin{aligned} E_-^s &:= \overline{\text{span}\{V_{k,1}(s_1), V_{k,4}(s_1); k \in \mathbb{Z} \setminus \{0\}\}}, \\ E_+^{cu} &:= \overline{\text{span}\{V_{k,2}(s_1), V_{k,3}(s_1), V_{k,4}(s_1); k \in \mathbb{Z}\}}, \end{aligned}$$

where the closures are taken in  $\overline{X}$ . It is easy to see that these spaces have the desired properties.  $\square$

Let  $Q$  be the projection in  $\overline{X}$  onto  $\text{Ran } \iota(\cdot, \cdot; \lambda_0, 0) = E_+^s + (E_-^u \oplus E_-^{cb})$  such that

$$\ker Q = E_+^{cu} \cap (E_-^{ca} \oplus E_-^s).$$

Note that  $\text{Ran } Q$  and  $\ker Q$  are closed subspaces of  $\overline{X}$ , and  $Q$  is therefore continuous.

Equation (62) is equivalent to the pair of equations

$$\begin{aligned} Q\iota(U_0^s, U_0^u + U_0^{cb}; \lambda, \tilde{\rho}) &= 0, \\ (I - Q)\iota(U_0^s, U_0^u + U_0^{cb}; \lambda, \tilde{\rho}) &= 0. \end{aligned} \quad (63)$$

**Lemma 23.** *For  $(\lambda, \tilde{\rho})$  in a neighbourhood of  $(\lambda_0, 0) \in \mathbb{R} \times \tilde{\mathcal{R}}$ , the first equation of (63) has a unique (up to constant multiples) nonzero solution  $(U_0^s, U_0^u + U_0^{cb})$  which depends smoothly on  $\lambda$  and  $\tilde{\rho}$  in this neighbourhood. We write  $U_0^s(\lambda, \tilde{\rho})$ ,  $U_0^u(\lambda, \tilde{\rho})$  and  $U_0^{cb}(\lambda, \tilde{\rho})$ . Furthermore,  $U_0^s(\lambda_0, 0) = U_*(s_1) = U_0^{cb}(\lambda_0, 0)$  and  $U_0^u(\lambda_0, 0) = 0$ .*

*Proof.* For  $(\lambda, \tilde{\rho})$  fixed,  $Q\iota$  is a linear mapping from  $E_+^s \times (E_-^u \oplus E_-^{cb})$  to  $\text{Ran } Q$ . It is clear that

$$\ker Q\iota(\cdot, \cdot; \lambda_0, 0) = \text{span}\{(U_*(s_1), U_*(s_1))\}.$$

By Lemma 20 and since  $\text{Ran } Q$  is closed, it follows that  $Q\iota$  is a smooth mapping in its arguments. Let  $D$  be an affine hyperplane of  $E_+^s \times (E_-^u \oplus E_-^{cb})$  such that  $D \cap \text{span}\{(U_*(s_1), U_*(s_1))\} = \{(U_*(s_1), U_*(s_1))\}$ . The implicit function theorem then implies that for  $(\lambda, \tilde{\rho})$  close to  $(\lambda_0, 0)$  the first equation of (63) has a unique solution  $(U_0^s, U_0^u + U_0^{cb}) = (U_0^s(\lambda, \tilde{\rho}), U_0^u(\lambda, \tilde{\rho}) + U_0^{cb}(\lambda, \tilde{\rho})) \in D$  in a neighbourhood of  $(U_*(s_1), U_*(s_1))$ . Moreover,  $U_0^s, U_0^u$  and  $U_0^{cb}$  are smooth in their arguments.  $\square$

For  $(\lambda, \tilde{\rho})$  in the neighbourhood obtained in Lemma 23, we let

$$\begin{aligned} F(\lambda, \tilde{\rho}) &:= (I - Q)\iota(U_0^s(\lambda, \tilde{\rho}), U_0^u(\lambda, \tilde{\rho}) + U_0^{cb}(\lambda, \tilde{\rho})) \\ &= (I - Q) \int_{-\infty}^{s_1} \Phi_-^{cs}(s_1, \tau; \lambda_0, 0) e^{2\tau} (\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 [U_0^{cb}(\tau; \lambda, \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda, \tilde{\rho}) U_0^u(\lambda, \tilde{\rho})] d\tau \\ &\quad + (I - Q) \int_{s_1}^{\infty} \Phi_+^{cu}(s_1, \tau; \lambda_0, 0) r'(\tau)^2 (\lambda - \lambda_0) B_0 \Phi_+^s(\tau, s_1; \lambda, 0) U_0^s(\lambda, \tilde{\rho}) d\tau \end{aligned}$$

where  $U_0^{cb}(s; \lambda, \tilde{\rho})$  corresponds to  $U^{cb}(s, \lambda, \tilde{\rho}, U_0^{cb}(\lambda, \tilde{\rho}))$  so that  $U_0^{cb}(s; \lambda_0, 0) = U_*(s)$ . We see that solving (63) is equivalent to solving  $F(\lambda, \tilde{\rho}) = 0$ .

On  $\text{Ran}(I - Q) \subset \bar{\mathcal{X}}$ , the  $\bar{\mathcal{X}}$ -norm is the same as the  $\mathcal{X}$ -norm, and so we solve  $F(\lambda, \tilde{\rho}) = 0$  in  $\mathcal{X}$ . For  $k \in \mathbb{Z} \setminus \{0\}$  let  $F_k(\lambda, \tilde{\rho}) := \langle W_{k,4}(s_1), F(\lambda, \tilde{\rho}) \rangle$ , and for  $k = 0$  and  $j = 3, 4$  we let  $F_{0,j}(\lambda, \tilde{\rho}) := \langle W_{0,j}(s_1), F(\lambda, \tilde{\rho}) \rangle$ . Define  $W_{k,4}(s)$  so that it satisfies the adjoint system (50) (and hence (51)) and passes through  $W_{k,4}(s_1)$  for  $s = s_1$ . As  $W_{k,4}(s_1) \in \text{Ran}(\Phi_{k,+}^{cu}(s_1, s_1)^*) \cap \text{Ran}(\Phi_{k,-}^s(s_1, s_1)^*) = \text{Ran}(\widehat{\Phi}_{k,+}^{cs}(s_1, s_1)) \cap \text{Ran}(\widehat{\Phi}_{k,-}^u(s_1, s_1))$ , we get for  $k \neq 0$

$$\begin{aligned} F_k(\lambda, \tilde{\rho}) &= \int_{-\infty}^{s_1} \langle W_{k,4}(s_1), \Phi_-^{cs}(s_1, \tau; \lambda_0, 0) e^{2\tau}(\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 [U_0^{cb}(\tau; \lambda, \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda, \tilde{\rho}) U_0^u(\lambda, \tilde{\rho})] \rangle d\tau \\ &\quad + \int_{s_1}^{\infty} \langle W_{k,4}(s_1), \Phi_+^{cu}(s_1, \tau; \lambda_0, 0) r'(\tau)^2 (\lambda - \lambda_0) B_0 \Phi_+^s(\tau, s_1; \lambda, 0) U_0^s(\lambda, \tilde{\rho}) \rangle d\tau \\ &= \int_{-\infty}^{s_1} \langle W_{k,4}(\tau), e^{2\tau}(\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 [U_0^{cb}(\tau; \lambda, \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda, \tilde{\rho}) U_0^u(\lambda, \tilde{\rho})] \rangle d\tau \\ &\quad + \int_{s_1}^{\infty} \langle W_{k,4}(\tau), r'(\tau)^2 (\lambda - \lambda_0) B_0 \Phi_+^s(\tau, s_1; \lambda, 0) U_0^s(\lambda, \tilde{\rho}) \rangle d\tau \end{aligned}$$

and for  $k = 0$  and  $j = 3, 4$  we have similarly

$$\begin{aligned} F_{0,j}(\lambda, \tilde{\rho}) &= \int_{-\infty}^{s_1} \langle W_{0,j}(\tau), e^{2\tau}(\lambda - \lambda_0 - \tilde{\rho}(\tau)) B_0 [U_0^{cb}(\tau; \lambda, \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda, \tilde{\rho}) U_0^u(\lambda, \tilde{\rho})] \rangle d\tau \\ &\quad + \int_{s_1}^{\infty} \langle W_{0,j}(\tau), r'(\tau)^2 (\lambda - \lambda_0) B_0 \Phi_+^s(\tau, s_1; \lambda, 0) U_0^s(\lambda, \tilde{\rho}) \rangle d\tau \end{aligned} \quad (64)$$

**Lemma 24.** *For  $(\lambda, \tilde{\rho})$  in a neighbourhood of  $(\lambda_0, 0) \in \mathbb{R} \times \tilde{\mathcal{R}}$ , the equation (63) has a nontrivial solution  $(U_0^s(\lambda, \tilde{\rho}), U_0^u(\lambda, \tilde{\rho}) + U_0^{cb}(\lambda, \tilde{\rho}), \lambda, \tilde{\rho}) \in E_+^s \times (E_-^u \oplus E_-^{cb}) \times \mathbb{R} \times \tilde{\mathcal{R}}$  if and only if  $F_k(\lambda, \tilde{\rho}) = 0$  for  $k \in \mathbb{Z} \setminus \{0\}$  and  $F_{0,j}(\lambda, \tilde{\rho}) = 0$  for  $j = 3, 4$ .*

*Proof.* Suppose that  $(U_0^s(\lambda, \tilde{\rho}), U_0^u(\lambda, \tilde{\rho}) + U_0^{cb}(\lambda, \tilde{\rho}), \lambda, \tilde{\rho}) \in E_+^s \times (E_-^u \oplus E_-^{cb}) \times \mathbb{R} \times \tilde{\mathcal{R}}$  solves (63). It is then clear from the definition of  $F_k$  that  $F_k(\lambda, \tilde{\rho}) = 0$  for  $k \in \mathbb{Z} \setminus \{0\}$  and  $F_{0,j}(\lambda, \tilde{\rho}) = 0$  for  $j = 3, 4$ . Conversely, let  $F_k(\lambda, \tilde{\rho}) = 0$  for  $k \in \mathbb{Z} \setminus \{0\}$  and  $F_{0,j}(\lambda, \tilde{\rho}) = 0$  for  $j = 3, 4$ . By Lemma 23, the first equation of (63) is satisfied, so it remains to check the second equation of (63). Recall that the basis vectors in  $\text{Ran}(I - Q)$  are  $V_{k,j}(s_1)$ , where  $j = 4$  for  $k \neq 0$  and  $j = 3, 4$  for  $k = 0$ . The coefficients of  $(I - Q)\iota(U_0^s, U_0^u + U_0^{cb}; \lambda, \tilde{\rho})$  with respect to this basis are then  $F_{0,j}$  ( $j = 3, 4$ ) and  $F_k$ ,  $k \in \mathbb{Z} \setminus \{0\}$ . Since all these coefficients vanish, the conclusion follows.  $\square$

**Lemma 25.** *The equation  $F_{0,3}(\lambda, \tilde{\rho}) = 0$  defines  $\lambda$  as a smooth function of  $\tilde{\rho}$  in a neighbourhood of  $\tilde{\rho} = 0$  such that  $\lambda(0) = \lambda_0$ . Furthermore,*

$$\lambda'(0)\tilde{\rho} = - \frac{\int_{-\infty}^{s_1} u_*(\tau)^2 \widehat{\rho}_0(\tau) e^{2\tau} d\tau}{\int_{-\infty}^{\infty} \bar{u}_*(\tau)^2 r(\tau) r'(\tau) d\tau}$$

where  $\widehat{\rho}_0$  is the Fourier coefficient of  $\tilde{\rho}$  corresponding to  $k = 0$ .

*Proof.* By Lemma 20 it follows that  $F_{0,3}$  is a smooth function of  $\lambda$  and  $\tilde{\rho}$  in a neighbourhood of  $(\lambda_0, 0)$ . Note that

$$\frac{\partial F_{0,3}}{\partial \lambda}(\lambda_0, 0) = \int_{-\infty}^{\infty} \langle W_{0,3}(\tau), r'(\tau)^2 B_0 U_*(\tau) \rangle d\tau = \int_{-\infty}^{\infty} \overline{w_{0,3}(\tau)} u_*(s) r'(\tau)^2 ds,$$

where we have used that  $U_0^s(\lambda_0, 0) = U_*(s) = U_0^{cb}(\lambda_0, 0)$  and  $U_0^u(\lambda_0, 0) = 0$ . We recall that  $W_{0,3}(s_1) = U_*^\perp(s_1)$  and that  $W_{0,3}(s)$  satisfies the adjoint system (50) for  $s \in \mathbb{R}$ . It can be verified that

$$W_{0,3}(s) = \frac{r(s)}{r'(s)} U_*^\perp(s)$$

for  $s \in \mathbb{R}$ , where  $U^\perp = (-u_4, u_3, -u_2, u_1)$ , and  $u_j$  are the components of  $U$ ,  $j = 1, \dots, 4$ . Hence

$$\frac{\partial F_{0,3}}{\partial \lambda}(\lambda_0, 0) = \int_{-\infty}^{\infty} u_*(s)^2 r(\tau) r'(\tau) ds > 0. \quad (65)$$

The last inequality follows since the integral is positive (using that  $u_*$  is an eigenfunction).

Since  $\partial F_{0,3}/\partial\lambda(\lambda_0, 0) \neq 0$ , we can solve the equation  $F_{0,3}(\lambda, \tilde{\rho}) = 0$  by the implicit function theorem for  $\lambda$  as a function of  $\tilde{\rho}$ , and this solution is a smooth function  $\lambda(\tilde{\rho})$ , defined in a neighbourhood of  $\tilde{\rho} = 0$ , such that  $\lambda(0) = \lambda_0$ , and  $\lambda'(0)$  is given by

$$\lambda'(0)\tilde{\rho} = -\frac{\partial F_{0,3}}{\partial\tilde{\rho}}(\lambda_0, 0)\tilde{\rho} / \frac{\partial F_{0,3}}{\partial\lambda}(\lambda_0, 0) = -\frac{\int_{-\infty}^{s_1} u_*(\tau)^2 \widehat{\tilde{\rho}}_0(\tau) e^{2\tau} d\tau}{\int_{-\infty}^{\infty} u_*(\tau)^2 r(\tau) r'(\tau) d\tau}$$

as claimed.  $\square$

Since we have solved  $F_{0,3} = 0$  for  $\lambda$  in terms of  $\tilde{\rho}$ , the remaining equation corresponding to  $k = 0$  is  $F_{0,4}(\lambda(\tilde{\rho}), \tilde{\rho}) = 0$ . We define

$$\begin{aligned} G_0(\tilde{\rho}) &:= \int_{-\infty}^{s_1} \langle W_{0,4}(\tau), e^{2\tau}(\lambda(\tilde{\rho}) - \lambda_0 - \tilde{\rho}(\tau)) B_0 [U_0^{cb}(\tau; \lambda(\tilde{\rho}), \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho}) U_0^u(\lambda(\tilde{\rho}), \tilde{\rho})] \rangle d\tau \\ &\quad + \int_{s_1}^{\infty} \langle W_{0,4}(\tau), r'(\tau)^2 (\lambda(\tilde{\rho}) - \lambda_0) B_0 \Phi_+^s(\tau, s_1; \lambda(\tilde{\rho}), 0) U_0^s(\lambda(\tilde{\rho}), \tilde{\rho}) \rangle d\tau, \end{aligned}$$

and for  $k \neq 0$ ,

$$\begin{aligned} G_k(\tilde{\rho}) &:= F_k(\lambda(\tilde{\rho}), \tilde{\rho}) \\ &= \int_{-\infty}^{s_1} \langle W_{k,4}(\tau), e^{2\tau}(\lambda(\tilde{\rho}) - \lambda_0 - \tilde{\rho}(\tau)) B_0 [U_0^{cb}(\tau; \lambda(\tilde{\rho}), \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho}) U_0^u(\lambda(\tilde{\rho}), \tilde{\rho})] \rangle d\tau \\ &\quad + \int_{s_1}^{\infty} \langle W_{k,4}(\tau), r'(\tau)^2 (\lambda(\tilde{\rho}) - \lambda_0) B_0 \Phi_+^s(\tau, s_1; \lambda(\tilde{\rho}), 0) U_0^s(\lambda(\tilde{\rho}), \tilde{\rho}) \rangle d\tau. \end{aligned}$$

**Lemma 26.** *The mapping  $\mathcal{G} : \tilde{\mathcal{R}} \rightarrow l_1^2$  defined by*

$$\mathcal{G}(\tilde{\rho}) = \{G_k(\tilde{\rho})\}_{k \in \mathbb{Z}}$$

*is smooth.*

*Proof.* We first verify that the range of  $\mathcal{G}$  belongs to  $l_1^2$ . To do this, we split the expression for  $G_k(\tilde{\rho})$  ( $k \neq 0$ ) into three terms, which we deal with separately:

$$\begin{aligned} G_k(\tilde{\rho}) &= (\lambda(\tilde{\rho}) - \lambda_0) \int_{-\infty}^{s_1} \langle W_{k,4}(\tau), e^{2\tau} U^{cu}(\tau) \rangle d\tau - \int_{-\infty}^{s_1} \langle W_{k,4}(\tau), e^{2\tau} \tilde{\rho}(\tau) U^{cu}(\tau) \rangle d\tau \\ &\quad + (\lambda(\tilde{\rho}) - \lambda_0) \int_{s_1}^{\infty} \langle W_{k,4}(\tau), r'(\tau)^2 U^s(\tau) \rangle d\tau, \end{aligned} \tag{66}$$

where we used the notation

$$\begin{aligned} U^{cu}(\tau) &:= [U_0^{cb}(\tau; \lambda(\tilde{\rho}), \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho}) U_0^u(\lambda(\tilde{\rho}), \tilde{\rho})], \\ U^s(\tau) &:= \Phi_+^s(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho}) U_0^s(\lambda(\tilde{\rho}), \tilde{\rho}). \end{aligned}$$

Then  $B_0 U^{cu}(\tau) \in \{0\} \times \{0\} \times \{0\} \times H^2(S^1)$  and  $B_0 U^s(\tau) \in \{0\} \times \{0\} \times \{0\} \times H^1(S^1)$ . Furthermore, by its construction, we have that  $W_{k,4}(s_1) \in \text{Ran}(\widehat{\Phi}_{k,-}^u(s_1, s_1)) \cap \text{Ran}(\widehat{\Phi}_{k,+}^{cs}(s_1, s_1))$  for all  $k \in \mathbb{Z} \setminus \{0\}$ . Thus Lemma 16 implies that for any  $\epsilon > 0$  there exists a constant  $K$  such that for every  $k \in \mathbb{Z} \setminus \{0\}$  and  $s \geq s_1$ ,

$$\|W_{k,4}(s)\|_{(\mathcal{X}^s)'} \leq K e^{\epsilon(s-s_1)}$$

as the norms on  $\mathcal{X}'$  and  $(\mathcal{X}^{s_1})'$  are equivalent and  $\|W_{k,4}(s_1)\|_{\mathcal{X}'} = 1$ .

Now observe that  $B_0 U^s(\tau)$  vanishes for all components except the last one, so we only need an estimate on the last component of  $W_{k,4}(s)$ , which we denote by  $w_{k,4}$ . Then the estimate above gives that there exists a constant  $K$  independent of  $k$  such that

$$|w_{k,4}(s)| \leq K e^{\epsilon(s-s_1)}, \text{ for all } k \in \mathbb{Z} \setminus \{0\} \text{ and } s \geq s_1, \tag{67}$$

as  $\mathcal{X}' \equiv H^{-1} \times L^2 \times H^{-1} \times L^2$ . Similarly, from  $W_{k,4}(s_1) \in \text{Ran}(\widehat{\Phi}_{k,-}^u(s_1, s_1))$ , Lemma 15,  $X' = H^{-2} \times H^{-1} \times H^{-1} \times L^2$  and  $\|W_{k,4}(s_1)\|_{\mathcal{X}'} \leq \|W_{k,4}(s_1)\|_{\mathcal{X}'} = 1$ , it follows that there is some constant  $K$  such that

$$|w_{k,4}(s)| \leq Ke^{|k|(s-s_1)} \text{ for all } k \in \mathbb{Z} \setminus \{0\} \text{ and } s \leq s_1. \quad (68)$$

First we look at the last integral in (66). Let  $\widehat{u}_k^s(\tau)$  be the first component of the  $k$ -th Fourier coefficient of  $\Phi_+^s(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho})U_0^s(\lambda(\tilde{\rho}), \tilde{\rho})$ , then the definition of  $B_0$  in (19) gives

$$\int_{s_1}^{\infty} \langle W_{k,4}(\tau), r'(\tau)^2 B_0 U^s(\tau) \rangle d\tau = \int_{s_1}^{\infty} r'(\tau)^2 \overline{w_{k,4}(\tau)} \widehat{u}_k^s(\tau) d\tau.$$

From Lemma 14, it follows that, for any  $\epsilon > 0$  and  $\tau \geq s_1$ ,

$$\|\Phi_+^s(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho})U_0^s(\lambda(\tilde{\rho}), \tilde{\rho})\|_{\mathcal{X}^\tau} \leq Ke^{-(\lambda(\tilde{\rho})^{1/4}-\epsilon)(\tau-s_1)} \|U_0^s(\lambda(\tilde{\rho}), \tilde{\rho})\|_{\mathcal{X}^{s_1}}.$$

This implies for the Fourier coefficients  $\widehat{u}_k^s$  that

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \left(1 + \frac{k^2}{\tau^2}\right) \left|\widehat{u}_k^s(\tau)\right|^2 \leq K^2 e^{-2(\lambda(\tilde{\rho})^{1/4}-\epsilon)(\tau-s_1)} \|U_0^s(\lambda(\tilde{\rho}), \tilde{\rho})\|_{\mathcal{X}^{s_1}}^2.$$

Combining this with (67), we see that for any  $\tilde{\rho}$ , we have

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2) \left( \int_{s_1}^{\infty} r'(\tau)^2 \overline{w_{k,4}(\tau)} \widehat{u}_k^s(\tau) d\tau \right)^2 &\leq C \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2) \left( \int_{s_1}^{\infty} Ke^{\epsilon(\tau-s_1)} |\widehat{u}_k^s(\tau)| d\tau \right)^2 \\ &\leq C \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \int_{s_1}^{\infty} e^{-\epsilon(\tau-s_1)} d\tau \right) \left( \int_{s_1}^{\infty} e^{3\epsilon(\tau-s_1)} (1+k^2) |\widehat{u}_k^s(\tau)|^2 d\tau \right) \\ &\leq \frac{C}{\epsilon} \int_{s_1}^{\infty} e^{3\epsilon(\tau-s_1)} \sum_{k \in \mathbb{Z} \setminus \{0\}} 2\tau^2 \left(1 + \frac{k^2}{\tau^2}\right) |\widehat{u}_k^s(\tau)|^2 d\tau \\ &\leq C \int_{s_1}^{\infty} \tau^2 e^{-2(\lambda(\tilde{\rho})^{1/4}-4\epsilon)(\tau-s_1)} \|U_0^s(\lambda(\tilde{\rho}), \tilde{\rho})\|_{\mathcal{X}^{s_1}}^2 d\tau \leq C, \end{aligned}$$

where  $C = C(\epsilon)$  denotes the different constants occurring.

Next, we look at the first integral in (66). Let  $\widehat{u}_k^{cu}$  be the first component of the  $k$ -th Fourier coefficient of  $U_0^{cb}(\tau; \lambda(\tilde{\rho}), \tilde{\rho}) + \Phi_-^u(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho})U_0^u(\lambda(\tilde{\rho}), \tilde{\rho})$ . The definition of  $B_0$  gives that the first integral can be written as

$$\int_{-\infty}^{s_1} \langle W_{k,4}(\tau), e^{2\tau} B_0 U^{cu}(\tau) \rangle d\tau = \int_{-\infty}^{s_1} e^{2\tau} \overline{w_{k,4}(\tau)} \widehat{u}_k^{cu}(\tau) d\tau.$$

As  $\Phi_-^u$  leads to solutions with an  $X$ -norm that is exponentially decaying at  $-\infty$  and  $U_0^{cb}$  is bounded in the  $X$ -norm, there exists a constant  $K$  such that the Fourier coefficients  $\widehat{u}_k^{cu}(\tau)$  satisfy

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2) \left|\widehat{u}_k^{cu}(\tau)\right|^2 \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2)^2 \left|\widehat{u}_k^{cu}(\tau)\right|^2 \leq K^2$$

for all  $\tau \leq s_1$ . Together with the fact that (68) implies that  $|w_{k,4}(\tau)| \leq K$  for all  $\tau \leq s_1$ , this gives

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2) \left( \int_{-\infty}^{s_1} e^{2\tau} \overline{w_{k,4}(\tau)} \widehat{u}_k^{cu}(\tau) d\tau \right)^2 &\leq K^2 \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2) \left( \int_{-\infty}^{s_1} e^{2s_1} e^{2(\tau-s_1)} |\widehat{u}_k^{cu}(\tau)| d\tau \right)^2 \\ &\leq K^2 e^{2s_1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \int_{-\infty}^{s_1} e^{2(\tau-s_1)} d\tau \right) \left( \int_{-\infty}^{s_1} (1+k^2) e^{2(\tau-s_1)} |\widehat{u}_k^{cu}(\tau)|^2 d\tau \right) \\ &\leq \frac{K^2 e^{2s_1}}{2} \int_{-\infty}^{s_1} e^{2(\tau-s_1)} \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2) |\widehat{u}_k^{cu}(\tau)|^2 d\tau \leq \frac{K^4 e^{2s_1}}{4}. \end{aligned}$$

Finally, let  $\nu(\tau)$  be the first component of  $\tilde{\rho}(\tau)U^{cu}(\tau)$ , so that  $\nu(\tau) = \tilde{\rho}(\tau)u^{cu}(\tau)$ . As  $u^{cu} \in H^2(S^1)$ , its  $H^2$  norm is uniformly bounded on  $(-\infty, s_1]$  and  $\tilde{\rho} \in L^2(J_-; H^{1/2}(S^1), e^{2s} ds)$ , Lemma 6 implies that  $\nu \in L^2(J_-; H^{1/2}(S^1), e^{2s} ds)$ .

Denote the Fourier coefficients of  $\nu$  by  $\widehat{\nu}_k$ . Then the second integral of (66) can be written as

$$\int_{-\infty}^{s_1} \langle W_{k,4}(\tau), e^{2\tau} \tilde{\rho}(\tau) U^{cu}(\tau) \rangle d\tau = \int_{-\infty}^{s_1} e^{2\tau} \overline{w_{k,4}(\tau)} \widehat{\nu}_k(\tau) d\tau$$

and the estimate (68) on the decay of  $w_{k,4}$  implies that there exists a constant  $C$  such that

$$\begin{aligned} \left( \int_{-\infty}^{s_1} e^{2\tau} \overline{w_{k,4}(\tau)} \widehat{\nu}_k(\tau) d\tau \right)^2 &\leq \left( \int_{-\infty}^{s_1} K^2 e^{2(|k|+1)(\tau-s_1)} \right) \left( \int_{-\infty}^{s_1} e^{2\tau} \widehat{\nu}_k(\tau)^2 d\tau \right) \\ &\leq \frac{K^2}{2(|k|+1)} \int_{-\infty}^{s_1} e^{2\tau} \widehat{\nu}_k(\tau)^2 d\tau \leq \frac{C}{\sqrt{1+k^2}} \int_{-\infty}^{s_1} e^{2\tau} \widehat{\nu}_k(\tau)^2 d\tau, \end{aligned}$$

and so

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2) \left( \int_{-\infty}^{s_1} e^{2\tau} \overline{w_{k,4}(\tau)} \widehat{\nu}_k(\tau) d\tau \right)^2 &\leq C \sum_{k \in \mathbb{Z} \setminus \{0\}} (1+k^2)^{1/2} \int_{-\infty}^{s_1} e^{2\tau} \widehat{\nu}_k(\tau)^2 d\tau \\ &= C \|\nu\|_{L^2(J_-, H^{1/2}(S^1), e^{2s} ds)}^2 < \infty. \end{aligned}$$

Hence the second term also belongs to  $l_1^2$ , and so the proof of the claim that the range of  $\mathcal{G}$  is contained in  $l_1^2$  is complete.

Smoothness follows since the integrands are smooth in  $\tilde{\rho}$  and since the derivatives of arbitrary order of the evolution operators  $\Phi_+^s(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho})$  and  $\Phi_-^s(\tau, s_1; \lambda(\tilde{\rho}), \tilde{\rho})$  belong to the same exponentially weighted space as the evolution operators themselves (see Theorem 2 and Lemma 14).  $\square$

Finally we consider  $\mathcal{G}'(0)$ . Since  $U_*$  is radially symmetric (and hence belongs to  $X_0$ ) we have for  $k \neq 0$  that

$$\begin{aligned} G'_k(0)\tilde{\rho} &= \int_{-\infty}^{\infty} \langle W_{k,4}(\tau), r'(\tau)^2 (\lambda'(0)\tilde{\rho} - \tilde{\rho}(\tau)) B_0 U_*(\tau) \rangle d\tau \\ &= - \int_{-\infty}^{s_1} \overline{w_{k,4}(\tau)} \widehat{\rho}_k(\tau) u_*(\tau) e^{2\tau} d\tau, \end{aligned}$$

where  $\widehat{\rho}_k$  is the  $k$ -th Fourier coefficient of  $\tilde{\rho}$ . For  $k = 0$  we have

$$G'_0(0)\tilde{\rho} = - \int_{-\infty}^{s_1} \overline{w_{0,4}(s)} \widehat{\rho}_0(s) u_*(s) e^{2s} ds - \frac{\int_{-\infty}^{\infty} \overline{w_{0,4}(\tau)} u_*(\tau) r'(\tau)^2 d\tau}{\int_{-\infty}^{\infty} u_*(\tau)^2 r(\tau) r'(\tau) d\tau} \int_{-\infty}^{s_1} \widehat{\rho}_0(s) u_*(s)^2 e^{2s} ds.$$

To rewrite the preceding expressions, we define

$$\eta_k(s) := \overline{w_{k,4}(s)} u_*(s) \chi_{(-\infty, s_1]}(s) \quad (69)$$

for  $k \in \mathbb{Z} \setminus \{0\}$ , and set

$$\eta_0(s) := \left[ \overline{w_{0,4}(s)} + u_*(s) \frac{\int_{-\infty}^{\infty} \overline{w_{0,4}(\tau)} u_*(\tau) r'(\tau)^2 d\tau}{\int_{-\infty}^{\infty} u_*(\tau)^2 r(\tau) r'(\tau) d\tau} \right] u_*(s) \chi_{(-\infty, s_1]}(s).$$

Then we may write

$$\mathcal{G}'(0)\tilde{\rho} = \left\{ - \int_{-\infty}^{s_1} e^{2\tau} \eta_k(\tau) \widehat{\rho}_k(\tau) d\tau \right\}_{k \in \mathbb{Z}}.$$

For any  $k \in \mathbb{Z}$ , we have  $\eta_k e^{ik\cdot} \in \tilde{\mathcal{R}}$ : indeed, (68) shows that  $|w_{k,4}(\tau)| \leq K e^{k|\tau-s_1|}$  for any  $\tau \leq s_1$  and  $k \in \mathbb{Z} \setminus \{0\}$ , while  $|w_{0,3}|$  and  $|w_{0,4}|$  are bounded on  $J_-$ , so that there exists a constant  $C$  such that

$$\int_{-\infty}^{s_1} |\eta_k(\tau)|^2 e^{2\tau} d\tau \leq \sup_{\tau \in (-\infty, s_1)} |u_*(\tau)|^2 C \int_{-\infty}^{s_1} e^{(2|k|+2)(\tau-s_1)} d\tau \leq \sup_{\tau \in (-\infty, s_1)} u_*(\tau)^2 \frac{C}{2|k|+2}. \quad (70)$$

From the definition of  $\mathcal{G}'(0)\tilde{\rho}$ , it follows immediately that  $\mathcal{G}'(0)\tilde{\rho} = 0$  if and only if

$$\int_{-\infty}^{s_1} e^{2\tau} \eta_k(\tau) \widehat{\rho}_k(\tau) d\tau = 0 \quad (71)$$

for all  $k \in \mathbb{Z}$ . Thus, if we define

$$\mathcal{M} := \overline{\text{span}\{\eta_k e^{ik\cdot}; k \in \mathbb{Z}\}},$$

where the closure is taken in  $\tilde{\mathcal{R}}$ , then it can be seen that  $\mathcal{M}$  is the orthogonal complement in  $\tilde{\mathcal{R}}$  of  $\ker \mathcal{G}'(0)$ , and so  $\tilde{\mathcal{R}} = \ker \mathcal{G}'(0) \oplus \mathcal{M}$ .

**Lemma 27.**  $\mathcal{G}'(0) : \mathcal{M} \rightarrow l_1^2$  is a linear homeomorphism. Furthermore, the spaces  $\ker \mathcal{G}'(0)$  and  $\mathcal{M}$  are both infinite-dimensional.

*Proof.* It is clear that  $\tilde{\mathcal{G}}'(0) : \tilde{\mathcal{R}} \rightarrow l_1^2$  is bounded since by Lemma 26,  $\mathcal{G}$  is smooth in a neighbourhood of 0.

We need to investigate the subspace  $\mathcal{M}$ . Let  $\eta \in \mathcal{M}$  be arbitrary, then

$$\eta(s, \varphi) = \sum_{k \in \mathbb{Z}} a_k \eta_k(s) e^{ik\varphi}. \quad (72)$$

The upper bound estimate (70) implies that

$$\|\eta\|_{\mathcal{R}}^2 = \sum_{k \in \mathbb{Z}} (1 + k^2)^{1/2} |a_k|^2 \int_{-\infty}^{s_1} |\eta_k(\tau)|^2 e^{2\tau} d\tau \leq C' \sum_{k \in \mathbb{Z}} |a_k|^2. \quad (73)$$

Next we derive a lower bound for  $\|\eta\|_{\mathcal{R}}^2$ . Since  $u_*(s_1) \neq 0$ , there exist  $\hat{\epsilon}$  and  $\delta > 0$  such that  $u_*(s) > \hat{\epsilon}^2$  for every  $s \in (s_1 - \delta, s_1)$ . Lemma 18 shows that, for  $k$  large,  $W_{k,4}$  is close to solutions of the system at infinity, both in the  $X$  and  $\mathcal{X}$  norms. This allows us to get a lower bound on  $|w_{k,4}(s)|$  for  $k$  large. Let  $\epsilon > 0$  and  $K$  as in Lemma 18. As  $W_{k,4}(s_1) \in \text{Ran}(\hat{\Phi}_k^u(s_1, s_1))$ , it follows that  $\hat{\Phi}_k^u(s, s_1)W_{k,4}(s_1) = W_{k,4}(s)$ , and hence

$$\|W_{k,4}(s) - e^{|k|(s-s_1)}(P_k^s)^*W_{k,4}(s_1)\|_{\mathcal{X}'} \leq \epsilon e^{|k|(s-s_1)}\|W_{k,4}(s_1)\|_{\mathcal{X}'} = \epsilon e^{|k|(s-s_1)}$$

for  $|k| > K$ . Thus we get for the fourth component  $w_{k,4}(s)$

$$\begin{aligned} |w_{k,4}(s)| &\geq e^{|k|(s-s_1)} (|(P_k^s)^*W_{k,4}(s_1)|_4 - |w_k(s) - e^{|k|(s-s_1)}((P_k^s)^*W_{k,4}(s_1))_4|) \\ &\geq e^{|k|(s-s_1)} (|(P_k^s)^*W_{k,4}(s_1)|_4 - \|W_{k,4}(s) - e^{|k|(s-s_1)}(P_k^s)^*W_{k,4}(s_1)\|_{\mathcal{X}'}) \\ &\geq e^{|k|(s-s_1)} (|(P_k^s)^*W_{k,4}(s_1)|_4 - \epsilon \|W_{k,4}(s_1)\|_{\mathcal{X}'}) \geq e^{|k|(s-s_1)} (|w_{k,4}(s_1)| - 2\epsilon \|W_{k,4}(s_1)\|_{\mathcal{X}'}) \\ &\geq e^{|k|(s-s_1)} (|w_{k,4}(s_1)| - 2\epsilon). \end{aligned}$$

To get a lower bound on  $w_{k,4}(s_1)$ , first note that Lemma 18 implies that  $\|W_{k,4}(s_1) - (P_k^s)^*W_{k,4}(s_1)\|_{\mathcal{X}'} \leq \epsilon$  and that  $\text{Ran}(P_k^s)^* = \text{span}\{(-|k|, 1, 0, 0)^T, (0, 0, -|k|, 1)^T\}$ . Thus there exist  $\alpha_k, \beta_k \in \mathbb{C}$  and  $W_k \in \mathcal{X}'_k$  with  $\|W_k\|_{\mathcal{X}'} \leq 1$  such that

$$W_{k,4}(s_1) = \alpha_k(-|k|, 1, 0, 0)^T e^{ik\cdot} + \beta_k(0, 0, -|k|, 1)^T e^{ik\cdot} + \epsilon W_k.$$

As  $V_{k,1}(s_1) \in E_{+}^s$ , its first component has to be a multiple of  $K_{|k|}(\lambda_0^{1/4}r_1)$  and hence

$$V_{k,1}(s_1) = C_k e^{ik\cdot} \begin{pmatrix} K_{|k|}(\lambda_0^{1/4}r_1) \\ r'(s_1)\lambda_0^{1/4}K'_{|k|}(\lambda_0^{1/4}r_1) \\ \lambda_0^{1/2}K_{|k|}(\lambda_0^{1/4}r_1) \\ r'(s_1)\lambda_0^{3/4}K'_{|k|}(\lambda_0^{1/4}r_1) \end{pmatrix} C_k e^{ik\cdot} \begin{pmatrix} \underline{v}_k \\ \lambda_0^{1/2}\underline{v}_k \end{pmatrix} \text{ with } \underline{v}_k = \begin{pmatrix} K_{|k|}(\lambda_0^{1/4}r_1) \\ r'(s_1)\lambda_0^{1/4}K'_{|k|}(\lambda_0^{1/4}r_1) \end{pmatrix}$$

where  $r_1 = r(s_1)$  and  $C_k$  is such that  $\|V_{k,1}(s_1)\|_{\mathcal{X}} = 1$ , i.e.,

$$C_k^2 = \frac{1}{(1 + \lambda_0) \left[ (k^2 + 1) \left( K_{|k|}(\lambda_0^{1/4}r_1) \right)^2 + \sqrt{\lambda_0} \left( r'(s_1)K'_{|k|}(\lambda_0^{1/4}r_1) \right)^2 \right]}.$$

Since  $\langle W_{k,4}(s_1), V_{k,1}(s_1) \rangle = 0$ , we have

$$0 = C_k \left( -|k|K_{|k|}(\lambda_0^{1/4}r_1) + \lambda_0^{1/4}r'(s_1)K'_{|k|}(\lambda_0^{1/4}r_1) \right) (\alpha_k + \beta_k \sqrt{\lambda_0}) + \epsilon \langle W_k, V_{k,1}(s_1) \rangle. \quad (74)$$

From (9.6.23) in [1], we see that  $K_{|k|}(z) > 0$  for any  $z > 0$  and (9.6.26) implies  $K'_{|k|}(z) = -K_{|k|-1}(z) - \frac{|k|}{z}K_{|k|}(z)$  for any  $z > 0$ , hence  $K'_{|k|}(z) < 0$  for any  $z > 0$ . So we see that  $-|k|K_{|k|}(\lambda_0^{1/4}r_1) < 0$  and  $\lambda_0^{1/4}r'(s_1)K'_{|k|}(\lambda_0^{1/4}r_1) < 0$ . A short analysis gives that

$$-2/\sqrt{1 + \lambda_0} \leq C_k \left( -|k|K_{|k|}(\lambda_0^{1/4}r_1) + \lambda_0^{1/4}r'(s_1)K'_{|k|}(\lambda_0^{1/4}r_1) \right) \leq -1/\sqrt{2(1 + \lambda_0)}.$$

Furthermore,  $|\langle W_k, V_{k,1}(s_1) \rangle| \leq \|W_k\|_{\mathcal{X}'} \|V_{k,1}(s_1)\|_{\mathcal{X}} \leq 1$ , and we can conclude from (74) that  $\alpha_k = -\beta_k \sqrt{\lambda_0} + \mathcal{O}(\epsilon)$ . Finally,  $\|W_{k,4}(s_1)\|_{\mathcal{X}'} = 1$  gives  $\frac{1+2k^2}{1+k^2} (\alpha_k^2 + \beta_k^2) = 1 - \mathcal{O}(\epsilon^2)$ , and hence  $(1 + \lambda_0) \beta_k^2 = \frac{1+k^2}{1+2k^2} - \mathcal{O}(\epsilon)$ . Thus there exists a  $C > 0$  such that  $|w_{k,4}(s_1)| > C$  for all  $|k| > K$ . This implies that there exists a  $\tilde{C} > 0$  such that  $|w_{k,4}(s)| > \tilde{C} e^{|k|(s-s_1)}$  for every  $s \leq s_1$  and  $|k| > K$ .

Combining the lower bounds on  $u_*(s)$  and  $w_{k,4}(s)$ , we find that there exists a  $\delta > 0$  such that for  $|k| > K$

$$\begin{aligned} \int_{-\infty}^{s_1} \eta_k(\tau)^2 e^{2\tau} d\tau &\geq \tilde{\epsilon}^2 \tilde{C}^2 \int_{s_1-\delta}^{s_1} e^{(2|k|+2)(\tau-s_1)} d\tau = \frac{\tilde{\epsilon}^2 \tilde{C}^2}{2|k|+2} \left(1 - e^{-(2|k|+2)\delta}\right) \\ &\geq \frac{\tilde{\epsilon}^2 \tilde{C}^2}{2|k|+2} (1 - e^{-2\delta}) \geq \frac{C}{(1+k^2)^{1/2}}, \end{aligned} \quad (75)$$

for some positive  $k$ -independent constant  $C$ . Since for all  $k \in \mathbb{Z}$ ,  $\int_{-\infty}^{s_1} \eta_k(\tau)^2 e^{2\tau} d\tau > 0$ , the constant  $C$  above can be modified so that

$$\int_{-\infty}^{s_1} \eta_k(\tau)^2 e^{2\tau} d\tau \geq \frac{C}{(1+k^2)^{1/2}}$$

also for  $|k| \leq K$ . Hence it follows that

$$\|\eta\|_{\tilde{\mathcal{R}}}^2 = \sum_{k \in \mathbb{Z}} (1+k^2)^{1/2} |a_k|^2 \int_{-\infty}^{s_1} \eta_k(\tau)^2 e^{2\tau} d\tau \geq C \sum_{k \in \mathbb{Z}} |a_k|^2.$$

The upper and lower bounds on  $\|\eta\|_{\tilde{\mathcal{R}}}^2$  show that  $\eta \in \mathcal{M}$  if and only if  $\eta$  is given by (72) and  $\{a_k\}_{k \in \mathbb{Z}} \in l^2$ .

As we have seen that the mapping  $\mathcal{G}'(0)$  is bounded above, it is sufficient to show that it is bounded below to conclude that  $\mathcal{G}'(0)$  is a linear homeomorphism from  $\mathcal{M}$  to  $l_1^2$ . From its definition, it follows that

$$\mathcal{G}'(0)\eta = \left\{ a_k \int_{-\infty}^{s_1} |\eta_k(\tau)|^2 e^{2\tau} d\tau \right\}_{k \in \mathbb{Z}},$$

and so by (75) and (73), we see that

$$\|\mathcal{G}'(0)\eta\|_{l_1^2}^2 = \sum_{k \in \mathbb{Z}} (1+k^2) |a_k|^2 \left( \int_{-\infty}^{s_1} |\eta_k(s)|^2 e^{2s} ds \right)^2 \geq \tilde{C} \sum_{k \in \mathbb{Z}} |a_k|^2 \geq C' \|\eta\|_{\tilde{\mathcal{R}}}^2.$$

It remains to show that the spaces  $\ker \mathcal{G}'(0)$  and  $\mathcal{M}$  have infinite dimension. For  $\mathcal{M}$ , this follows directly from its definition. Next, consider the characterization of  $\ker \mathcal{G}'(0)$  given in (71). We proved above that the functions  $w_{k,4}(s)$  that appear in the definition (69) of  $\eta_k$  satisfy  $|w_{k,4}(s_1)| \geq C$  uniformly in  $|k| \geq K$ , which implies that the space  $\ker \mathcal{G}'(0)$  has infinite dimension as claimed.  $\square$

We are now ready to complete the proof of Theorem 1.

*Proof of Theorem 1.* By Lemma 21, if  $(\lambda, \tilde{\rho})$  is sufficiently close to  $(\lambda_0, 0)$  then  $\lambda$  is an embedded eigenvalue for  $\Delta^2 + \tilde{\theta} + \tilde{\rho}$  if and only if (62) holds. We have also seen that (62) is equivalent to  $F(\lambda, \tilde{\rho}) = 0$ , which allowed us to solve for  $\lambda$  as a function of  $\tilde{\rho}$  and finally obtain the equation  $\mathcal{G}(\tilde{\rho}) = 0$ , where  $\mathcal{G} : \tilde{\mathcal{R}} \rightarrow l_1^2$ . By Lemma 27,  $\tilde{\mathcal{R}} = \ker \mathcal{G}'(0) \oplus \mathcal{M}$ , and  $\mathcal{G}'(0) : \mathcal{M} \rightarrow l_1^2$  is a linear homeomorphism. Hence for  $\tilde{\rho} \in \tilde{\mathcal{R}}$  we may write  $\tilde{\rho} = \xi + \eta$ , where  $\xi \in \ker \mathcal{G}'(0)$  and  $\eta \in \mathcal{M}$ . By the implicit function theorem, we can solve for  $\eta$  in terms of  $\xi$ , and this equation defines a smooth manifold in a neighbourhood of 0 with infinite dimension and codimension, since the spaces  $\ker \mathcal{G}'(0)$  and  $\mathcal{M}$  are infinite-dimensional by Lemma 27.  $\square$

## 8 Conclusions and open problems

In this paper, we considered the planar bilaplacian with a smooth, radially symmetric and compactly supported potential  $\theta$  and described the set of perturbations of the potential in the space  $\mathcal{R} = L^2([0, r_1]; H^{1/2}(S^1), r dr)$

for which an embedded eigenvalue persists. We expect that the space  $\mathcal{R}$  can be replaced by the Sobolev space  $H^{1/2}(B_{r_1}(0))$  of  $H^{1/2}$ -functions of two variables that have support in the ball  $B_{r_1}(0)$ .

One restriction of our work is that we consider only potentials with compact support: The reason is that we were forced to work with different function spaces of solutions for  $r$  small and for  $r$  large. For  $r$  small, we have some freedom in choosing the space, as any space of the form  $X = H^{1+\alpha} \times H^\alpha \times H^1 \times L^2$ , with  $0 < \alpha \leq 1$ , ensures that an exponential dichotomy exists. For  $r$  large, due to the structure of the equations, there is no such freedom, the regularity on the first two components has to be same as the regularity of the last two. So it is unclear whether there exists an exponential dichotomy when the support of  $\rho$  is not compact. It would be interesting to see whether our hypothesis that  $\rho$  has compact support could be replaced by an appropriate decay condition on  $\rho$ .

For the original potential  $\theta$ , we see no obstacles in removing the condition that  $\theta$  has compact support. It should be possible to replace this condition by the long range condition  $|\theta'(r)| \leq C(1+r)^{-1-\beta}$  for some  $\beta > 0$ . It should also be possible to remove the condition that  $\theta$  is radially symmetric, although considerably more work will be needed without this condition.

We believe that the methods put forward in this paper can be used to study other operators. In particular, the exponential-dichotomy results established in [18] are for systems of reaction-diffusion equations, so we believe that the only obstacle for extending our results to selfadjoint systems are the presence of nonsmooth potentials. For other operators, it might not be possible to modify the function spaces involved to prove the existence of exponential dichotomies. These are difficult problems that have to be studied in future work.

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