Spin relaxation in lateral quantum dots: Effects of spin-orbit interaction

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We report results of calculations of the effect of spin-orbit interaction on electron spin relaxation in a lateral quantum dot. Our study is motivated by puzzling results of high source-drain transport measurements of singlet-triplet transitions of two electrons in lateral and vertical devices that show a strong asymmetry as a function of the applied magnetic field. Using exact diagonalization techniques, we investigate the influence of the spin-orbit interaction on the energy levels of a two-electron droplet and we show that the spin-orbit interaction strongly affects the expectation values of the total and z-projection spins of the two-electron system. We then evaluate the energy relaxation rates for the two-electron droplet through the emission of longitudinal acoustic phonons and show that they are strongly dependent on the spin energy levels involved in the process. Our study shows that the spin-orbit interaction provides an effective coupling between the spin-polarized triplet states and the singlet state. However, the transition involving the spin singlet and the unpolarized triplet component is very weak even in the presence of spin-orbit interaction. The calculated scattering rates from the excited states to the ground state of the two-electron system clearly confirm this picture and reveal a micro-second time scale for the single-triplet relaxation through spin-orbit-mediated acoustic phonon emission and the relaxation mechanism presents a built-in magnetic field asymmetry, in qualitative agreement with experimental findings.

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I. INTRODUCTION

Semiconductor quantum dots generate strong interest brought about by the possibility of engineering in a controllable fashion their electronic properties. These “artificial atoms” are created by three-dimensional confinement potentials and present well-defined (atomic-like) quantized energy levels.1 The lateral quantum dots which are studied in this paper are created within a two-dimensional electron gas (2DEG), formed at the interface of two III-IV semiconductor materials (typically, GaAs/GaAlAs) with an additional lateral confinement produced by electrodes. The characteristics of the quantum dot system can be engineered with a high degree of accuracy: the number of electrons in a dot can be continuously tuned from thousands of electrons to a single electron by tuning the gate voltages that control the dot coupling to the surrounding two- (three-)dimensional bath of unconfined electrons;2 the symmetry of the electronic wave function can be altered by changes in the topology of the confining potential,3 etc.

The ability to create and manipulate charge and spin populations in low-dimensional quantum systems by tailoring the structural and electronic environment of localized electrons generates a wide class of applications ranging from the spin field-effect transistor4 that uses the spin-orbit coupling of electrons in a 2DEG to the use of quantum dots as basic building blocks in the implementation of a quantum computer.5 In both spintronics and quantum computation, of special importance is the preservation of phase coherence among the spins and thus the minimization of decoherence and dissipation generated by the interaction of the confined spins with the environment. In this context, understanding spin-mixing and spin-relaxation phenomena caused by, among other mechanisms, the spin-orbit coupling, is of central importance as it determines the characteristic times and length scales over which coherent physics phenomena are observed. In lateral III-IV semiconductor quantum dots, the spin-orbit coupling is generated by the twofold asymmetry of the quantum structure: the inversion asymmetry of the bulk dielectric backbone material (with a zinc-blende symmetry) gives rise to the Dresselhaus term6 in the system Hamiltonian, whereas the asymmetry of the macroscopic vertical confinement potential produces the Rashba term.7

Previous theoretical studies have addressed the magnetic-field-driven singlet-triplet transition in the two-electron interacting problem, in both the single-5 and double-dot6 architectures. The nonparabolicity of the confining potential in a lateral quantum dots determines a strong renormalization of the low-magnetic-field spectrum,8 whereas in the presence of spin-orbit coupling important changes in the g factor10 or intrinsic spin flips11 may occur. Despite a recent resurgence of interest12-16 the analysis of spin relaxation mechanisms in quantum dots is limited, and a direct comparison of theoretical predictions with experimental results is a difficult task. Consequently, the theoretical and experimental characterization of spin-relaxation mechanisms in quantum dot systems is still an open subject. Among previous studies, we note the advances in Ref. 17, in which spin-flip processes in quantum dots mediated by several mechanisms, including spin-orbit coupling and hyperfine interaction, are discussed. In Ref. 18 a novel spin-phonon coupling arising from the interface motion is put forward, and in Ref. 19 exact diagonalization techniques were used to elucidate the influence of spin-orbit coupling on the spin relaxation in single-electron systems. Recently,20 the interplay of the Rashba and Dresselhaus spin-orbit interactions on the phonon-induced spin-relaxation rate has been investigated.

In the present work, we provide in-depth analysis of the effects of spin-orbit interaction on the spin relaxation in a two-electron lateral quantum dot following on our previous
FIG. 1. High source-drain transport spectroscopy of the two-electron droplet. The curves correspond to fluctuations between $N$ = 1 and 2 electrons and represent the lowest-lying energy levels of the two-electron droplet as a function of the magnetic field (from Ref. 8).

Our analysis includes the effects of the nonparabolicity of the confining potential and the electron-electron interaction, and is motivated by results of high source-drain transport measurements of singlet-triplet transitions of two electrons in lateral8,21 and vertical22 devices: on the low-magnetic-field side of the singlet-triplet transition both the singlet ground state and the excited state are observed, whereas once the triplet becomes the ground state (for higher values of the magnetic field), the singlet excited state is no longer observed, as depicted in Fig. 1.

In Sec. II, we introduce the model Hamiltonian of the system we analyze. In Sec. III, we investigate the effects of spin-orbit interaction and involving the emission of a longitudinal acoustic phonon.15 Section V B presents a simplified approach, in which we consider a restricted Hilbert space of the two-electron droplet, while in Sec. V C we evaluate the relaxation rates of the two-electron system through exact numerical signalization of the full two-electron Hamiltonian. In Sec. VI, we discuss the results of the study presented in this paper.

II. SYSTEM DESCRIPTION AND MODEL HAMILTONIAN

We consider a two-electron droplet in a lateral ($x$-$y$ plane) confining potential consisting of both a parabolic piece and a nonparabolic piece,8 a lateral quantum dot, and placed in vertical external magnetic field $\mathbf{B}=(0,0,B)$. This system is an extension of the Fock-Darwin model (a 2D harmonic oscillator with spin$^{23-25}$), to which we add effects associated

with the nonparabolicity of the confining potential and the spin-orbit interaction. In the effective mass approximation, the Hamiltonian describing the system is

$$H = \sum_{i=1,2} (H_{sp}^{(i)} + V_c^{(1,2)}),$$

where $H_{sp}^{(i)}$ is the single-particle Hamiltonian, and $V_c^{(1,2)}$ represents the two-body Coulomb interaction.

The single-particle Hamiltonian is given by

$$H_{sp} = \sum_{i=1,2} (H_{kin}^{(i)} + H_{conf}^{(i)} + H_{so}^{(i)}),$$

where $H_{kin}^{(i)}$ is the kinetic Hamiltonian, $H_{conf}^{(i)}$ represents the confining potential, and $H_{so}^{(i)}$ is the Hamiltonian involving the spin of the two-electron droplet.

The kinetic term is given by

$$H_{kin}^{(i)} = \frac{\pi^2}{2m^*}.$$  

Here, for simplicity, we have dropped the particle index. $\pi = -i\hbar \nabla + eA$ is the canonical momentum. We choose to work in the symmetrical gauge with the electromagnetic potential given by $A=(1/2)(\mathbf{B} \times r)/B/2(-y,x,0)$: $m^*$ and $\epsilon$ are the effective mass and the charge of the electron, respectively.

The confining potential

$$H_{conf} = H_p + H_w$$

consists of a conventional parabolic piece

$$H_p = \frac{1}{2}m^* \omega_0^2 (x^2 + y^2)$$

and a nonparabolic contribution, modeled as a parabolic semimcircular wire of “diameter” $D$, which intersects the quantum point contacts and the center of the dot,

$$H_w = \gamma \frac{m^* \omega_0^2 (x^2 - \frac{y^2}{2D})^2}{2D}.$$  

Here, $\omega_0$ is the confining potential frequency, $\gamma$ a parameter controlling the strength of the confining potential nonparabolicity, and $D$, the wire diameter, a parameter that controls the shape of the nonparabolic piece of the confining potential. We present in Fig. 2 a contour plot of the total two-dimensional confining potential for some typical values of the parameters used in this paper.

The spin Hamiltonian presents two contributions

$$H_s = H_z + H_{so}.$$  

The usual Zeeman term $H_z$ is given by

$$H_z = \frac{1}{2} g^* \mu_B B \sigma,$$

where $\mu_B$ is the Bohr magneton, $g^*$ is the effective Lande factor ($g=-0.44$ for GaAs), and $\sigma$ is the z projection of the electron spin, with $\sigma = \pm 1$. Introducing the cyclotron frequency $\omega_c = eB/m^*$, the magnitude of the Zeeman energy is then
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The Bohr radius, and Ry is given by 

where \( \omega_c = 1 \) meV and the parameters characterizing the nonparabolic piece of the confining potential are \( \gamma = 1, D = 10a_0. \)

The spin-orbit interaction term \( H_{SO} \) is given by

\[
H_{SO} = H_{SO}^{(D)} + H_{SO}^{(R)},
\]

with \( H_{SO}^{(D)} \) the Dresselhaus term originating from the bulk inversion asymmetry of the semiconductor material, and \( H_{SO}^{(R)} \) the Rashba term, arising from the asymmetry of the quantum well that confines the 2DEG. The explicit form of the two contributions (the Dresselhaus spin-orbit interaction Hamiltonian includes also a cubic term in the electron momentum which, for simplicity, is neglected here) is cast in the form\(^{27}\)

\[
H_{SO}^{(D)} = -\frac{\beta}{\hbar} (\sigma_x \pi_y - \sigma_y \pi_x),
\]

\[
H_{SO}^{(R)} = \frac{\alpha}{\hbar} (\sigma_x \pi_y - \sigma_y \pi_x),
\]

where \( \alpha \) and \( \beta \) are the Rashba and Dresselhaus coefficients, respectively, and \( \pi_{x,y} \) are the Cartesian components of the two-dimensional canonical momentum \( \pi \). The Dresselhaus spin-orbit coupling parameter is given by\(^{15}\)

\[
\beta = \gamma_{SO} \frac{\hbar}{\sqrt{2m^*}} \frac{a_0^2}{d^2} \text{Ry},
\]

where \( E_G \) is the semiconductor band gap energy, \( \gamma_{SO} \) is the spin-orbit coupling constant, \( a_0 = \hbar^2/\mu_e^2 \) is the effective Bohr radius, and \( \text{Ry} = m^* e^4/2e^2\hbar^2 \) is the effective Rydberg, respectively. The parameter \( d \) is inversely proportional to the average of the \( z \) component of the wave vector \( d^{-2} = \langle \psi^*_f | \hat{\nabla}^2 | \psi_f \rangle \), where \( \psi_f(z) \) is the ground-state wave function of the vertical quantum well. The Dresselhaus term is strongly dependent on the growth geometry (orientation of the heterointerface with respect to the symmetry axes of the crystal). Here, we consider the electrons confined in the \((1,0,0)\) GaAs plane. The Dresselhaus parameter may have a large range of values depending on the materials forming the interface, and, for GaAs/AlGaAs structures, it is of order \( 1 - 50 \) meV Å.\(^{28-30}\)

Generally, the strength of the Rashba term is assumed to be proportional to the average electric field across the interface, \( \langle E \rangle = -(1/\epsilon \partial \partial \zeta) (E_x + V) \). However, it has been shown that the band-edge profile and the applied electrostatic profile play different roles in determining the strength of this term.\(^{31}\)

Similarly to the Dresselhaus term, the Rashba term may also take a large range of values, depending on the shape of the confining quantum well along the \( z \) direction and the 2DEG electron density, and its magnitude can be varied using proper gating configurations.\(^{32}\) Direct Raman scattering measurements of the spin splitting in GaAs/AlGaAs quantum wells have revealed that for low electron densities the Rashba and Dresselhaus contributions may become comparable.\(^{33}\) For completeness sake, in the present study we consider both spin-orbit mechanisms, and assume that they have similar strength (the typical numerical values used are \( \beta = 10 \) meV Å and \( \alpha = 1 \) meV Å, which are close to the ones predicted for GaAs/AlGaAs structures).

The Coulomb interaction Hamiltonian is given by

\[
V^{(1,2)} = \frac{\kappa e^2}{\varepsilon r_{12}},
\]

where \( \varepsilon \) is the dielectric constant (\( \varepsilon = 12.4 \) for GaAs), \( \kappa = 1/4 \pi \sigma_{\theta} \), and \( r_{1,2} = (x_1, y_1) \) and \( r_{12} = |r_1 - r_2| \) are the electron positions and their relative separation, respectively.

### III. THE SINGLE-ELECTRON CASE

In analyzing the effects of spin-orbit interaction on the single-particle excitation spectrum, we start by introducing the Fock-Darwin energy levels. In the absence of nonparabolicity and of the spin-orbit interaction (\( \gamma = \alpha = \beta = 0 \)), the single-particle Hamiltonian is given by

\[
H_{FD} = H_{kin} + H_P + H_Z
\]

where \( H_{kin}, H_P, \) and \( H_Z \) are the kinetic, parabolic confinement, and Zeeman interaction contributions, respectively, and are given in Eqs. (2.3), (2.8), and (2.8).

In order to transform the Hamiltonian (3.1) to a two-oscillator form, we introduce the creation and annihilation operators

\[
a = \frac{\hat{x} - i\hat{y}}{2} + \frac{\hat{x} - i\hat{y}}{2}, \quad b = \frac{\hat{x} + i\hat{y}}{2} + \frac{\hat{x} + i\hat{y}}{2}
\]

FIG. 2. Contour plot of the two-dimensional confining potential. The parabolic piece of the confining potential is characterized by \( \omega_0 = 1 \) meV and the parameters characterizing the nonparabolic piece of the confining potential are \( \gamma = 1, D = 10a_0. \)

Here, \( \varepsilon \) is the dielectric constant (\( \varepsilon = 12.4 \) for GaAs), \( \kappa = 1/4 \pi \sigma_{\theta} \), and \( r_{1,2} = (x_1, y_1) \) and \( r_{12} = |r_1 - r_2| \) are the electron positions and their relative separation, respectively.
The lengths have been scaled by $l = \sqrt{\hbar/m\omega_0}$, the Fock-Darwin radius, $(\hat{x}, \hat{y}) = (x, y)/l$ and $(\hat{\rho}_z, \hat{\rho}_\theta) = (l \hat{\rho}_z, \hat{\rho}_\theta)$, and $\omega_0 = \sqrt{\omega_0^2 + \omega_c^2}/4$. The operators $a, a^\dagger, b, b^\dagger$ satisfy the algebra of two independent harmonic oscillators: $[a, a^\dagger] = [b, b^\dagger] = 1$, $[a, b] = [a^\dagger, b^\dagger] = [a, b^\dagger] = [b, a] = 0$.

In terms of the $a$ and $b$ operators, the Hamiltonian (3.1) becomes

$$H_{FD} = \hbar \omega_0 \left( \frac{a^\dagger a}{2} + \frac{1}{2} \right) + \hbar \omega_c \left( \frac{b^\dagger b}{2} + \frac{1}{2} \right) + \frac{1}{2} g \mu_B \sigma$$

(3.3)

where $\omega_0 = \omega_0 \pm \omega_c/2$ are the frequencies of the oscillators. The single-particle levels are given by the usual Fock-Darwin spectrum $^{23-25}$ (a 2D harmonic oscillator with spin, characterized by two-oscillator quantum numbers $m, n = 0, 1, \ldots$, and a quantum number for spin, $\sigma = \pm 1$)

$$|m, n, \sigma\rangle = \frac{1}{\sqrt{n! m!}} (a^\dagger)^m (b^\dagger)^n |0, 0 \rangle \otimes \chi_{\sigma},$$

(3.4a)

$$\epsilon_{m n \sigma} = \hbar \omega_0 \left( n + \frac{1}{2} \right) + \hbar \omega_c \left( m + \frac{1}{2} \right) + \frac{1}{2} g \mu_B \sigma,$$

(3.4b)

where $\chi_{\sigma}$ are the usual spin-$1/2$ spinors. For strong magnetic fields, the energy spectrum consists of manifolds labeled by the quantum number $n$ and separated from each other by $\hbar \omega_0$. The manifold components are labeled by $m$ and separated by $\hbar \omega_c$. As the magnetic field increases, the manifolds converge asymptotically to the Landau energy levels.

Finally, we express the dimensionless position, spatial derivative, and canonical momentum operators in terms of the harmonic oscillator operators. Using Eq. (3.2a), we have

$$\hat{x} = \frac{a + a^\dagger + b + b^\dagger}{2}, \quad \hat{y} = \frac{i \hat{\rho}_x - \hat{\rho}_y}{2},$$

(3.5a)

$$\hat{\rho}_z = \frac{a - a^\dagger + b - b^\dagger}{2}, \quad \hat{\rho}_\theta = \frac{i \hat{\rho}_x + \hat{\rho}_y}{2},$$

(3.5b)

$$\hat{\rho}_z = -i \hat{\rho}_\theta - \theta_0 \hat{\rho}_x, \quad \hat{\rho}_\theta = -i \hat{\rho}_\theta + \theta_0 \hat{\rho}_x,$$

(3.5c)

where $(\hat{\rho}_z, \hat{\rho}_\theta) = (\hat{\rho}_x, \hat{\rho}_y)/(\hbar/l)$ and $\theta_0 = (\omega_0^2/2)/\omega_0$.

In the absence of the nonparabolic confinement potential and spin-orbit interaction, the system presents circular symmetry and angular momentum projection along the symmetry axis, $\hat{L}_z = x\hat{\rho}_y - y\hat{\rho}_x$, is conserved. In the Fock-Darwin basis, the dimensionless form of the $z$ component of the angular momentum, $\hat{L}_z = x\hat{\rho}_y - y\hat{\rho}_x$, is diagonal and its matrix elements are given by

$$\langle m, n | \hat{L}_z | m', n', \sigma\rangle = (m - n) \delta_{m, m'} \delta_{n, n'} \delta \sigma \sigma'.$$

(3.6)

The circular symmetry makes convenient a different representation of the Fock-Darwin energy levels. Introducing the principal quantum number and the azimuthal quantum number defined by $N = m + n$ and $R = m - n$, respectively, the energy levels are given by

$$\epsilon_{N R \sigma} = \hbar \omega_0 (N + 1) - \frac{1}{2} \hbar \omega_R + \frac{1}{2} g \mu_B \sigma.$$  

(3.7)

This relabeling of the Landau levels is practical for low magnetic fields, when the energy spectrum consists of $N$-labeled manifolds separated by $\hbar \omega_0$, with components labeled by the angular momentum quantum number $L$ and separated by $\hbar \omega_c/2$.

The nonparabolic confinement potential in Eq. (2.6) presents a quadratic functional dependence on the position operator $x$ and a quartic dependence on the position operator $y$. Using Eq. (3.5a), the explicit dependence of the nonparabolic confinement potential on the creation (annihilation) operators $a, b(a^\dagger, b^\dagger)$ can be easily obtained. The nonparabolicity of the confining potential breaks the radial symmetry and mixes states of different angular momentum. Since the position operators are linear combinations of the oscillator ladder operators, and given the dependence of the confining potential on the creation and annihilation operators, the mixing occurs among states with angular momentum differing at most by four units. The effects of nonparabolicity on the single-particle energy levels had been investigated in Ref. 8, where it has been shown that, at low fields it leads to a strong renormalization of the shell structure, whereas beyond the $\nu = 2$ line its effects are minimal.

The spin-orbit interaction effects on the energy spectrum can also be inferred from the functional dependence on the position $x, b(a^\dagger, b^\dagger)$ operators. The spin-orbit Hamiltonian can be written as

$$H_{SO}^{(D)} = -\beta \langle \sigma_z \pi_+ + \sigma_- \pi_- \rangle,$$

(3.8a)

$$H_{SO}^{(R)} = i \alpha \langle \sigma_z \pi_- - \sigma_- \pi_+ \rangle,$$

(3.8b)

where we introduced $\sigma_z = (\sigma_x \pm i \sigma_y)/2$, $\pi_\pm = \hat{\pi}_x \pm i \hat{\rho}_y$, and $\pi_z = i [(a^\dagger - b) + \theta_0 (a^\dagger + b)]$, $\pi_+ = -i [(a - b^\dagger) + \theta_0 (a + b^\dagger)]$.

(3.9a)

(3.9b)

In terms of the $a, b, a^\dagger, b^\dagger$ operators, the spin-interaction Hamiltonian takes the form $^{29}$

$$H_{SO}^{(D)} = \sigma_z \pi_+^{(D)} + \sigma_- \pi_-^{(D)},$$

(3.10a)

$$H_{SO}^{(R)} = \sigma_z \pi_-^{(R)} + \sigma_- \pi_+^{(R)},$$

(3.10b)

where we have introduced the operators

$$\pi_\pm^{(D,R)} = \lambda^{(D,R)} \pi_\pm - \lambda^{(D,R)} a \mp \lambda^{(D,R)} b^\dagger,$$

(3.11a)

$$\pi_\pm^{(D,R)} = [\pi_\pm^{(D,R)}]^{-1}$$

(3.11b)

and the parameters
\[ \lambda_\pm^{(d)} = i \beta \frac{1 \pm \theta_0}{l}, \quad (3.12a) \]
\[ \lambda_\pm^{(r)} = \alpha \frac{1 \pm \theta_i}{l}. \quad (3.12b) \]

The Dresselhaus spin-orbit coupling (3.10a) couples the \(|m, n, \sigma\rangle\) state to \(|m, n \pm 1, 0, \pm \sigma\rangle\) and \(|m = \pm 1, n, \sigma \pm 1\rangle\) states, whereas the Rashba term Eq. (3.10b) couples the \(|m, n, \sigma\rangle\) state to \(|m, n \pm 1, \sigma \pm 1\rangle\) and \(|m \pm 1, n, \sigma \pm 1\rangle\) states. In other words, the Dresselhaus coupling causes mixing among states in the Fock-Darwin spectrum for which \(\Delta N = 1\) (\(\Delta N = -1\)) and \(\Delta R = \pm 1 = -\Delta \sigma\), whereas the Rashba coupling mixes states for which \(\Delta N = 1\) (or \(\Delta N = -1\)) and \(\Delta R = \pm 1 = \Delta \sigma\). Depending on the sign and the relative magnitude of the \(\alpha\) and \(\beta\) coefficients, the overall effect of the spin-orbit interaction on the spectrum is to shift and split the levels and to cause anticrossing whenever Fock-Darwin levels satisfying the spin-orbit coupling selection rules cross.\(^{11}\)

In the second quantization formalism, the single-particle total Hamiltonian is given by

\[ H_{\text{SP}} = \sum_{i, \sigma} c_{i\sigma}^\dagger c_{i\sigma} + \gamma \sum_{i,j} h_{ij}^{(W)} c_{i\sigma}^\dagger c_{j\sigma} + \alpha \sum_{i,j,\sigma,\sigma'} h_{ij}^{(R)} c_{i\sigma}^\dagger c_{j\sigma'}^\dagger + \beta \sum_{i,j,\sigma,\sigma'} h_{ij}^{(D)} c_{i\sigma}^\dagger c_{j\sigma'}, \quad (3.13) \]

where \(i, j\) are composite indices used to denote the triplet of quantum numbers \(m, n, \sigma\). The creation (annihilation) operator \(c_{i\sigma}^\dagger \rightarrow c_{m\sigma\alpha} (c_{i\sigma})\) creates (annihilates) a particle in the state \(|m\sigma\alpha\rangle\), with energy \(e_{m\sigma\alpha}\), \(z\) component of the angular momentum \(R = (m-n)\), and \(z\) projection of the spin \(\sigma/2\). The second term in Eq. (3.13) is the second-quantized version of (2.6), with \(h_{ij}^{(W)} = \langle i | H^{(W)} | j \rangle \rightarrow \langle m\sigma\alpha | H^{(W)} | m'\sigma'\alpha' \rangle\). Analogously, the third and fourth terms in (3.13) are the second-quantized version of the spin-orbit interaction Hamiltonians (2.11a) and (2.11b), \(h_{ij}^{(D)} = \langle i | H^{(D)} | j \rangle \rightarrow \langle m\sigma\alpha | H^{(D)} | m'\sigma'\alpha' \rangle\), and the last one is the Zeeman interaction.

To obtain the single-particle energy levels, the Hamiltonian (3.13) is numerically diagonalized in the 2D harmonic oscillator basis [Eq. (3.4a)], using a truncated single-particle Hilbert space (states above a cutoff energy level are omitted). The dimension of the Hilbert space is gradually increased until the numerical procedure becomes independent (within a prescribed error) of the energy cutoff. To help understand the changes in the single-particle spectrum, we begin with a simplified perturbational analysis of the effects of the spin-orbit interaction. In the presence of only one spin-coupling mechanism and in the absence of the nonparabolic potential, it is possible to obtain a second-order perturbation (with the spin-orbit coupling strength as expansion parameter) solution for energy levels and eigen-vectors of the single-particle problem.\(^{34}\) For simplicity, we consider \(B = 0\) and neglect the nonparabolicity of the confining potential and the spin-orbit coupling due to the Rashba term. We also assume that \(H_0 \gg H_{SO}^{(D)}\), and keep terms up to second order in \(\beta\). The Hamiltonian \(H = H_{\text{kin}} + H_{SO}^{(D)}\) is diagonalized using the following unitary transformation:

\[ U = \exp \left( -i \beta \frac{m}{\hbar} \left( \sigma_{x} y - \sigma_{y} x \right) \right). \quad (3.14) \]

with the transformed Hamiltonian given by\(^{34}\)

\[ \tilde{H} = U^\dagger H U = \left( \frac{\pi^2}{2m} + \frac{1}{2} m^* \omega_0^2 (x^2 + y^2) \right) + \beta^2 \frac{m^*}{\hbar^2} (x \sigma_x - y \sigma_y) \sigma_z - \beta^2 \frac{m^*}{\hbar^2} \tilde{L}_z + O(\beta^3), \quad (3.15) \]

i.e. (after dropping the additive constant by redefining the zero energy level),

\[ \tilde{H} = H_0 + \beta^2 \frac{m^*}{\hbar^2} \tilde{L}_z \sigma_z. \quad (3.16) \]

where \(\tilde{L}_z\) is the orbital angular momentum. It follows that, in the transformed frame, the Dresselhaus term does not mix the different spin states and the Hamiltonian is diagonal in the eigenstates of \(H_0\). We denote these wave functions by \(\Psi_{m,n,\sigma}(r) = \Phi_{m,n}(r) \chi_\sigma(\sigma)\), where \(\Phi_{m,n}(r)\) are the Fock-Darwin wave functions.\(^{25}\) In the rotated frame, the eigenstates are characterized by well-defined values of the orbital momentum and spin, i.e., \(\tilde{L}_z \Phi_{m,n}(r) = (m-n) \Phi_{m,n}(r)\) and \(\sigma \chi_\sigma = \pm \chi_\sigma\). In this approach, the mixing of the spatial and spinorial degrees of freedom is revealed by transforming back to the nonrotating frame of reference. While the energy levels are unchanged (unitary transformation)

\[ \epsilon_{m\sigma\alpha} = \hbar \omega_0 (m + n + 1) + \beta^2 \frac{m^*}{\hbar^2} (m-n) \sigma - \beta^2 \frac{m^*}{\hbar^2}, \quad (3.17) \]

a two-dimensional representation of the new wave functions (which we still associate with spin up and down, since we expect the deviations from the \(\pm 1\) values are small, of order \(\beta\))

\[ \xi_{m,n,\sigma}(r) = \Phi_{m,n}(r) \begin{pmatrix} 1 \\ -\beta m^* \hbar^2 (y + i x) \end{pmatrix}, \quad (3.18a) \]
\[ \xi_{m,n,\sigma}(r) = \Phi_{m,n}(r) \begin{pmatrix} \beta m^* \hbar^2 (y - i x) \\ 1 \end{pmatrix}. \quad (3.18b) \]

shows that actually the two-dimensional “spinors” act on both spatial and spin degrees of freedom of the system. As an example, we present in Fig. 3 a schematic diagram of the zero-magnetic-field energy levels. It shows that, in agreement with the time-reversal invariance, the levels are still doubly degenerate: for example, the \(p\)-shell level with \(\sigma = 1\), \(m = 1\) has the same energy as the \(\sigma = -1\), \(m = -1\) level, while the \(p\) levels \(\sigma = 1\), \(m = 1\) and \(\sigma = 1\), \(m = -1\) have the same energy as well. To facilitate comparison with the “exact” results obtained through numerical diagonalization of the single-particle Hamiltonian, we also note that the single-particle energies can be written in the form
The Rashba term is absent, and the labels \( s, p, d, e \) are related to the values of principal quantum number \( N = 0, 1, 2, 3, \ldots \). The confinement is assumed circular with magnetic field for the circular confinement and in the presence of Dresselhaus spin-orbit interaction and \( \hbar \omega_0 = \beta^2 m^* / \hbar^2 \), \( \hbar \omega_0 \) is the circular confinement frequency, the energy levels carry three indices \( |m, n, \sigma \rangle \), and \( \epsilon_{mn\sigma} = \hbar \omega_0 (n + m + 1) + \beta m^* (m - n - 1) \sigma \), (3.19)

which shows that the \( p \)-level splitting \( \Delta \epsilon_p = \epsilon_{1,0,\uparrow} - \epsilon_{0,1,\uparrow} = \epsilon_{0,1,\downarrow} - \epsilon_{0,1,\uparrow} \) in zero field is \( 2 \beta^2 m^* / \hbar^2 \) and the upper states of the \( p \) doublet have the same energy as in the absence of the spin-orbit interaction [as a result of the cancellation taking place in the last parenthesis of Eq. (3.19) for \( m - n = 1 \)], while the lower states are pushed down from the unperturbed values by \( 2 \beta^2 m^* / \hbar^2 \).

In Fig. 4 we plot the energy levels as function of the magnetic field for the circular confinement and in the presence of only the Dresselhaus spin-orbit interaction, obtained through exact numerical diagonalization of the corresponding Hamiltonian. Figure 5 presents an expanded view of the low-magnetic-field splitting of the levels originating from the same shell, and fully confirms the perturbative results discussed above (see also Ref. 34). In a given shell, energy levels with the same value for \( I, \sigma \) remain degenerate and the splitting is accompanied by a global lowering of the energy levels. In Fig. 6 we plot the energy levels for the full Hamiltonian including the nonparabolic term and both Rashba and Dresselhaus contributions to the spin-orbit interaction. We note the important effects of the nonparabolicity on the shell structure at low magnetic field (the \( p \)-shell level splitting is about 1/4 of the \( s-P \) level separation) and the high magnetic field “oscillatorylike” behavior of the levels, a combined effect of the Zeeman term (which favors level crossings) and the spin-orbit coupling (which favors the anticrossings of levels).

In the following we discuss the influence of the spin-orbit coupling on the expectation values of electron spin. Since the magnitude of the spin-orbit coupling is rather small, we expect it to have a sizable contribution only near (anti)crossover points of the energy levels (the energy level separation may become comparable to the spin-orbit-induced splitting of the energy levels). In order to analyze the lowest Zeeman-interaction-induced crossings we first estimate the critical magnetic field at which the crossing would occur in the absence of spin-orbit interaction (if we neglect the spin-orbit interaction and in the absence of the Coulomb interaction, this critical magnetic field will also correspond to the two-electron single-triplet transition point). A simple evaluation of the critical field necessary to obtain the first crossing is given by the condition that the level separation in the Landau level manifold, \( \omega_\perp \), is equal to the Zeeman splitting energy \( g \mu_B B \). Using Eq. (2.9), this condition yields \( \omega_\perp = \omega_0 / \sqrt{\gamma_\perp^2 + \gamma_\parallel^2} \), and using the connection between \( \omega_\perp \) and \( B \), and appropriate values for GaAs, we obtain \( B_{\text{crit}} (\omega_0) = 6 \text{ T/meV} \). Therefore, for \( \omega_0 = 1 \text{ meV} \) we obtain \( B_{\text{crit}} = 6 \text{ T} \). In Fig. 7 we plot the levels involved in the first three Zeeman-mediated crossings. Since the spin-orbit interaction couples states of different spin that differ at most by one unit in the angular momentum, the first two crossings between the levels \( \frac{1}{2} \) [which originates from the Fock-Darwin (FD) level \( (0,0,\uparrow) \)] and \( \frac{3}{2} \) [originating from the FD level \( (0,1,\downarrow) \)], and \( \frac{1}{2} \) [originating from the FD level \( (0,2,\downarrow) \)] are transformed into anticrossings, while the third one between \( \frac{3}{2} \) and \( \frac{5}{2} \) levels preserves its crossing character. This picture is confirmed by the spin \( z \)-projection expectation value behavior, presented in Fig. 8, where it is shown that for spin-orbit coupled levels there is only a gradual change in the spin expectation value (anticrossing), in contrast with the usual, purely Zeeman-induced crossing, where the change in the spin expectation values occurs very rapidly. The apparently unexpected large region over which the spin transition takes place is consistent with the relative increase in importance of the spin-orbit interaction for two closely spaced levels. The validity of the spin-orbit coupling selection rules is also surprising, given the coupling of different angular momentum levels through...
the nonparabolic confinement potential. However, the influence of the effects induced by the nonparabolicity of the confinement potential is rather minimal for the magnetic fields under consideration, well above the υ=2 line.

IV. ENERGY LEVELS OF THE TWO-ELECTRON DROPLET

For the two-particle problem, in the second quantization, the total Hamiltonian takes the form

\[ H = \sum_{i,\sigma} \epsilon_{i\sigma} c_{i\sigma}^{\dagger} c_{i\sigma} + \gamma \sum_{i,\sigma'\sigma} h_{ij}^{(W)} c_{i\sigma'}^{\dagger} c_{j\sigma} + \beta \sum_{i,\sigma'\sigma} h_{ij}^{(D)} c_{i\sigma'}^{\dagger} c_{i\sigma} + \frac{1}{2} \sum_{i,\sigma} \sigma c_{i\sigma}^{\dagger} \sigma c_{i\sigma} \]

FIG. 5. Excited energy levels for low magnetic field in the presence of spin-orbit interaction (only the Dresselhaus term). The plots correspond to the ρ, d, f, and g-shell level splitting [from (a) to (d)], and the rest of the parameters are the same as in Fig. 4.

FIG. 6. The energy levels as function of the magnetic field in the presence of both spin-orbit coupling mechanisms (β=10 meV Å, α=6 meV Å). The nonparabolic piece of the confining potential is characterized by γ=1, D=10α0, with α0 the effective Bohr radius (α0=98 Å in GaAs).

\[
H = \sum_{i,\sigma} \epsilon_{i\sigma} c_{i\sigma}^{\dagger} c_{i\sigma} + \gamma \sum_{i,\sigma'\sigma} h_{ij}^{(W)} c_{i\sigma'}^{\dagger} c_{j\sigma} + \beta \sum_{i,\sigma'\sigma} h_{ij}^{(D)} c_{i\sigma'}^{\dagger} c_{i\sigma} + \frac{1}{2} \sum_{i,\sigma} \sigma c_{i\sigma}^{\dagger} \sigma c_{i\sigma} \]

\[
\epsilon_{i\sigma} = \epsilon_{i0\sigma} + \frac{1}{2} \sum_{\beta=1}^{N} R_{i\beta} \epsilon_{\beta\sigma} + \frac{1}{2} \sum_{\beta=1}^{N} R_{i\beta} \epsilon_{\beta\sigma} + \frac{1}{2} \sum_{\beta=1}^{N} R_{i\beta} \epsilon_{\beta\sigma} \]

\[
\gamma = \gamma_0 \sum_{\sigma'\sigma} \epsilon_{i\sigma}(W) c_{i\sigma'}^{\dagger} c_{j\sigma} + \frac{1}{2} \sum_{i,\sigma} \sigma c_{i\sigma}^{\dagger} \sigma c_{i\sigma} \]

\[
\beta = \beta_0 \sum_{i,\sigma'\sigma} h_{ij}^{(D)} c_{i\sigma'}^{\dagger} c_{i\sigma} + \frac{1}{2} \sum_{i,\sigma} \sigma c_{i\sigma}^{\dagger} \sigma c_{i\sigma} \]

\[
\alpha = \alpha_0 \sum_{i,\sigma'\sigma} V_{ij}^{(D)} c_{i\sigma'}^{\dagger} c_{i\sigma} + \frac{1}{2} \sum_{i,\sigma} \sigma c_{i\sigma}^{\dagger} \sigma c_{i\sigma} \]

The many-body effects are induced by the Coulomb interaction [the last term in Eq. (3.13)] with the two-body matrix element \( V_{ij}^{DD} = \langle i; j | \epsilon | r^2 / 2 | k; l \rangle \). An explicit expression for the Coulomb matrix elements can be found by Fourier expansion of 1/r and further use of the algebra of the ladder operators \( a, a^\dagger, b, b^\dagger \), and reads

\[
V_{ij}^{DD} = \langle m_{2L}, n_{2L}; m_{1L}, n_{1L}; 1, 2 | V_{ij}^{DD} | m_{1R}, n_{1R}; m_{2R}, n_{2R} \rangle
\]

\[
= \frac{E_0}{\sqrt{2\pi}} \sum_{k_1=0}^{\min(m_{1L}, m_{1R})} k_1 ! \left( \begin{array}{c} m_{1L} \\ k_1 \end{array} \right) \left( \begin{array}{c} m_{1R} \\ k_1 \end{array} \right) \]

\[
\times \sum_{k_2=0}^{\min(n_{1L}, n_{1R})} k_2 ! \left( \begin{array}{c} n_{1L} \\ k_2 \end{array} \right) \left( \begin{array}{c} n_{1R} \\ k_2 \end{array} \right) \]

\[
\times \sum_{k_3=0}^{\min(m_{2L}, m_{2R})} k_3 ! \left( \begin{array}{c} m_{2L} \\ k_3 \end{array} \right) \left( \begin{array}{c} m_{2R} \\ k_3 \end{array} \right) \]

\[
\times \sum_{k_4=0}^{\min(n_{2L}, n_{2R})} k_4 ! \left( \begin{array}{c} n_{2L} \\ k_4 \end{array} \right) \left( \begin{array}{c} n_{2R} \\ k_4 \end{array} \right) \times \left( -\frac{1}{2} \right)^{k_1+k_2+k_3+k_4} \left( \begin{array}{c} k_1+k_2+k_3+k_4 \\ 1 \end{array} \right) \]

\[
\times \Gamma \left( k_1 + k_2 + k_3 + k_4 + \frac{1}{2} \right). \]
with $E_0 = \sqrt{\pi} \kappa e^2 / l_0$. The Kronecker $\delta$ function enforces the conservation of the angular momentum $R_L = (m_{1L} - n_{1L}) + (m_{2L} - n_{2L})$, $R_R = (m_{2R} - n_{2R}) + (m_{1R} - n_{1R})$. The parameter $\alpha_c$ is introduced here as a heuristic constant to take into account the screening of the Coulomb interaction by the surrounding metallic gates ($\alpha_c \leq 1$).

A generic two-particle basis vector is $|k_1, k_2\rangle = \mathcal{A}[|k_1\rangle|k_2\rangle]$, with $|k_1\rangle$ single-particle states and $\mathcal{A}$ the two-particle antisymmetrizing operator. Using this two-particle state basis, a numerical representation of the two-electron Hamiltonian Eq. (4.1) is built. In principle, there are several choices for constructing a two-electron basis set. One possibility is to use the unaltered Fock-Darwin single-particle states the single-particle problem in the absence of nonparabolicity and spin-orbit interaction. In this basis, we evaluate then the nonparabolic potential Hamiltonian matrix, the spin-orbit and Coulomb interaction Hamiltonian matrix elements, and numerically find the eigenvalues and eigenvectors of the matrix form of the Schrödinger equation. This method has the advantage that, for a fixed number of basis states, it requires the least computational time. On the other hand, it requires a relatively large basis set to achieve the required convergence in the example presented in this section, we required convergence within 1% for the first 100 energy levels. A second option is to solve the parabolically confined two-electron problem, and then use the obtained interacting two-body states to treat the nonparabolic piece of the confining potential and the spin-orbit interaction. This method requires the rotation in the two-particle Hilbert space of the nonparabolic and spin-orbit Hamiltonians (both single-body operator matrices), and is computationally more intensive. Finally, another possibility is to solve first the full single-particle problem (including nonparabolicity and spin-orbit interactions) and build the two-particle basis starting from these single-particle states. This procedure provides a “natural” two-particle basis, and, implicitly, requires the least number of states in the basis. This procedure has the disadvantage that it requires the rotation in the two-particle Hilbert space of the Coulomb interaction Hamiltonian (a two-body operator), which proves to be a rather computationally intensive operation. We have investigated all three methods and found that, while the first method is the least efficient,
the last two have comparable efficiency, with the balance slightly inclined in favor of the last one.

In the absence of spin-orbit interaction, the two-electron states are characterized by well-defined values of the spin and its projection, and, at low magnetic fields, the ground state is a singlet, $|S=0, S_z=0\rangle$, while the first excited states correspond to a degenerate triplet manifold, $|S=1, S_z=\pm 1, 0\rangle$, that is split by the Zeeman interaction (for negative values of $g^*_z$, the state $|S=1, S_z=1\rangle$ becomes the state with the lowest energy in the triplet manifold). The next step is to include the spin-orbit interaction. Consistent with the weak magnitude of the spin-orbit interaction, we expect its effects to be significant only when the two-electron levels under consideration originate from single-particle states that are coupled by spin-orbit interaction ($|\Delta R|=1, |\Delta \sigma|=1$) and their energy separation is of the same order as the spin-orbit energy splitting. The ideal region to notice these effects is near the singlet-triplet transition point presented in Fig. 9.

While outside the $B$ field region surrounding the transition point the characteristics of the energy level diagram are almost unchanged by the presence of spin-orbit interaction, near the transition point the singlet-triplet energy diagram presents an anticrossing behavior clearly determined by the coupling of the spin and orbital spatial degrees of freedom. Given the large number of parameters in the problem ($\alpha, \beta, \gamma, \alpha_s$), we have limited ourselves to a set of parameters that determines the critical magnetic field $B_c=0.92$ T observed in experiments.21

V. SPIN RELAXATION OF THE TWO-ELECTRON DROPLET

In this section we investigate the spin-dephasing and energy-relaxation rates for the two-electron droplet. We assert that the main relaxation channel is related to an inelastic spin-flip relaxation mechanism mediated by spin-orbit interaction and involving the emission of a longitudinal acoustic phonon.15 We note that electron-phonon scattering processes involving a spin flip are forbidden in the absence of spin-orbit interaction or the presence of other spin-relaxation mechanisms, such as hyperfine interaction35. This is because, while providing the necessary energy-relaxation mechanism, the interaction with phonons cannot change the electronic spin. In the presence of spin-orbit coupling, the spin of two-electron droplet relaxes through the admixture mechanism brought about by spin-orbit interaction (the spin and its $z$ projection are no longer good quantum numbers, and therefore the spin-relaxation processes do not require the conservation of spin anymore), while its energy is relaxed through the emission of a longitudinal acoustic phonon. The decay rate is dependent on both electron-phonon coupling and the magnitude of the spin-orbit interaction. To simplify the analysis, we neglect the influence of the confining geometry (interface and gates) on the phonon modes,18 and assume that the electronic system interacts with bulk phonons. Since LO-phonon scattering is strongly suppressed in quantum dots, we limit our analysis to the interaction with longitudinal acoustic (LA) phonons.

A. Relaxation rates of the two-electron droplet

We assume that the electron and phonon interact by means of a deformation potential, with the interaction Hamiltonian given by

$$H_{e-ph} = \sum_{i,j,q} M_{ij}(q)(b_q + b_q^\dagger)c_i^\dagger c_j. \tag{5.1}$$

Here, $c_i$ ($c_i^\dagger$) are electron annihilation (creation) operators, and $b_q$ ($b_q^\dagger$) annihilates (creates) a phonon in a momentum state $|q\rangle$. The magnitude of the electron-phonon interaction is characterized by the matrix elements

$$M_{ij}(q) = \Lambda(q) \int dr \, \psi_i^*(r) e^{iqr} \psi_j(r), \tag{5.2}$$

where, for coupling of electrons to LA phonons through a deformation potential $D$, the form factor is given by36 (assuming a linear dispersion relation for the LA phonons, $E_q = \hbar c_q q$)

$$\Lambda(q) = \sqrt{\frac{D^2 \hbar}{2pc_s q^2}}. \tag{5.3}$$

For GaAs the material parameters present in Eq. (5.3) have the values $\rho=5300$ kg/m$^3$, $c_s=3700$ m/s, and $D=8.6$ eV.
Using the Fermi golden rule, the phonon scattering rate of a given two-electron state |in⟩ by emission of a phonon is given by
\[ \Gamma_{in} = \sum_f \Gamma_{in-f}, \]  
\[ (5.4a) \]
with
\[ \Gamma_{in-f} = \frac{2\pi}{\hbar} \sum_{q, ij} |M_{ij}(-q)(f|c_i^\dagger c_j|in)|^2 \delta(\epsilon_f + \hbar \omega_q - \epsilon_m) \]
\[ \times [n_q(\hbar \omega_q, T) + 1]. \]  
\[ (5.4b) \]
Here, the summation is over all phonon wave vectors q, and \( \epsilon_{in}, \epsilon_f, \) and \( \hbar \omega_q \) are the initial and the final energies of the electron, and the energy of the phonon involved in the transition, respectively; \( n_q(\hbar \omega_q, T) \) is the thermal occupation number of the phonon state with energy \( \hbar \omega_q \) at the temperature \( T \). We assume in our calculations \( T \rightarrow 0 \). A detailed calculation of the relaxation rate \( \Gamma_{in-f} \) is presented in the Appendix A.

**B. Spin-relaxation rates of the two-electron droplet: A simplified approach**

In this section, we evaluate the dephasing times of the two-electronic droplet in a simplified model, in which we consider only four single-particle energy levels (generated by the Fock-Darwin levels \([0,0,\sigma]\) and \([0,1,\sigma']\), with \( \sigma, \sigma' = \pm 1 \). Since the most relevant feature of the phonon-mediated dephasing process is the admixture relaxation mechanism brought about by spin-orbit interaction, we also neglect here the effects of the nonparabolicity of the confining potential. The six two-electron configurations generated from the lowest four single-electron states can be written in a compact form as
\[ |1⟩ = c_{0,0,-1}^\dagger c_{0,1,0}^\dagger |0⟩, \]  
\[ (5.5a) \]
\[ |2⟩ = c_{0,0,-1}^\dagger c_{0,1,0}^\dagger |0⟩, \]  
\[ (5.5b) \]
\[ |3⟩ = c_{0,0,-1}^\dagger c_{0,1,1}^\dagger |0⟩, \]  
\[ (5.5c) \]
\[ |4⟩ = c_{0,0,0}^\dagger c_{0,1,-1}^\dagger |0⟩, \]  
\[ (5.5d) \]
\[ |5⟩ = c_{0,0,1}^\dagger c_{0,1,0}^\dagger |0⟩, \]  
\[ (5.5e) \]
\[ |6⟩ = c_{0,1,-1}^\dagger c_{0,1,1}^\dagger |0⟩, \]  
\[ (5.5f) \]
where \( c_{m,n,\sigma}^\dagger \) are the creation operators introduced previously and |0⟩ is the two-electron vacuum state. In the absence of nonparabolicity of the confining potential and of the spin-orbit interaction, the total spin and z-projection of the spin are conserved and the two-electron levels separate into singlet and triplet manifolds. However, some elements of the truncated configuration-interaction basis in Eq. (5.5) are not eigenvectors of total spin and spin z-projection operators and in order to construct a common basis for the total Hamiltonian and the total and z-projection spin operators we need to perform a rotation in the two-electron Hilbert space. The simplest example of such a basis is four dimensional and consists of the conventional singlet (\( |S=0, S_z=0⟩ \)) and triplet (\( |S=1, S_z=0, \pm 1⟩ \)) manifolds
\[ |1⟩ = |1⟩, \]  
\[ (5.6a) \]
\[ |2⟩ = |2⟩, \]  
\[ (5.6b) \]
\[ |3⟩ = |3⟩ + |4⟩, \]  
\[ (5.6c) \]
\[ |4⟩ = |5⟩, \]  
\[ (5.6d) \]
or, equivalently,
\[ |1⟩ = |Φ_s⟩ \otimes |S=0, S_z=0⟩, \]  
\[ (5.7a) \]
\[ |2⟩ = |Φ_A⟩ \otimes |S=1, S_z=−1⟩, \]  
\[ (5.7b) \]
\[ |3⟩ = |Φ_A⟩ \otimes |S=1, S_z=0⟩, \]  
\[ (5.7c) \]
\[ |4⟩ = |Φ_A⟩ \otimes |S=1, S_z=1⟩. \]  
\[ (5.7d) \]
Here,
\[ Φ_s = |m=0, n=0⟩ \]  
\[ (5.8a) \]
\[ = |m=0, n=0⟩ \]  
\[ (5.8b) \]
and
\[ |S=0, S_z=0⟩ = \frac{|1,1⟩ − |1,−1⟩}{\sqrt{2}}, \]  
\[ (5.9a) \]
\[ |S=1, S_z=−1⟩ = |1,−1⟩, \]  
\[ (5.9b) \]
\[ |S=1, S_z=0⟩ = \frac{|1,−1⟩ + |1,1⟩}{\sqrt{2}}, \]  
\[ (5.9c) \]
\[ |S=1, S_z=1⟩ = |1,−1⟩ \]  
\[ (5.9d) \]
are the usual symmetric and antisymmetric two-particle wave functions and the corresponding singlet and triplet manifold spinors, respectively.

The two-electron spin-orbit Hamiltonian is given by
\[ H_{SO} = -\frac{1}{\hbar} \sum_{i=1,2} \left[ (β \tilde{π}_i^{(i)} - iα \tilde{π}_i^{(i)} ) \sigma_+^{(i)} + (β \tilde{π}_i^{(i)} + iα \tilde{π}_i^{(i)} ) \sigma_-^{(i)} \right] \]  
\[ (5.10) \]
where \( \tilde{π}_s^{(i)}, \sigma_s^{(i)} \) are the scaled canonical momentum and spin projection operators of the individual electrons defined in (3.9).

In the truncated basis introduced above, the total Hamiltonian
can be easily diagonalized. We emphasize that the structure of the spin-orbit part of the Hamiltonian, which is ultimately responsible for the relaxation of the electronic spin, offers a very intuitive path in understanding different relaxation processes of the two-electron droplet spin. We argue that the spin-unpolarized triplet state \( |3⟩ \) is not mixed with the singlet state or the other triplet components even in the presence of spin-orbit interaction. In fact, it is possible to show without any involved calculations that the spin-unpolarized triplet state \( |3⟩ \) is not coupled to the singlet state \( |1⟩ \) by the spin-orbit interaction. The evaluation of the matrix element \( \langle 3 | H_{SO} | 1⟩ \) involves expressions such as \( a_{x}^{(i)} | 1, \uparrow \rangle \), \( a_{x}^{(i)} | 1, \downarrow \rangle \), etc. (with \( i = 1, 2 \)). Since the operators \( a_{x}^{(i)} \) are switching the spin of the given electron act upon, we can argue that the matrix element that connects the unpolarized triplet state \( |3⟩ \) to the singlet state \( |1⟩ \)

\[
\langle 3 | H_{SO} | 1⟩ = 0,
\]

vanishes. The \( a_{x}^{(i)} \) operators acting on the right-hand state,

\[
\langle a_{x}^{(i)} | 1, \uparrow \rangle - | 1, \downarrow \rangle \sqrt{2} \]

will flip the spin of at least one electron. As a result, the new right-hand state will either contain aligned spins or will be the two-electron vacuum state (if the spin cannot be flipped). Consequently, the scalar product of the new right state with the left state (which contains combinations of opposite spins) will be zero. Therefore, we have

\[
\langle 3 | H_{SO} | 1⟩ = 0,
\]

and we conclude that regardless of the magnitude of the spin-orbit coupling, the triplet state \( |3⟩ \) is not mixed with the singlet state \( |1⟩ \). The situation is different for the other matrix elements, since they involve states containing aligned spins, and by flipping one of them the matrix elements \( \langle S = 1, S_z = ± 0 | a_{x}^{(i)} | S = 1, S_z = ± 1 \rangle \), \( \langle S = 1, S_z = ± 1 | a_{x}^{(i)} | S = 0, S_z = 0 \rangle \) may become nonzero. However, as the detailed calculations presented in the remainder of this section show, the spin unpolarized state \( |3⟩ \) remains also uncoupled to the other triplet manifold states \( \langle 3 | H_{SO} | 2⟩ = \langle 3 | H_{SO} | 4⟩ = 0 \) due to the vanishing spatial part of matrix elements, whereas the spin-polarized states \( |2⟩, |4⟩ \) are mixed by the spin-orbit interaction with the singlet state \( |1⟩ \).

The explicit form of the matrix of the spin-orbit interaction is given by

\[
\tilde{H}_{SO} = \begin{pmatrix}
0 & H_{SO}^{(1,2)} & 0 & H_{SO}^{(1,4)} \\
H_{SO}^{(2,1)} & 0 & H_{SO}^{(2,3)} & 0 \\
0 & H_{SO}^{(3,2)} & 0 & H_{SO}^{(3,4)} \\
H_{SO}^{(4,1)} & 0 & H_{SO}^{(4,3)} & 0
\end{pmatrix},
\]

with the shortcut notations

\[
H_{SO}^{(1,2)} = (H_{SO}^{(2,1)})^\dagger = \frac{\sqrt{2}}{2l} \langle \Phi_{3} | a_{x}^{(1,2)} | \Phi_{A} ⟩,
\]

\[
H_{SO}^{(1,4)} = (H_{SO}^{(4,1)})^\dagger = \frac{\sqrt{2}}{2l} \langle \Phi_{3} | a_{x}^{(1,4)} | \Phi_{A} ⟩,
\]

\[
H_{SO}^{(2,3)} = (H_{SO}^{(3,2)})^\dagger = \frac{\sqrt{2}}{2l} \langle \Phi_{A} | h_{Z} | \Phi_{3} ⟩,
\]

\[
H_{SO}^{(3,4)} = (H_{SO}^{(4,3)})^\dagger = \frac{\sqrt{2}}{2l} \langle \Phi_{A} | h_{Z} | \Phi_{3} ⟩,
\]

and

\[
h_{(1,2)} = - \sum_{i=1,2} (1)^{-1} (b_{\uparrow} a_{\uparrow} - i a_{\uparrow} b_{\uparrow}'),
\]

\[
h_{(1,4)} = \sum_{i=1,2} (1)^{-1} (b_{\downarrow} a_{\downarrow} - i a_{\downarrow} b_{\downarrow}'),
\]

\[
h_{(2,3)} = - \sum_{i=1,2} (b_{\uparrow} a_{\uparrow} - i a_{\uparrow} b_{\uparrow}'),
\]

\[
h_{(3,4)} = - \sum_{i=1,2} (b_{\downarrow} a_{\downarrow} + i a_{\downarrow} b_{\downarrow}').
\]

These matrix elements can be evaluated using the transformation of the spatial variables in terms of the two-dimensional harmonic oscillator ladder operators given by (3.9). After some straightforward calculations, we obtain

\[
h_{(1,2)} = i \{ (1 + \theta_{h}) [b_{a_{1} - a_{2}} - i a_{a_{1} - a_{2}}] \\
- (1 - \theta_{h}) [b_{(a_{1} - a_{2})} - i a_{(a_{1} - a_{2})}] \},
\]

\[
h_{(1,4)} = i \{ (1 + \theta_{h}) [b_{a_{1} - a_{2}} + i a_{a_{1} - a_{2}}] \\
- (1 - \theta_{h}) [b_{(a_{1} - a_{2})} + i a_{(a_{1} - a_{2})}] \},
\]

\[
h_{(2,3)} = - i \{ (1 + \theta_{h}) [b_{a_{1} + a_{2}} - i a_{a_{1} + a_{2}}] \\
- (1 - \theta_{h}) [b_{(a_{1} + a_{2})} - i a_{(a_{1} + a_{2})}] \},
\]

\[
h_{(3,4)} = - i \{ (1 + \theta_{h}) [b_{a_{1} + a_{2}} + i a_{a_{1} + a_{2}}] \\
- (1 - \theta_{h}) [b_{(a_{1} + a_{2})} + i a_{(a_{1} + a_{2})}] \},
\]

The main feature of the operators in Eqs. (5.18) is that in the presence of only one spin-orbit coupling mechanism (\( \alpha = 0 \) and \( \beta \neq 0 \) or \( \alpha \neq 0 \) and \( \beta = 0 \)) they do not mix creation and annihilation operators of the same kind (\( a \) or \( b \)).

For the choice of \( \Phi_{A} \) in Eqs. (5.8), the spin-orbit interaction Hamiltonian becomes...
We first analyze the two-electron energy levels and the spin behavior as the system passes through the singlet-triplet transition. In Fig. 10, we plot the energy levels and the excitation energy for the two-electron droplet in the four-state approximation. The plots correspond to $\omega_0 = 1\text{ meV}$, $\alpha = 1$, $\beta = 10\text{ meV Å}$, and $\alpha = 6\text{ meV Å}$.

$$\hat{H}_{SO} = \frac{1 - \theta_h}{l} \begin{pmatrix} 0 & \alpha & 0 & i\beta \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i\beta & 0 & 0 & 0 \end{pmatrix}.$$ 

(5.19)

We first analyze the two-electron energy levels and the spin-polarized triplet state $|S=1, S_z = -1\rangle$ is coupled to the singlet state due to the presence of the Rashba contribution to the spin-orbit Hamiltonian, whereas the other spin-polarized triplet state $|S=1, S_z = 1\rangle$ is coupled to the singlet state due to the presence of the Dresselhaus contribution to the spin-orbit Hamiltonian.

This feature already suggests an important asymmetry of the energy diagram for the two-electron droplet: below the transition point, when the triplet states constitute an excited manifold, there is a state $|S=1, S_z = 0\rangle$ for which spin-flip processes remain forbidden even in the presence of spin-orbit coupling. Therefore, in this region, both $|S=1, S_z = 0\rangle$ (excited state) and $|S=0, S_z = 0\rangle$ (ground state) should be experimentally observable. After the transition point, the triplet states have lower energy than the singlet state. It is clear that the singlet state, which is now the most excited state, may decay through a spin-flip process, since it is coupled to both $|S=1, S_z = \pm 1\rangle$ triplet levels. Therefore, in this region only the triplet manifold should be experimentally observed. We also mention that coupling of different spin-polarized triplet states to the singlet state is associated with different terms in the spin-orbit Hamiltonian. For example, if only the Dresselhaus term is considered, then only the $|S=1, S_z = 1\rangle$ triplet state is coupled to the singlet state. However, if both Dresselhaus and Rashba contributions are taken into account, both $|S=1, S_z = 1\rangle$ and $|S=1, S_z = -1\rangle$ triplet states are coupled to the singlet state. This behavior is determined by the different functional dependence of the Dresselhaus and Rashba term on the individual spins of the electrons.

We have also investigated the spin and spin projection as a function of the magnetic field. As shown in Fig. 11, the spin of the two-electron levels coupled by spin-orbit interaction changes gradually as the system goes through the singlet-triplet transition. The spin of the unpolarized triplet state $|S=1, S_z = 0\rangle$ is unaffected throughout the range of magnetic field considered here. The next step is to evaluate the spin dephasing rates of the two-electron states. In the simplified approach developed in this section, we consider only the decay processes involving the emission of a LA phonon and the direct transitions to the ground state. Since the electron-phonon interaction is symmetric with respect to particle exchange and does not change the total spin, the only nonzero matrix elements of the electron-phonon coupling Hamiltonian are the diagonal ones $\langle \Phi_3 | H_{e-ph} | \Phi_3 \rangle$, $\langle \Phi_4 | H_{e-ph} | \Phi_4 \rangle$. To take full advantage of the simplified model analyzed here, we evaluate the electron-phonon scattering rates by direct numerical integration of Eq. (5.4). As shown in Fig. 12, in agreement with our previous assertion, the triplet state $|S=0, S_z = 0\rangle$ is stable against LA-phonon deexcitation process. This long-lived state should be experimentally observed regardless of the magnitude of magnetic field. Since spin-flip processes are allowed in the second order of perturbation in the spin-orbit coupling strength, the dephasing rates increase in the neighborhood of the transition point (the level separation becomes comparable to the spin-orbit-induced splitting). The difference between the decay rates of the levels originating from different triplet states $|S=1, S_z = 1\rangle \rightarrow |S=0, S_z = 0\rangle$, $|S=1, S_z = -1\rangle \rightarrow |S=0, S_z = 0\rangle$ is determined by the different origin of the admixture mechanism for these states, respec-
relatively the interplay between the Dresselhaus and Rashba contributions. In principle, this relative difference in the dephasing rates of the triplet states may be used to assert the relative importance of the Dresselhaus and Rashba terms in a given quantum dot.

The general features of the scattering rate dependence on the magnetic field are determined by both the relative magnitude of the spin-orbit interaction and its functional dependence on the spinorial degrees of freedom. Since the phonon has zero spin, the spin relaxation through emission of a LA phonon conserves the total spin and spin $z$ projection of the electron. Consequently, the spin relaxation (taking place between electron levels that are an admixture of states with definite spin values) is maximized when the relative contribution of the same-spin states to the levels involved in the transition is maximized. This condition determines the critical field at which the scattering rates peak. In a first order of approximation, this condition is met when the difference between the expectation values of the total spin and the spin $z$ projection for transition levels is the smallest, which happens approximately at the singlet-triplet “crossing” critical point (see Fig. 11). The magnitude of the spin-orbit interaction also determines the magnetic field range over which the spin mixing has a sizable magnitude (from Fig. 11, this interval is roughly from 0.5 to 1.4 T), which, in turn, determines the width of the scattering rate curves in Fig. 12. The values of the spin relaxation rates (on order of microseconds in our simulations) are determined by both the magnitudes of phonon-electron interaction (which we consider to be mediated only by a deformation potential interaction) and spin-orbit interaction.

C. Spin-relaxation rates of the two-electron droplet

In this section we present the results for the relaxation rates of the first three excited states of the two-electron system obtained through the numerical diagonalization of the full two-electron Hamiltonian (4.1). The size of the basis set of two-particle Hilbert space is varied until we achieve convergence within 1% for the first 20 energy levels. Within the framework of the third numerical method described above, this convergence is achieved for an 861-dimensional basis. The calculation of the LA-phonon-mediated spin-relaxation rates, although it involves a similar path to the case presented in Sec. V B, becomes now very computationally intensive. For an 861-dimensional two-electron Hilbert space, the direct numerical evaluation of the relaxation rates for the first three excited states including all terms appearing in Eq. (A1) is beyond our computational resources. Consequently, we introduce a selective procedure in the summation in Eq. (A1), and discard states whose contribution to the initial and final states involved in the transition is smaller than a prescribed numerical value $e_{err}$, for which $|A_{k_i,k_f}|^2 < e_{err}$.

Figure 13 shows the spin-relaxation rates for the first three excited states of the two-electron droplet as a function of the applied magnetic field.
FIG. 13. Electron–LA-phonon scattering rates of the first three excited states of the two-electron system as a function of the applied magnetic field. The inset shows a magnified curve corresponding to the relaxation of the state |2⟩, which is of unobservable magnitude in the main plot (note the different y-axis scaling in the main plot and the inset).

From the experimental observations of the excited-state transport measurements, and predicts an accelerated decay of the excited levels in the vicinity of the single-triplet transition.

We have assumed that the electrons are subject to a scattering mechanism associated with the emission and absorption of longitudinal acoustic phonons. Using exact diagonalization techniques, we have evaluate the magnetic field evolution of the energy spectrum and total spin of the two-electron droplet in the presence of spin-orbit interaction. The spin-orbit interaction mixes singlet and three triplet components. This mixing, combined with electron-phonon interaction, is responsible for spin relaxation. We have shown that the relaxation rates for the two spin-polarized triplet states depend strongly on the SO mechanism involved and this allows for the separate measurement of the Dresselhaus and of the Rashba contribution.

The spin-relaxation rates due to electron-phonon scattering were evaluated here in first-order perturbation theory. In the absence of the spin-orbit interaction, the transitions between the singlet and triplet states through the relaxation channel involving the emission of an acoustic phonon are forbidden by the spin-conservation rules. Our study shows that the spin-orbit interaction provides an effective coupling between the spin-polarized triplet states \( |S=1, S_z = \pm 1 \rangle \) and the singlet state \( |S=0, S_z = 0 \rangle \). However, the transition involving spin-unpolarized singlet and triplet states \( |S=1, S_z = 0 \rangle \leftrightarrow |S=0, S_z = 0 \rangle \) is extremely weak. The calculated scattering rates from the excited states to the ground state of the two-electron system clearly confirm this picture and reveal a microsecond time scale for the single-triplet relaxation through spin-orbit-mediated acoustic phonon emission.

We then argue that if the dwell time of the second electron in the dot (the effective lifetime of the two-electron droplet) is longer than the relaxation times predicted by our calculations, spin-orbit coupling may provide a direct interpretation of the unusual transport measurements. For magnetic fields below the single-triplet transition point, the spin-unpolarized state from the triplet manifold of excited states cannot relax to the ground state, and should be observed in transport measurements. For higher magnetic fields, the excited state, now a singlet, has two available relaxation channels \( |S=0, S_z = 0 \rangle \leftrightarrow |S=1, S_z = \pm 1 \rangle \), and may relax to the ground state before being experimentally resolved, thus causing a quenching of the measured excited-state transport current.

In conclusion, our study offers a possible interpretation of the experimental observations of the excited-state current.
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APPENDIX A: EVALUATION OF THE RELAXATION RATES FOR THE TWO-ELECTRON DROPLET

In this appendix we evaluate the relaxation rates of the energy levels of the two-electron system described in Sec. IV. The initial and final levels involved in the transition are obtained through numerical diagonalization as a finite superposition of the two-particle basis introduced in Sec. IV,

\[ |\text{in}\rangle \equiv \sum_{k_1k_2} A_{k_1k_2}^{in} |k_1k_2\rangle, \quad (A1a) \]

\[ |\text{f}\rangle \equiv \sum_{k_1k_2} A_{k_1k_2}^{f} |k_1k_2\rangle. \quad (A1b) \]

Then

\[ \Gamma_{\text{in} \rightarrow \text{f}} = \frac{2\pi}{\hbar} \sum_{k} \sum_{q} M_{ij}(-q) A_{k_1k_2}^{f} A_{k_1k_2}^{in} \times \langle k_1k_2|c_j^{\dagger}c_i|k_1k_2\rangle^2 \delta(e_{in} - e_{f} - \hbar \omega_q). \quad (A2) \]

Using the matrix elements of \( c_j^{\dagger}c_i \) (a one-body operator)

\[ \langle k_1k_2|c_j^{\dagger}c_i|k_1k_2\rangle = \delta_{k_1, k_2} \delta_{k_2, j} \delta_{k_1, i} \delta_{k_1, 1} + \delta_{k_2, k_2} \delta_{k_1, j} \delta_{k_2, i} \delta_{k_2, 1} \\
- \delta_{k_1, k_2} \delta_{k_2, j} \delta_{k_1, i} \delta_{k_1, 1} - \delta_{k_2, k_2} \delta_{k_2, j} \delta_{k_1, i} \delta_{k_2, 1}, \]

the scattering rate (A2) becomes

\[ \Gamma_{\text{in} \rightarrow \text{f}} = \frac{2\pi}{\hbar} \sum_{q} \sum_{k} M_{ij}(-q) A_{k_1k_2}^{f} A_{k_1k_2}^{in} \times \left[ A_{k_2, j}^{f} A_{k_1, i}^{in} - A_{k_1, j}^{f} A_{k_2, i}^{in} \right] \delta(e_{in} - e_{f} - \hbar \omega_q). \quad (A3) \]

Furthermore, by introducing the matrix \( G_{\text{in} \rightarrow \text{f}} \), defined by

\[ G_{\text{in} \rightarrow \text{f}} = \sum_{k} \left( A_{k_1k_2}^{f} A_{k_1k_2}^{in} + A_{k_1k_2}^{f} A_{k_1k_2}^{in} - A_{k_1k_2}^{f} A_{k_1k_2}^{in} - A_{k_1k_2}^{f} A_{k_1k_2}^{in} \right), \quad (A5) \]

Eq. (A4) yields

\[ \Gamma_{\text{in} \rightarrow \text{f}} = \frac{2\pi}{\hbar} \sum_{q} \sum_{ij} M_{ij}(-q) G_{\text{in} \rightarrow \text{f}} \delta(e_{in} - e_{f} - \hbar \omega_q). \quad (A6) \]

We now evaluate the matrix element \( M_{ij}(-q) \) defined by

\[ M_{ij}(-q) = \lambda \sqrt{q} \int \text{d}r \, \psi_1^*(r)e^{-i\mathbf{r} \cdot \mathbf{q}} \psi_2(r), \quad (A7) \]

with \( \lambda = D(h/2Vc_x)^{1/2} \).

In the approximation that the vertical confinement is much stronger than the lateral confinement, the basis of the electronic Hilbert space \( \{ \psi_1(r) \} \) contains factorized wave functions

\[ \psi_1(r) = \hat{\phi}_x^{(i)}(z) \hat{\phi}_y^{(m,n)}(x,y), \quad (A8) \]

with \( \hat{\phi}_y^{(m,n)}(x,y) \) two-dimensional harmonic oscillator wave functions. Under these conditions, Eq. (A7) becomes

\[ M_{ij}(-q) = \lambda \left( q_z^2 + q_{xy}^2 \right)^{1/4} \int \text{d}z \, \hat{\phi}_x^{(i)}(z) \hat{\phi}_y^{(m,n)}(x,y), \quad (A9) \]

with

\[ \int \text{d}r \, \psi_1^{(m,n)}(r_1)e^{-i\mathbf{r}_1 \cdot \mathbf{q}_1} \psi_2^{(m,n)}(r_2), \quad (A11) \]

and \( r_i \) and \( q_i \) are the in-plane \((xy)\) components of the three-dimensional vectors \( \mathbf{r} \) and \( \mathbf{q} \).

Furthermore, we assume that the vertical confinement is so strong that the electron remains always in its “vertical” ground state. In the infinite-well approximation, the \( z \) integration can be easily performed, and we obtain

\[ F_z(q_z) = \frac{\sin(q_zL/2)}{q_zL/2} \left( 1 - (1/\pi^2)(q_zL/2)^2 \right). \quad (A12) \]

In evaluating \( F_{xy}^{(m,n,p,m,n)}(q_z) \), we use the same method as the one introduced in the evaluation of the Coulomb matrix elements, and obtain

\[ F_{xy}^{(m,n,p,m,n)}(q_z) = \frac{e^{-|q_z|^2}}{[m_1! n_1! m_1! n_1!]^{1/2}} \sum_{p_1} \sum_{p_2} \left( \begin{array}{c} n_1 \\ m_1 \\ p_1 \end{array} \right) \left( \begin{array}{c} m_1 \\ n_1 \\ p_2 \end{array} \right) p_1! p_2! \times (-iQ_z)^{p_1+m_1-p_1-p_2} \]
\( \times (-iQ_{||})^{m_{||}+m_{\perp}-1} \cdot P_{z} \),

(A13)

with \( q_{||} = (q_{z}, \phi_{z}) \) and

\[ |Q_{||}| = \frac{|f_{0}(q_{||})|}{\sqrt{2}}, \]

(A14)

\[ Q_{||} = |Q_{||}| e^{i\phi_{z}}. \]

(A15)

The scattering rate by (A4) is now given by

\[ \Gamma_{in-f} = \lambda^{2} \sum \sum (q_{z}^{2} + q_{||}^{2})^{1/2} F_{q}(q_{z})^{2} \]

\[ \times \left| \sum F_{m_{||}, m_{\perp}}^{(m_{||}, m_{\perp})}(q_{||}) G_{m_{||}, m_{\perp}}^{in-f} \right|^{2} \]

\[ \times \delta(\Delta \epsilon_{in,f} - \hbar \omega_{q}), \]

(A16)

where \( \Delta \epsilon_{in,f} = \epsilon_{in} - \epsilon_{f} \). Here, we first perform the \( z \) integral

\[ I_{z}(q_{z}) = \int dq_{z} (q_{z}^{2} + q_{||}^{2})^{1/2} F_{q}(q_{z})^{2} \delta(\Delta \epsilon_{in,f} - \hbar \omega_{q}) \]

\[ = \frac{1}{(\hbar c_{s})^{2}} \left[ \frac{\Delta \epsilon_{in,f}^{2}}{(\hbar c_{s} q_{||})^{2}} \left| F_{m_{||}, m_{\perp}}^{(m_{||}, m_{\perp})}(q_{||}) \right|^{2} \right]^{1/2}. \]

(A17)

Moreover, the angular in-plane integration in

\[ \Gamma_{in-f} = \lambda^{2} \sum \sum I_{z}(q_{z}) \left| \sum F_{m_{||}, m_{\perp}}^{(m_{||}, m_{\perp})}(q_{||}) G_{m_{||}, m_{\perp}}^{in-f} \right|^{2} \]

(A18)

can also be performed analytically, and we obtain

\[ \Gamma_{in-f} = \lambda^{2} \int dq_{z} \int_{0}^{2\pi} d\phi_{z} F_{q}(q_{z}) \]

\[ \times \left| \sum F_{m_{||}, m_{\perp}}^{(m_{||}, m_{\perp})}(q_{||}) G_{m_{||}, m_{\perp}}^{in-f} \right|^{2} \]

\[ = 2 \pi \lambda^{2} \int dq_{z} F_{q}(q_{z}) \]

\[ \times \left| \sum F_{m_{||}, m_{\perp}}^{(m_{||}, m_{\perp})}(q_{||}) G_{m_{||}, m_{\perp}}^{in-f} \right|^{2} \]

\[ \times \delta[(l_{f} + l_{i}) - (l_{f} + l_{i})], \]

(A19)

where \( l = m - n \) is the quantum number for angular momentum of the electron in the states \( |m, n, \sigma, \rangle \), and we introduced the notation

\[ \Gamma_{xy}^{(m_{||}, m_{\perp})}(q_{||}) = \frac{e^{-|Q_{||}|^{2}}}{\left[ |m_{||}! n_{1}! m_{\perp}! n_{f}! \right]^{1/2}} \]

\[ \times \sum_{p_{1}} \sum_{p_{2}} \left( \frac{n_{i}}{p_{1}} \frac{n_{j}}{p_{2}} \frac{m_{i}}{p_{1}} \frac{m_{j}}{p_{2}} \right) p_{1}! p_{2}! \]

\[ \times (-iQ_{||})^{m_{||}+m_{\perp}-1} \cdot P_{z} \]

(A20)

However, the integral over the in-plane phonon momentum magnitude \( q_{||} \) in Eq. (A20) has to be performed numerically. We note that the integrand possesses a weak (square root) integrable singularity, which will give rise to a maximum in the scattering rate when the electron level energy difference matches the energy of the emitted photon \( \Delta \epsilon_{in,f} = \hbar \omega_{q} \).
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40 The vertical confinement is approximated by a quantum well of width $L_z$ centered on the origin, and of infinite depth. The wave functions are given by $\psi^0_i = \sqrt{2/L_z} \sin(i\pi L_z/(z-L_z/2))$.  

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