Theoretical Models of Eddy Current Interaction with Defects

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Summary

Quantitative nondestructive evaluation (NDE) using eddy current techniques should be based on a good physical understanding and good physical models. Once these models have been developed, they can be applied to the two broad classes of tasks in NDE: the forward problem, predicting the probe response from a known physical situation, and the inverse problem, predicting some of the physical parameters from a measured probe response. The theory and the numerical methods used to find approximate solutions to these problems are presented in this thesis for a broad class of geometries, probes and defects.

The structure of typical workpieces divides naturally into two groups, planar and cylindrical. The planar media types are presented first. One of the contributions of this thesis is the general description of the field interactions for planar media with any number of layers, with the source any layer. This general theory developed for planar types is then extended in a consistent way to include cylindrical structures. Once in place, this theory was used to develop a layered media forward model which predicts probe responses to a large class of stratified conductors. Validation exercises are then presented, along with a brief applications section.

The theory developed for unblemished media is extended to model probe/flaw interactions in planar and cylindrical stratified conductors based on a simple flaw model. The volume integral method is then used to find approximate solutions to the three-dimensional forward problem. The implementation of this general purpose forward model, capable of predicting probe responses to a wide variety of defects: intergranular attack (IGA), surface breaking cracks, embedded cracks, inclusions, etc., represents one of the major contributions of this work. The model has been validated by comparing model predictions with analytical results, experimental results and international benchmarks. The power of the models is then presented in a brief applications section.

A physical model of IGA corrosion is then presented, along with a novel approach for solving the inverse problem, that is, predicting the depth of corrosion from a measured probe response. The analytical and numerical analysis used in this inverse model is presented. Model validation work based on both experimental and numerical data is presented and indicates that this approach is both robust and accurate.
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Chapter 1

Introduction

_The chess-board is the world; the pieces are what we call the laws of Nature. The player on the other side is hidden from us. We know that his play is always fair, just, and patient. But also we know, to our cost, that he never overlooks a mistake, or makes the smallest allowance for ignorance._

Thomas Henry Huxley 1825–1895

1.1 Nondestructive evaluation

Nondestructive evaluation (NDE), in the ideal sense, lets one see inside materials to locate and size defects. All NDE procedures have an illumination source: visible light, ultrasonic waves, gamma rays, electromagnetic waves, etc., and some sort of media on which the illuminated image is projected or recorded: retina, transducers, film, impedance analyzers, etc. It is this projected image, scattered from the defect, that is the basis of NDE.

Not all NDE inspection involves defect discrimination. Many NDE applications involve the inspection of variations of layer or plate thicknesses, which could be important for quality control in a manufacturing setting. Other applications might be the monitoring of changes in the stresses in materials by monitoring changes in their material properties, for instance inspecting the tensile and compressive loads which occur in rail road rails due to environmental temperature changes.

There are two distinct aspects of NDE, qualitative and quantitative. Qualitative NDE is concerned only with detection of a defect or whether some threshold level has been violated. On the other hand, quantitative NDE is concerned with locating and sizing defects or provided numerical measures of depth or material properties. Historically,
CHAPTER 1. INTRODUCTION

Qualitative NDE has been the most widely used. However, if more information is needed than just the mere presence of cracks or corrosion, then quantitative techniques must be applied.

Figure 1.1: Conceptual model of IGA in an austenitic stainless steel

Defects can be combinations of inclusions, cracks, voids, corrosion or variations in material properties. Figure 1.1, for example, schematically shows a cross-section of the micro-crack structure of intergranular attack (IGA) in an austenitic stainless steel. This is a form of environmentally assisted cracking that occurs in the grain structure. A feature of this type of cracking is that it tends to form as layers achieving a mature depth. In practice multiple initiation sites lead to isolated patches which can then interlink.

To some extent the type of defect determines the NDE technique to be used, because the defect must scatter the illumination source in a detectable way. In dye penetrant testing, the dye interacts with the crack structure on the surface of the material shown in the Figure, indicating the extent and possibly some measure of the density of the corrosion. Eddy current inspection, on the other hand, can interact with the entire volume of the crack structure, and therefore contains information about all of the corrosion.

We will be interested in eddy current nondestructive examination. Eddy current techniques are based on the physical phenomenon that the eddy currents induced in a test-piece by an eddy current probe causes the impedance of the probe to change. The resistive losses which occur in the test-piece increase the resistive loss in the coil. The magnetic flux arising from the induced current opposes the magnetic flux generated by the current in the coil, hence causing a change in the inductive component of the coil. Variations in the electromagnetic properties of the test-piece, due to a defect, variations in geometry, etc., will cause the probe's impedance to vary as it is scanned over the workpiece. These changes in signal are the basis of eddy current NDE. Ideally, an eddy current probe would provide a field measurement at a large number of points for each probe position, instead it gives one complex number for each probe position. The NDE inspector only gets to see a "shadow", a 1D projection, of this induced field. It is this nature of eddy current inspection which makes it hard to use the measurements in a quantitative way.

Figure 1.2 schematically represents a cross-section of a surface breaking crack and
Figure 1.2: Conceptual model of surface-breaking and embedded cracks with IGA in an austenitic stainless steel

an embedded crack within an area of IGA which could result from manufacture or from stress corrosion cracking induced by significant surface stresses. Quantitative NDE should distinguish between Figures 1.1 and 1.2. If eddy current NDE is used, the eddy currents induced in the workpieces in Figure 1.2 are deflected by these large defects in a complicated, but predictable way, causing an increase in the impedance $\Delta Z$ in the probe. This scattering is what illuminates the flaws.

There are two distinct types of problems in NDE:

1. For a known flaw and probe, what is $\Delta Z$?
2. For a known $\Delta Z$ and probe, what is the flaw?

These distinct problems are the forward and inverse problems, respectively, and are discussed in the next two sections.

### 1.1.1 The forward problem

The interaction of eddy currents with defects is quite a complex phenomenon, but if a firm understanding can be developed then one has taken a large step towards quantitative eddy current testing. Lord and Palanisamy[37] describe this very well:

> Although many eddy current tests are carried out to determine composition, hardness, dimensions, and other properties of metal parts, the major barrier to further development of eddy current and, indeed, all electromagnetic testing methods at this time, is the lack of a viable theoretical model capable of predicting the complex field/defect interactions which are the very essence of any sound defect characterization scheme.

The theoretical model is often referred to as the *forward problem* or *forward model* and is the heart of any quantitative scheme.

The *forward problem* is describing the response of some system when all the inputs into the system are known. The forward problem is well-posed if the system response to
a fixed set of inputs is always the same, that is to say the system response is a *function* of its inputs. This function is represented diagrammatically in Figure 1.3. The system response can be multi-dimensional, but must be completely determined by the input parameters.

In this work the forward problem of interest predicts a probe response due to a work-piece. In the framework of Figure 1.3, the inputs consists of all the dimensional and material parameters to describe the probe, work-piece and flaw with the driving force being the alternating current in the probe. The response is the change in impedance $\Delta Z$ in the coil due to the workpiece. The system, in this example, governs the relationship between the probe and the flaw in the workpiece. Understanding the interactions of induced eddy currents in the workpiece is based on an appropriate *physical model*, *mathematical model*, and *numerical model*.

### 1.1.2 The inverse problem

The *inverse problem* can be described as predicting the input of some system which has caused a known system response. This problem is well-posed, if for a particular response, there is a unique set of system parameters. This requirement, referred to as *one-to-one*, is a much more stringent requirement for the inverse problem to be well-posed than for the forward problem. The inverse problem is represented diagrammatically in Figure 1.4.

The inverse problem to be discussed involves computing the material properties of the flaw, when the impedance change $\Delta Z$ is known. It may be necessary to include several frequencies for this problem to be well-posed.
1.2 Modelling

Modelling is an attempt to describe the system behaviour with a simple set of rules. The accuracy of the model is dependent on how well these rules describe the system. For example, if the system to be studied is the population of rabbits and foxes in a closed ecological system, the rules could be: foxes eat rabbits, rabbits don't eat foxes, rabbits eat grass, grass dies if there are too many rabbits, foxes die if there are too few rabbits. With these simple rules, an attempt can be made to model the population changes inside this closed ecological system. This model as stated is not quantitative, it just sets down rules or assumptions about the behaviour of the system. It is the physical model.

1.2.1 Physical models

A physical model attempts to describe the system by setting out some simplifying assumptions, not necessarily quantitative ones, that the system follows. In the rabbit and fox system described above, the rules seem quite intuitive and would help build understanding of the population variations of the two species. However, if there was a high speed motorway through this closed system, the model, as stated, would be unlikely to predict population changes accurately, because of the failure of the physical model to fit the actual situation. By these simplifying assumptions, it is hoped that the system can then be described mathematically so that quantitative information can be obtained. This is the mathematical model.

1.2.2 Mathematical models

A mathematical model describes the physical model in quantitative terms, it is the theoretical description of the system. For example, the population of rabbits \( r(t) \) and foxes \( f(t) \) at time \( t \) could be described as

\[
\begin{align*}
r(t) &= c_0 \frac{\partial r}{\partial f} + c_1 r; \quad r(0) = r_0 \\
f(t) &= c_2 \frac{\partial f}{\partial r} + c_3 f; \quad f(0) = f_0
\end{align*}
\]  

(1.1)

for parameters \( c_0, \ldots, c_3 \). By choosing the parameters appropriately, all the assumptions in the physical model can be realized and the mathematical model will be able to predict the population at any time \( t \) by solving the mathematical system.

1.2.3 Numerical models

If a mathematical model is available, then the forward problem can be solved, if all the inputs are known. In the rabbit and fox system modelled by the equations in (1.1), if
CHAPTER I. INTRODUCTION

the parameters $c_0, \ldots, c_3, r_0$ and $f_0$ are known, an attempt can be made to solve for $r(t)$ and $f(t)$. However, there may not be a closed form expression for the solution to this problem, in which case the solution needs to be approximated by numerical techniques. The *computational model* or *numerical model* entails finding approximate solutions to the mathematical model. Similarly, if the populations $r(t_j)$ and $f(t_j)$ are known at several times $t_j, j = 0, 1, \ldots, n$, the inverse problem would amount to solving for, or at least approximating, the coefficients $c_0, \ldots, c_3$.

It is important to notice at this stage that there are two distinct types of modelling errors. The first is the difference between the numerical model approximation and the exact solution to the mathematical model. These are *numerical errors*. The other is the difference between the exact solution of the mathematical model and *real world* response of the system. These are *modelling errors*. A good model is one that is efficient to compute, but has small modelling and numerical errors in its solution.

1.3 Review of eddy current NDE

Electromagnetic NDE has a long history in England, and played an important role during the Second World War in two important areas: radar and mine sweeping. Both applications involved detection, with some discrimination. Although both areas of research were in their infancy, they had won over major support from scientific and political arenas. In a meeting at the Air Defence Research committee in July, 1935, Robert Watson-Watt, inventor of radio direction finding (RDF)\(^1\), presented his paper ‘Detection and Location of Aircraft by Radio Methods’ for the first time to Sir Winston Churchill, who was to become a strong advocate. At the meeting he passed a four line pencilled note across the table saying:\[22\]

‘Seeking, Finding, Following, Keeping,
Is he sure to bless?
Angels, Martyrs, Prophets, Virgins,
Answer: ”M, Yes”.’

The detection of enemy aircraft by radar was one of the major success stories of the war, firmly establishing the utility of this electromagnetic technique.

While the Royal Air Force was busy looking for enemy aircraft, enemy mines were busy looking for the Royal Navy and Allied shipping. Enemy mines were equipped with primitive electromagnetic detections systems for sensing the passing magnetic field of ships and then exploding. In fact, the Thames estuary was closed to shipping for over a year because of this danger. The Royal Navy developed an successful experimental electromagnetic technique for projecting an image of a ship ahead of its bow to trigger these

\(^1\)The American term for RDF, Radio Direction and Ranging, was later adopted in an effort to standardize on nomenclature, hence RADAR
1.3. REVIEW OF EDDY CURRENT NDE

mines, unfortunately for the crew, it was not too far ahead[40]. This started a tradition of expertise in mine sweeping continued to this day. The Royal Navy also developed electromagnetic counter-measures for its fleet to avoid detection by these mines.

In the following decade, industry began using electromagnetic techniques for inspection. Most of these techniques were empirical. In an early effort, General Motors[23] was using a quasi-theoretical approach to measure overlay thicknesses. However; the equipment used was analog and based on calibrated standards, advances in computers still restricted quantitative NDE to theoretical work. The theoretical basis upon which quantitative NDE stands, was being laid down at this time. Work by Tai[60] and Wait[64] to mention a few.

By the mid-60's, advances in computer science had made computational electromagnetics possible and affordable for large corporations or government bodies. Therefore, theoretical results could now be approximated numerically as well as verified by experiment. The theoretical work continued with important contributions on the analytical and computational complexities of the free-space Green's function by Fikioris[21] an Van Bladel[62]. In their important paper Dodd and Deeds[18] provided a theoretical basis for understanding the interaction between and eddy current probe and planar and cylindrical conductors with up to two layers. This work also showed that numerical results were in good agreement with experiment for a large range of normalized probes geometries. Contributions were also being made from geophysics, where geophysicists were faced with similar problems. This can be seen in the work of Weaver[68].

The theoretical efforts continued into the next decade. The seventies saw the important work of Dodd and Deeds[17] and Wait[65] on cylindrical conductors. The geophysicists were now looking at volume integral techniques to solve their problems, with important contributions from Raiche and Coggon[47]. During this period quantitative eddy current NDE became well established as an accepted inspection technique in the field; being used especially in safety related areas in aerospace, oil and power generation industries. In the field calibration standards and manual inspection still were the norm, with very little numerical analysis of data.

There has been an explosion of effort on the computational side of eddy current NDE since 1980 as computation speed has increased and the price of hardware decreased. Early in this decade there were several results attempting to come to grips with the nature of the singularity in the heart of the Green's function used for electromagnetic analysis when using integral techniques by Lee[35], Yaghjian[69], Wang[66] and Beissner[5]. Finite element techniques were also applied to NDE by Lord[36] and Vérité[63] and others. Boundary elements techniques were developed by Beissner[6] and others.

With the increase in computational speed, the possibility of solving NDE inverse problems becomes more feasible. A major contribution of this work was the solving of the inverse problem for predicting the depth of corrosion in steam pipes and is presented
Experimental work for the validation of theory and computer codes has been carried out by Burke[14,16]. In an effort to establish experimental and theoretical benchmarks for these complicated codes two international organizations were established: Testing Electromagnetic Analysis Methods (TEAM) and Applied Computational Electromagnets (ACES). Some of the validation exercises in this work are compared against these benchmarks.

The application of the volume integral technique to eddy currents is the main contribution of this work. This method has been applied by Sabbagh and Sabbagh[51], M'Kirdy[37] and Bowler[12], to cite a few. The volume integral approach divides only the flaw region into small elements and uses a Green's function approach to predict how each element interacts with all the others. A different Green's function is used for each of the simple stratified media examined, so the boundary conditions need not be imposed externally, so there is no need to subdivide the un-flawed media. Approaches in the past have relied on Fourier techniques to compute the inter-element interaction. But this technique proved expensive in both memory and computer time and gave poor results when the cell aspect ratio increased much above 4. The accuracy of computing these interactions is critical in finding a good result[28]. In Chapter 6 a physical space integration technique is described that is precise, robust, fast and uses very little memory. This approach has allowed the implementation of the volume integral technique to run on personal computers on desk tops and is one of the most exciting aspects of this work.

The whole framework of electromagnetic NDE fits inside the framework of Maxwell's equations which are discussed next.

### 1.4 Maxwell's equations

Maxwell's equations in differential form are

\[
\nabla \times \mathbf{E}(r, t) = -\frac{\partial \mathbf{B}(r, t)}{\partial t}, \\
\nabla \times \mathbf{H}(r, t) = \mathbf{J}(r, t) + \frac{\partial \mathbf{D}(r, t)}{\partial t}, \\
\n\nabla \cdot \mathbf{D}(r, t) = \rho(r), \\
\n\nabla \cdot \mathbf{B}(r, t) = 0, \\
\]

where \( \mathbf{H} \) is the magnetic field, \( \mathbf{B} \) the magnetic flux density, \( \mathbf{E} \) the electric field intensity, \( \mathbf{J} \) the current density, \( \mathbf{D} \) is the distribution current and \( \rho \) is the electric charge density.

In order to find solutions to these equations, some simplifying assumptions about the physical problems under investigation are needed. Assume that the materials present are
1.5. POTENTIAL THEORY

ideal in some way. Assume that the permeability \( \mu(\mathbf{r}) \) and permittivity \( \varepsilon(\mathbf{r}) \) are isotropic and homogeneous, that is to say that there exist constants \( \mu_i \) and \( \varepsilon_i \) such that

\[
D(\mathbf{r}, t) = \varepsilon(\mathbf{r})E(\mathbf{r}, t), \quad B(\mathbf{r}, t) = \mu(\mathbf{r})H(\mathbf{r}, t).
\] (1.3)

Assume further that the materials are linear with respect to conductivity \( \sigma(\mathbf{r}) \), which implies that

\[
J(\mathbf{r}, t) = \sigma(\mathbf{r})E(\mathbf{r}, t).
\] (1.4)

Now assume that the source term, \( J(\mathbf{r}, t) \), hence all the fields, are time harmonic with frequency \( \omega \); the time dependence is of the form \( e^{-i\omega t} \). Suppressing the time dependence, one can rewrite equation (1.2) as

\[
\nabla \times E(\mathbf{r}) = -i\omega \mu_0 H(\mathbf{r})
\]

\[
\nabla \times H(\mathbf{r}) = [\sigma(\mathbf{r}) - i\omega \varepsilon_0] E(\mathbf{r}).
\] (1.5)

By eliminating \( H \) in equation (1.5), we obtain the vector wave equation

\[
\nabla \times \nabla \times E - k^2 E = i\omega \mu_0 J,
\] (1.6)

where

\[
k = \sqrt{i\omega \mu_0 \sigma_i} = \frac{1 + i}{\delta}
\] (1.7)

with \( \delta \) being the skin depth: the free-space wavelength. The skin depth is an important parameter, since all electromagnetic interactions are limited to ranges of up to two or three skin depths.

1.5. Potential theory

The field \( H \) may be determined by introducing a magnetic vector potential. Generally a field vector can be defined (apart from a trivial constant) by specifying its curl and its divergence. The magnetic vector potential is conventionally defined such that \( B = \nabla \times A \), therefore for a nonmagnetic region (\( B = \mu_0 H \)),

\[
H = \frac{1}{\mu_0} \nabla \times A,
\] (1.8)

where \( \mu_0 \) is the permeability of free-space. From equation (1.5)

\[
E = i\omega A - \nabla \phi,
\] (1.9)

for any scalar potential \( \phi \), since \( \nabla \times (\nabla \phi) \equiv 0 \). Substituting equations (1.8) and (1.9) back into equation (1.6) to find

\[
\nabla \times \nabla \times A = \mu_0 J + k^2 A + i\omega \mu_0 \varepsilon_0 \nabla \phi.
\] (1.10)
Apply the identity \( \nabla \times \nabla \phi \equiv \nabla \nabla \cdot - \nabla^2 \) and restrict the choice of the scalar potential by imposing the Lorentz gauge: \( \nabla \cdot A = i \omega \mu_0 \epsilon_0 \phi \), to equation (1.10) to find

\[
\nabla^2 A + k^2 A = -\mu_0 J, \tag{1.11}
\]

which is a vector Helmholz equation. A solution to (1.11) can be written simply as

\[
A(r) = -\mu_0 \int \frac{G(r|r')J(r')}{r} \, dr', \tag{1.12}
\]

where \( G(r|r') \) is the scalar Green's function discussed in the next section. Once the vector potential \( A \) has been determined, the electric field \( E \) is known, since

\[
E(r) = i\omega \left[ A(r) + \frac{1}{k^2} \nabla \nabla \cdot A(r) \right]. \tag{1.13}
\]

Before going any further there are some straightforward but significant things to notice about (1.11) and the proposed form of its solution. Firstly (1.11) is of the form \( L(A) = -\mu_0 J \), where \( L \) is a linear operator, for it has the property \( L(A + B) = L(A) + L(B) \). The fact that it is linear allows us to use the Green's function method. Secondly because \( L \) is a scalar operator (\( \nabla^2 + k^2 I \) in this case) and the free-space boundary condition on \( A \) does not introduce coupling between components, the Cartesian components of \( A \) are independent. This means we can formally express the solution of (1.11) as \( A = L^{-1}(-\mu_0 J) \), where \( L^{-1} \) is a scalar operator. Here we are writing the solution of a vector problem as an scalar integral operator acting on a vector source. However, it is not always possible to use a scalar operator for this purpose. In general, an elementary vector source gives rise to a field having a different direction. In addition components of the field may be coupled both through the governing equation and through the boundary conditions. In dealing with these complications, dyadic Green's functions are used but for the present we shall continue to use scalar Green's functions.

### 1.6 Green's function approach

The Green's function approach is introduced by looking at one-dimensional heat flow in a rod. For appropriate physical constants, the heat distribution \( u(x) \) can be expressed as

\[
\frac{d^2 u}{dx^2} = f(x), \quad 0 < x < 1; \quad u(0) = 0, \quad u(1) = 0, \tag{1.14}
\]

where \( f(x) \) is the forcing function. Equation (1.14) is a linear, inhomogeneous differential equation. The Green's function approach allows for the solution to this equation for a particular forcing function in a form that can be reused for other forcing functions. This can be achieved using the superposition principle: If \( u_1(x) \) is a solution with forcing function \( f_1(x) \) and \( u_2(x) \) is a solution with forcing function \( f_2(x) \), then \( u_1(x) + u_2(x) \)
1.7. SCALAR GREEN'S FUNCTIONS

is a solution with forcing function \( f_1(x) + f_2(x) \). Therefore, the superposition allows
the decomposition of complicated data into simpler parts, solving each simpler boundary condition, and then reassembling these solutions to find a solution to the original problem.

Solving (1.14) illustrates this approach. Decompose the interval \([0, 1]\) into \(n\) equal
parts of length \(\Delta x'\) and on each subinterval assign a point source which approximates
\(f(x)\) over the subinterval. Using the superposition principle, the temperature at \(x\) for
all concentrated sources is

\[
\sum_{i=0}^{n} g(x|x')f(x')\Delta x',
\]

which as \(n \to \infty\), tends to

\[
u(x) = \int_0^1 g(x|x')f(x')dx'.
\]

It can be shown that this limit is unique and depends continuously on the forcing
function[56].

The Green's function can be constructed by solving the special case when the forcing
function is a unit source at \(x = x'\). On the intervals \([0, x')\) and \((x', 1]\) \(f(x) = 0\), hence
\(g''(x) = 0\) and taking into account that \(g\) vanishes at the endpoints, we find that

\[
g(x|x') = Ax, \quad 0 < x < x'; \quad g(x|x') = B(1 - x), \quad x' < x < 1.
\]

The one-dimensional character of the problem means that no heat flows through the
lateral surface, so

\[
-g'|_{x=x'+\epsilon} + g'|_{x=x'-\epsilon} = 1
\]

which, as \(\epsilon\) tends to 0, leads to the jump condition for \(g'\):

\[
g'|_{x=\epsilon^+} - g'|_{x=\epsilon^-} = -1
\]

Combining equations (1.17) and (1.19), the Green's function is found to be

\[
g(x|x') = \begin{cases} 
(1 - x')x & 0 \leq x < x' \\
(1 - x)x' & x' < x \leq 1
\end{cases}
\]

1.7 Scalar Green's functions

The \textit{steady diffusion equation} in a three-dimensional medium for a concentrated steady
unit source at the origin can be written as[56]

\[
-\nabla^2 u + q^2 u = \delta(x), \quad x \in \Omega,
\]
where \( \Omega \) is assumed to be the whole space. A mass balance shows that the flux through a small sphere about the source must equal the input in the ball, that is,

\[
\lim_{\epsilon \to 0} - \int_{|z| = \epsilon} \frac{\partial u}{\partial n} \, dS = 1. \tag{1.22}
\]

Impose the condition that \( u \) vanishes at infinity. The concentration will depend only on the radial coordinate \( r \), so on switching to cylindrical coordinates equation (1.21) can be written

\[
- \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{du}{dr} \right) + q^2 u = 0, \quad r > 0,
\]

while equation (1.22) becomes

\[
-1 = \lim_{\epsilon \to 0} 4\pi \epsilon^2 \left( \frac{du}{dr} \right)_{r=\epsilon}. \tag{1.24}
\]

Introduce the change of variable \( v = u/r \) into (1.23) to find

\[
-v'' + q^2 v = 0,
\]

which has as general solution \( Ae^{-qr} + Be^{qr} \). The required behaviour at infinity implies that \( B = 0 \), while imposing (1.24) gives \( A = 1/4\pi \). Therefore, the solution is

\[
u = \frac{e^{-qr}}{4\pi r}.
\]

The effect of a source at \( x' \) is obtained by translation. The concentration due to such a source is what is called the scalar Green's function:

\[
G(x|x') = \frac{e^{-q|x - x'|}}{4\pi |x - x'|}. \tag{1.27}
\]

The scalar Green’s function needed in equation (1.12) is of the same form\[59\]

\[
G(r|r') = \frac{e^{ik|r - r'|}}{4\pi |r - r'|}. \tag{1.28}
\]

and is referred to as the dynamic scalar Green’s function for time harmonic sources in free-space. If the frequency of the source goes to zero, the static scalar Green’s function is obtained:

\[
G_0(r|r') = \frac{1}{4\pi |r - r'|}. \tag{1.29}
\]

These Green’s functions play a fundamental role in the following chapters.
1.8 Objectives

The objectives of this work are:

1. The creation and validation of theoretical and numerical models for predicting probe responses for a large class of planar and cylindrical stratified conductors: the \textit{layered forward model}.

2. The creation and validation a valid theoretical and numerical models for predicting probe responses from defects of general shape and material properties in planar or cylindrical stratified materials: the \textit{3D forward model}.

3. The creation and validation of theoretical and numerical models for predicting depth and material properties of a layer on a conducting half-space from eddy current probe signals: the \textit{layered half-space inverse model}.

4. Validation of the layered half-space inverse model for IGA in austenitic steel.

The basic theory needed to analyze stratified conductors is presented in the next two chapters: the first for conductors with planar boundaries, the second for conductors with cylindrical boundaries.
Chapter 2

Basic Theory: Planar Interfaces

*By viewing nature, nature’s handmaid art,*
*Makes mighty things from small beginnings grow: …*

John Dryden 1631–1700

2.1 Introduction

The first step in modelling complex physical behaviour is to develop some sort of physical model, with simplifying assumptions that will make analysis possible. It is then hoped that the behaviour of the model will substitute for the behaviour of the more complex system. For NDE, we must develop models for the test piece, and possibly flaws inside the test piece, and a model of the probe. We can then analyze the interaction between these models to predict the complex probe/material interaction.

We divide the type of analysis into two types: simple layered conductors without flaws and a full three-dimensional analysis. The simple layered analysis is further divided by considering conductors having planar and cylindrical geometries. The material properties are assumed constant in each layer. Only in the full three-dimensional analysis will the material properties be allowed to vary in all three coordinate directions. Of course we could study the two dimensional case, when material properties vary only in two coordinate directions, but this analysis does not cover very many practical physical problems in NDE.

In this chapter, the basic theory for analyzing simple planar layered conductors is presented. The theory is an extension of the work done by Dodd and Deeds at Oak Ridge Laboratory[17]. Their work treated three special cases of conductors with layered structures. The theory presented in this chapter extends the theory to conductors with
any number of layers, with an electric source in any region; therefore, layers on both sides of the source region are considered. Although this situation does not arise in practice, the inclusion of these additional conducting regions allows for a consistent notation and theory to be used when the source region under consideration changes from region to region. Conductors with cylindrical geometries will be considered in Chapter 3.

Once this theoretical machinery is in place, it is applied to the planar physical models that are the subject of this chapter. The models correspond to layered materials with a certain type of planar geometry or media type. The simplifying assumptions and the parameters used for each physical model are presented. Results for each of the different planar types of geometries are presented in detail, even though they follow quite naturally from the theory presented, because the computation is non-trivial.

### 2.2 Free-space

![Figure 2.1: Free-space physical model](image)

Assuming an infinite linear, isotropic space, as shown schematically in Figure 2.1, various electromagnetic quantities can be determined when an electric source is introduced into this space. For example, if a point source is introduced, the electric field can be determined; or if an air-cored axially symmetric coil is introduced, the impedance in the coil can also be determined.

The conductivity \( \sigma_0 \) and the permeability \( \mu_0 \) are everywhere constant. The physical model is just an idealization of the material properties to allow investigation. This physical model can be used, for example, in the study of the free-space inductance of a coil in air, in which case \( \sigma_0 \) should be set to zero. The magnetic vector potential and coil impedance will be determined for this simple model.

#### 2.2.1 Free-space vector potential

Let the current density \( \mathbf{J}(r) \) be axially symmetric, having the form

\[
\mathbf{J}(r) = j \hat{r} (\rho', z') \hat{r}'.
\]  

Converting the dynamic scalar Green's function from equation (1.28) into cylindrical coordinates by the substitutions

\[
x = \rho \cos \phi, \quad y = \rho \sin \phi, \\
u = \alpha \cos \beta, \quad v = \alpha \sin \beta,
\]  

(2.2)
2.2. FREE-SPACE

then

\[ u(x - x') + v(y - y') = \alpha [\rho \cos(\phi - \beta) - \rho' \cos(\phi' - \beta)] \]
\[ du \ dv = \alpha \ d\alpha \ d\beta \]
\[ \phi' = \hat{\phi} \cos(\phi - \phi') + \hat{\phi} \sin(\phi - \phi'). \]  

(2.3)

One can show by integration that\[19\]

\[ G(r|r'|) = \frac{\epsilon^{ik|r-r'|}}{4\pi|r - r'|} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha|z-z'|} e^{iu(z-z') + iv(y-y')} \ du \ dv, \]

(2.4)

the vector potential can be determined by employing equations (1.12) and (2.4). Therefore,

\[ A(r) = \hat{A}(r) \hat{\phi} = \mu_0 \hat{\phi} \int \tilde{G}(\rho, z|\rho', z') J(\rho' z') \rho' d\rho' dz', \]

(2.5)

where

\[ \tilde{G}(\rho, z|\rho', z') = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \tilde{G}(z|z') e^{i\alpha[\rho \cos(\phi - \beta) - \rho' \cos(\phi' - \beta)]} \]
\[ \cdot \cos(\phi - \beta) \cos(\phi' - \beta) \alpha \ d\alpha \ d\phi' d\beta, \]

(2.6)

and

\[ \tilde{G}(z|z') = \frac{1}{2\alpha} e^{-\alpha|z-z'|}. \]

(2.7)

Using the standard integral\[1\]

\[ J_1(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} e^{i\alpha \cos \theta} \cos \theta \ d\theta, \]

(2.8)

integration of (2.6) with respect to \( \phi' \) and \( \beta \) gives

\[ \tilde{G}(\rho, z|\rho', z') = \int_0^{\infty} \tilde{G}(z|z')J_1(\alpha \rho)J_1(\alpha \rho') \alpha \ d\alpha. \]

(2.9)

In order to describe the vector potential due to an axially symmetric current density, start with the current density produced by a delta-function coil of radius \( \rho_j \) at height \( l_j \), so \( \hat{\mathbf{j}}(\rho', z') = \delta(\rho' - \rho_j)\delta(z' - l_j) \). The axially symmetric form of equation (1.12) can then be used to describe the vector potential due to this delta coil as

\[ \hat{A}_j(\rho, z) = -\mu_0 \int \tilde{G}(\rho, z|\rho', z') \hat{\mathbf{j}}(\rho', z') \ d\rho' \ dz'. \]

(2.10)

The vector potential created from the rectangular cross-section air-cored coil in the probe represented schematically in Figure 2.2 can be found by using superposition if certain simplifying assumptions are made about the coil. The coil is assumed to be axially symmetric and that the current density is uniformly distributed over the cross-section of the coil and that each loop has the same phase and amplitude. At high frequencies this
assumption breaks down and the coil can resonate; therefore, the operating frequency of the probe is assumed to be well below this condition. This type of probe is usually much wider than it is tall and is sometimes referred as a \textit{pancake} coil. For the cylindrical conductors, the axis of the coil is assumed to be coaxial to the media and is often referred to as a \textit{bobbin} coil.

Applying superposition to equation (2.10) yields

\[
\mathbf{A} (\rho, z) = -\mu_0 \sum_{j \in \mathbb{N}} \mathbf{A}_j (\rho, z) = -\mu_0 \sum_{j \in \mathbb{N}} \mathbf{G} (\rho, z|r_j, l_j) \mu_j r_j,
\]

where \( \mu_j \) is the applied current density of the delta-function coil of radius \( r_j \). This describes the vector potential for coils of any cross-section. If we let the current distribution of the delta-function coils approach a continuous current distribution, we obtain the vector potential due to the entire coil

\[
\mathbf{A} (\rho, z) = -\mu_0 \int_S \mathbf{G} (\rho, z|r, l) \mu(r, l) dS.
\]

Assume the coil has an applied current density \( \mu_0 \) that is constant over the dimensions of the coil; therefore,

\[
\mathbf{A} (\rho, z) = -\mu_0 \int_{r_0}^{r_1} \int_0^{l_1} \mathbf{G} (\rho, z|r, l) r dl dr.
\]

Combining equations (2.9) and (2.13), after reversing the order of integration, yields

\[
\mathbf{A} (\rho, z) = \frac{\mu_0 \mu_0}{2} \int_0^{\infty} \int_{r_0}^{r_1} \int_0^{l_1} r J_1(\alpha r) J_1(\alpha \rho) e^{-\alpha z - \alpha l} dl d\alpha.
\]
Integrating with respect to the coil parameters $r$ and $l$ gives the vector potential $\hat{A}_{\text{above}}(\rho,z)$ for the region above the coil, $z > l_1$, as
\begin{equation}
\hat{A}_{\text{above}}(\rho,z) = -\frac{\mu_0 l_0}{2} \int_0^\infty \frac{\Psi(\alpha,r_0,r_1)}{\alpha} J_1(\alpha \rho) \left[ e^{\alpha(l_1-z)} - e^{\alpha(l_0-z)} \right] d\alpha,
\end{equation}

where
\begin{equation}
\Psi(\alpha,r_0,r_1) = \int_{r_0}^{r_1} rJ_1(\alpha r) dr.
\end{equation}

Similarly, for the region below the coil, $z < l_0$ we find
\begin{equation}
\hat{A}_{\text{below}}(\rho,z) = -\frac{\mu_0 l_0}{2} \int_0^\infty \frac{\Psi(\alpha,r_0,r_1)}{\alpha} J_1(\alpha \rho) \left[ e^{\alpha(z-l_0)} - e^{\alpha(z-l_1)} \right] d\alpha.
\end{equation}

The vector potential $\hat{A}(\rho,z)$ we desire for $l_0 \leq z \leq l_1$ can be found by combining the vector potentials for the regions above and below the coil[17]. For $z$ in this region, we use $\hat{A}_{\text{above}}$ from $z$ down to $l_0$ and $\hat{A}_{\text{below}}$ from $z$ up to $l_1$. Therefore, by substituting $z$ for $l_1$ in equation (2.15) and $z$ for $l_0$ in equation (2.17) and adding we have
\begin{equation}
\hat{A}_{\text{coil}}(\rho,z) = -\frac{\mu_0 l_0}{2} \int_0^\infty \frac{\Psi(\alpha,r_0,r_1)}{\alpha} J_1(\alpha \rho) \left[ 2 - e^{\alpha(z-l_1)} - e^{\alpha(z-l_0)} \right] d\alpha.
\end{equation}

This result has been derived for a coil in free-space, but similar integrals arise in calculating the vector potential due to axially symmetric coils above planar stratified conductors. In the presence of these conductors, one must modify the Green's function to take into account the wave interactions caused by structure of the material.

### 2.2.2 Coil impedance

We now have sufficient theory to compute an air-cored probe response in air. For an axially symmetric coil with a single loop of radius $r$ the induced voltage $V$ is given by[17]
\begin{equation}
V = -2\pi i \omega r \hat{A}_{\text{coil}}(r,z).
\end{equation}

The total voltage induced by a coil can then be computed as the sum over all of the single loops. This sum can be computed as an integral over the cross sectional area of the coil times the turn density. Since we assume that our coils have a constant number of turns per unit cross-sectional area $A_c = (l_1 - l_0)(r_1 - r_0)$, the induced voltage can be written
\begin{equation}
V_{\text{coil}} = -\frac{2\pi i \omega n}{A_c} \int_{l_0}^{l_1} \int_{r_0}^{r_1} \rho \hat{A}_{\text{coil}}(\rho,z) d\rho dz,
\end{equation}

where $n$ is the total number of turns.

Using the induced coil voltage $V_{\text{coil}}$, the coil impedance is computed simply using the relation
\begin{equation}
Z = V/I.
\end{equation}
2.3 Simple planar interfaces

Inhomogeneity can be caused by many factors: variations in material properties, defects, edges, joints, etc. To limit the types of non-homogeneous media under investigation, in this chapter, only materials without flaws or defects will be considered. Flawed medias will be considered in Chapter 5. This chapter will analyze only the anomalies caused from boundaries between simple homogeneous planar regions.

To motivate the general planar stratified eddy current interface problem, a simple one-dimensional planar interface problem is considered first.

2.3.1 1D planar interface problem

Imagine we have two regions, 0 and 1. In each there is an associated medium and a steady-state wave-like disturbance described by a scalar potential $\phi_j$, $j = 0, 1$ satisfying

$$ \left[ \frac{d^2}{dz^2} + k_j^2 \right] \phi_j(z) = 0, \quad (2.23) $$

where $k_j$, for $j = 0, 1$ is the wave number for each region, respectively. Figure 2.3 shows the situation schematically. Suppose that at the boundary, $z = 0$, we have the following interface conditions

$$ \frac{d\phi_0}{dz} = \frac{d\phi_1}{dz}. \quad (2.24) $$

Solutions to (2.23) are of the form

$$ \phi_j = A_j e^{ik_jz} + B_j e^{-ik_jz}, \quad (2.25) $$
where $A_j$ and $B_j$, $j = 0, 1$ are constants representing the amplitude of the incident waves travelling from the left and right, respectively. Assume that there is an incident wave travelling from the left, of amplitude $a$, possibly coming from a source some way off in the negative $z$-direction. This situation is represented by the solution

$$
\phi_0(z) = a[e^{-i\alpha_0 z} + \Gamma_1 e^{i\alpha_0 z}],
$$

$$
\phi_1(z) = a\Gamma_1 e^{-i\alpha_1 z},
$$

where $\Gamma_1$ is the reflection coefficient, which describes the magnitude and phase of the wave reflected from the interface. $\Gamma_1$ is the transmission coefficient, which describes the magnitude and phase of the wave transmitted through the interface. To evaluate these coefficients the interface conditions in (2.24) are imposed on the general solution (2.25) to find

$$
\mu_0(1 + \Gamma_1) = \mu_1 \Gamma_1, \quad (2.27)
$$

$$
\alpha_0(1 + \Gamma_1) = -\alpha_1 \Gamma_1, \quad (2.28)
$$

where $\alpha_j = -i\alpha_j$, $j = 0, 1$. Solving for the reflection and transmission coefficients yields

$$
\Gamma_1 = \frac{\mu_1\alpha_0 - \mu_0\alpha_1}{\mu_0\alpha_1 + \mu_1\alpha_0},
$$

$$
\Gamma_1 = \frac{2\mu_0\alpha_0}{\mu_0\alpha_1 + \mu_1\alpha_0}. \quad (2.29)
$$

These coefficients will be re-computed in a more general setting later. It is important to note that these equations for the reflection and transmission coefficients are a consequence of the continuity conditions at the interface.

### 2.3.2 Planar interface problem: half-space conductors

![Figure 2.4: Half-space physical model](image)

Turn now to the three-dimensional planar interface problem by introducing the simplest media type with an interface, the half-space media type. Figure 2.4 schematically represents the half-space physical model. This model represents a probe over a linear and
isotropic half-space of material of infinite extent in both the \( x \) and \( y \) coordinate directions and for all values of \( z \geq 0 \), where the positive \( z \) direction is assumed down and \( z = 0 \) at the top of the half-space.

This physical model can be used for studying most 'flat' surfaces. Flat, here, meaning that any variation or curvature of the surface is small relative to the coil parameters. Therefore, this model could even be applied to bore-hole, tube or pipe inspections, if the probe is normal to the inside or outside surface and is small relative to the radius of curvature. If the work piece has a depth of at least 2 skin depths, the interaction with the lower surface will be small; therefore, this physical model should only be used to study 'thick' materials.

Assume there is a half-space of air above a half-space of some conducting material. Suppose the source is in the air region, region 0, the region where \( z < 0 \). Suppose that there exists a potential \( \phi_j(r) \), \( j = 0, 1 \) in each region satisfying

\[
(\nabla^2 + k_0^2)\phi_0(r) = J(r) \\
(\nabla^2 + k_1^2)\phi_1(r) = 0.
\]

At the boundary between the regions we have for \( z = 0 \),

\[
\frac{\partial \phi_0}{\partial z} = \frac{\partial \phi_1}{\partial z}.
\]

Because similar continuity conditions were specified in the one-dimensional case, we can expect to find similar transmission and reflection coefficients here.

Our solution uses scalar Green's functions satisfying

\[
(\nabla^2 + k_0^2)G_0(r|r') = \delta(r - r') \\
(\nabla^2 + k_1^2)G_1(r|r') = 0
\]

with the interface conditions on the Green functions chosen to be the same as on the potentials.

This three-dimensional problem can be turned into the form of the one-dimensional problem we just saw by taking two-dimensional Fourier transforms of equation (2.32) giving

\[
\left( \frac{\partial^2}{\partial z^2} - \alpha_0^2 \right) \tilde{G}_0(z|z') = \delta(z - z') \\
\left( \frac{\partial^2}{\partial z^2} - \alpha_1^2 \right) \tilde{G}_1(z|z') = 0
\]

where

\[
\alpha_j = (\alpha^2 - k_j^2)^{\frac{1}{2}}, \quad \alpha^2 = u^2 + v^2,
\]
and where the root with a positive real part is assumed. The form of the solutions will be similar to those given in equation (2.26),

\[
\tilde{G}(z|z') = a[e^{z|z'-z'} + A_0 e^{z|z'+z'} + B_0 e^{-z|z'+z'}] \\
\tilde{G}(z|z') = a[A_1 e^{z|z'+z'} + B_1 e^{-z|z'+z'}]
\]

Now we have a quasi-one-dimensional problem. The results found previously can be used. Writing down the general form of the solution and imposing the interface conditions gives

\[
\tilde{G}_0(z|z') = \frac{1}{2\alpha_0} [e^{-z|z'-z'} + \Gamma e^{z|z'+z'}] \\
\tilde{G}_1(z|z') = \frac{1}{2\alpha_0} \Gamma e^{-z|z'-z'}
\]

Note the correspondence with the solution in the one-dimensional case given in equation (2.29). As before the interface conditions are used to determine the transmission and reflection coefficients and they are given by (2.29); only the definition of \( \alpha_j, j = 0, 1 \) is now given by (2.34).

The vector potential could be derived in either region using these reflection and transmission coefficients in equation (2.35); however, the vector potential will be derived more generally later. First general planar stratified media are analyzed.

### 2.4 General planar interface problem

![Figure 2.5: 3D interface problem for any stratified conductor](image)

We now turn to the general planar stratified conductor case. Suppose there are \( m \) regions, separated by planar boundaries to the left of the source region, region 0, and there are \( n \) similar regions to the right of the source region. This situation is illustrated in Figure 2.5. The interfaces are parallel to the \( xy \)-plane and located at the \( n+m \) points

\[
0 = z_{-m} < z_{-m+1} < \ldots < z_{-1} \leq z' < z_0 < \ldots < z_{n-2} < z_{n-1}
\]

Therefore, in all regions \( j \neq -m \) exponential terms with negative arguments can be thought of as waves propagating to the right in the Figure, while exponential terms with
positive arguments propagate to the left. There is only a left propagating wave in region 
\(-m\), since the source occurs in region 0 to the right, hence \(\Gamma_{-m-1} \equiv 0\). Similarly, there is 
only a right propagating wave in region \(n\) and \(\Gamma_{n+1} \equiv 0\). In each of the remaining regions 
\(\pm j\), there is an associated reflection coefficient \(\Gamma_j \pm \Gamma_{j+1}\) and a transmission coefficient \(T_{\pm j}\),
except in region 0, where there are two reflection coefficients \(\Gamma_{-1}\) and \(\Gamma_1\).

In each of the regions \(-j\), where \(-j < 0\), the Green's function has the form

\[
\tilde{G}_{-j}(r|\tau') = \frac{1}{2\alpha_0} \left[ \Gamma_{-j-1} e^{-\alpha_{-j} z-\alpha_0 \tau'} + \Gamma_{-j} e^{\alpha_{-j} z-\alpha_0 \tau'} \right],
\]

where \(\Gamma_{-m-1} \equiv 0\). In the source region, region 0, the solution has the form

\[
\tilde{G}_0(r|\tau') = \frac{1}{2\alpha_0} \left[ e^{-\alpha_0 |z'-z|} + \Gamma_{-1} e^{-\alpha_0 (z+z')} + \Gamma_{1} e^{\alpha_0 (z+z')} \right].
\]

In each of the region \(j\), where \(j > 0\), the Green's function has the form

\[
\tilde{G}_j(r|\tau') = \frac{1}{2\alpha_0} \left[ \Gamma_{j+1} e^{\alpha_j z+\alpha_0 \tau'} + \Gamma_{j} e^{-\alpha_j z+\alpha_0 \tau'} \right],
\]

where \(\Gamma_{n+1} \equiv 0\). Each Green's function is a solution of

\[
(\nabla^2 + k_j^2) \tilde{G}_j(r|\tau') = \delta_0 \delta(r - \tau').
\]

Imposing the same interface conditions as before

\[
\mu_j \tilde{G}_j(z_j|z') = \mu_{j+1} \tilde{G}_{j+1}(z_j|z'),
\quad \frac{\partial \tilde{G}_j}{\partial z} \bigg|_{(z_j|z')} = \frac{\partial \tilde{G}_{j+1}}{\partial z} \bigg|_{(z_j|z')},
\]

for \(-m \leq j \leq n - 1\).

For regions \(-j < -1\), to the left of the source region in the Figure, we have the two relations

\[
\mu_j \left[ \Gamma_{-j} e^{\alpha_{-j} z-\alpha_0 \tau'} + \Gamma_{-j-1} e^{-\alpha_{-j} z-\alpha_0 \tau'} \right] = \\
\mu_{j+1} \left[ \Gamma_{-j+1} e^{\alpha_{-j+1} z-\alpha_0 \tau'} + \Gamma_{-j} e^{-\alpha_{-j+1} z-\alpha_0 \tau'} \right]
\]

\[
\alpha_j \left[ \Gamma_{-j} e^{\alpha_{-j} z-\alpha_0 \tau'} - \Gamma_{-j-1} e^{-\alpha_{-j} z-\alpha_0 \tau'} \right] = \\
\alpha_{j+1} \left[ \Gamma_{-j+1} e^{\alpha_{-j+1} z-\alpha_0 \tau'} - \Gamma_{-j} e^{-\alpha_{-j+1} z-\alpha_0 \tau'} \right].
\]

For region \(j > 1\), to the right of the source region in the Figure 2.5, we have two similar
relations

\[
\mu_j \left[ \Gamma_j e^{\alpha_j z_j+\alpha_0 \tau'} + \Gamma_{j+1} e^{\alpha_j z_j+\alpha_0 \tau'} \right] = \\
\mu_{j+1} \left[ \Gamma_{j+1} e^{\alpha_{j+1} z_j+\alpha_0 \tau'} + \Gamma_{j+2} e^{\alpha_{j+1} z_j+\alpha_0 \tau'} \right]
\]

\[
\alpha_j \left[ -\Gamma_j e^{-\alpha_j z_j+\alpha_0 \tau'} + \Gamma_{j+1} e^{\alpha_j z_j+\alpha_0 \tau'} \right] = \\
\alpha_{j+1} \left[ -\Gamma_{j+1} e^{-\alpha_{j+1} z_j+\alpha_0 \tau'} + \Gamma_{j+2} e^{\alpha_{j+1} z_j+\alpha_0 \tau'} \right].
\]
Define the matrices

\[
A_j = \begin{bmatrix}
\mu_j e^{\alpha_j z_j \pm \alpha_0 z'} \\
\alpha_j e^{\alpha_j z_j \pm \alpha_0 z'} \\
\end{bmatrix}, \quad A_0 = \begin{bmatrix}
\mu_0 e^{\alpha_0 (z_0 \pm z')} \\
\alpha_0 e^{\alpha_0 (z_0 \pm z')} \\
\end{bmatrix}, \quad (2.44)
\]

\[
B_j = \begin{bmatrix}
\mu_{j+1} e^{\alpha_{j+1} z_j \pm \alpha_0 z'} \\
\alpha_{j+1} e^{\alpha_{j+1} z_j \pm \alpha_0 z'} \\
\end{bmatrix}, \quad B_{-1} = \begin{bmatrix}
\mu_0 e^{\alpha_0 (z_0 \pm z')} \\
\alpha_0 e^{\alpha_0 (z_0 \pm z')} \\
\end{bmatrix}, \quad (2.45)
\]

where the positive sign is used for \( j > 0 \) and the negative sign is used for regions where \( j < 0 \). Also define corresponding solution vectors as

\[
X^{(j)} = \begin{cases}
[T_j \quad \Gamma_{j-1}]^T & \text{if } j < 0 \\
[T_1 \quad \Gamma_{-1}]^T & \text{if } j = 0 \\
[T_{j+1} \quad \Gamma_j]^T & \text{otherwise}
\end{cases}
\] \quad (2.46)

Using these matrices the relationships in (2.42) and (2.43) can be rewritten in matrix form for any two regions not including the source region, that is for \(-m < j < -2\) and \(1 \leq j \leq n-1\), as

\[
A_j X^{(j)} = B_j X^{(j+1)}
\] \quad (2.47)

For the 3 regions involving the source term, we have the following two matrix relationships

\[
\begin{align*}
A_{-1} X^{(-1)} &= B_{-1} X^{(0)} + Y^{(-1)} \\
B_0 X^{(1)} &= A_0 X^{(0)} + Y^{(0)}
\end{align*}
\] \quad (2.48)

where

\[
Y^{(-1)} = e^{-\alpha_0 (z_0 - z_l)} \begin{bmatrix} \mu_0 \\ \alpha_0 \end{bmatrix}, \quad Y^{(0)} = e^{-\alpha_0 (z_0 - z')} \begin{bmatrix} \mu_0 \\ -\alpha_0 \end{bmatrix}.
\] \quad (2.49)

So far, we have assumed that \( m > 0 \) and \( n > 0 \). If \( m = 0 \), the source region is a half-space, since there are no regions to the left. Therefore, there will be no transmission term \( T_{-1} \), so \( X^{(-1)} \equiv Y^{(-1)} \equiv 0 \). Similarly, if \( n = 0 \), then \( X^{(1)} \equiv Y^{(0)} \equiv 0 \). If both \( m \) and \( n \) are zero, then \( X^{(0)} \equiv 0 \), which corresponds to the free-space case. Equations (2.47) and (2.48) together imply that the system of equations is block tri-diagonal. If each of the blocks is non-singular, the whole system is singular[25], therefore a solution exists if \( \alpha_j \neq 0 \) and \( \mu_j \neq 0 \) for all \( j \).

Using equation (2.47) we can define the transformation matrix from region \(-j\) to region \(-j+1\), where \(-j < 0\), as

\[
T^{(-j)} = B_{-j}^{-1} A_{-j} = \frac{1}{2\mu_{-j+1} \alpha_{-j+1}}
\]

\[
\begin{bmatrix}
(\mu_{-j} \alpha_{-j+1} + \mu_{-j+1} \alpha_{-j}) e^{(\alpha_{-j} - \alpha_{-j+1}) z_{-j}} \\
(\mu_{-j} \alpha_{-j+1} - \mu_{-j+1} \alpha_{-j}) e^{(\alpha_{-j} + \alpha_{-j+1}) z_{-j}}
\end{bmatrix}
\]

\[
\begin{bmatrix}
(\mu_{-j} \alpha_{-j+1} - \mu_{-j+1} \alpha_{-j}) e^{-(\alpha_{-j} + \alpha_{-j+1}) z_{-j}} \\
(\mu_{-j} \alpha_{-j+1} + \mu_{-j+1} \alpha_{-j}) e^{(\alpha_{-j+1} - \alpha_{-j}) z_{-j}}
\end{bmatrix},
\] \quad (2.50)
where \(-j < -1\) and

\[
T^{(-1)} = B^{-1}_1 A_{-1} = \frac{1}{2 \mu_0 \alpha_0} .
\]

\[
\begin{pmatrix}
(\mu_1 \alpha_0 + \mu_0 \alpha_1) e^{(\alpha - \alpha_0) z_1 - 2\alpha z'} & (\mu_1 \alpha_0 - \mu_0 \alpha_1) e^{(\alpha + \alpha_0) z_1 - 2\alpha z'} \\
(\mu_1 \alpha_0 - \mu_0 \alpha_1) e^{(\alpha + \alpha_0) z_1} & (\mu_1 \alpha_0 + \mu_0 \alpha_1) e^{(\alpha - \alpha_0) z_1 - 2\alpha z'}
\end{pmatrix},
\]

(2.51)

The transformation matrix from region \(-m\) to region \(-j\) is the product of these 2 \times 2 transformation matrices, so we can define this transformation as

\[
V(j) = T(j-1) T(j-2) ... T^{(-m)}.
\]

(2.52)

The transformation from region \(j\) to itself is just the identity matrix, hence \(V^{(-m)} = I\).

The transformation matrix from region \(j+1\) to region \(j\), where \(j \geq 0\), is

\[
T(j) = A^{-1}_j B_j = \frac{1}{2 \mu_j \alpha_j}.
\]

\[
\begin{pmatrix}
(\mu_{j+1} \alpha_j + \mu_j \alpha_{j+1}) e^{(\alpha_{j+1} - \alpha_j) z_j} & (\mu_{j+1} \alpha_j - \mu_j \alpha_{j+1}) e^{-(\alpha_{j+1} + \alpha_j) z_j} \\
(\mu_{j+1} \alpha_j - \mu_j \alpha_{j+1}) e^{(\alpha_{j+1} + \alpha_j) z_j} & (\mu_{j+1} \alpha_j + \mu_j \alpha_{j+1}) e^{-(\alpha_{j+1} - \alpha_j) z_j}
\end{pmatrix},
\]

(2.53)

where \(j > 0\) and

\[
T(0) = A^{-1}_j B_0 = \frac{1}{2 \mu_0 \alpha_0}.
\]

\[
\begin{pmatrix}
(\mu_1 \alpha_0 + \mu_0 \alpha_1) e^{(\alpha_1 - \alpha_0) z_0} & (\mu_1 \alpha_0 - \mu_0 \alpha_1) e^{-(\alpha_0 + \alpha_1) z_0} \\
(\mu_1 \alpha_0 - \mu_0 \alpha_1) e^{(\alpha_0 + \alpha_1) z_0 + 2\alpha z'} & (\mu_1 \alpha_0 + \mu_0 \alpha_1) e^{(\alpha - \alpha_0) z_0 + 2\alpha z'}
\end{pmatrix},
\]

(2.54)

The transformation matrix from region \(n\) to region \(j\) can be defined as

\[
U(j) = T(j) T(j+1) ... T^{(n-1)},
\]

(2.55)

where \(U^{(n)} = I\).

By using (2.47) recursively along with (2.48), the relationship can be expressed between \(X^{(-m)}\), \(X^{(n)}\) and \(X^{(0)}\) as

\[
X^{(0)} = V^{(0)} X^{(-m)} - B^{-1}_1 Y^{(-1)}
\]

\[
X^{(0)} = U^{(0)} X^{(n)} - A^{-1}_0 Y^{(0)}.
\]

(2.56)

Define

\[
Y = U^{(0)} X^{(n)} - V^{(0)} X^{(-m)} = A^{-1}_0 Y^{(0)} - B^{-1}_1 Y^{(-1)}.
\]

(2.57)

Since \(X^{(-m)}\) and \(X^{(n)}\) have only one unknown each, we can rewrite (2.57) as

\[
\begin{pmatrix}
-Y_0^{(0)} & U^{(0)}_1 \\
-Y_2^{(0)} & U^{(0)}_2
\end{pmatrix}
\begin{bmatrix}
\eta \\
\zeta
\end{bmatrix}
=
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix},
\]

(2.58)
2.4. GENERAL PLANAR INTERFACE PROBLEM

where

\[ \eta = \begin{cases} \gamma_{-m} & \text{if } m > 0 \\ \gamma_1 & \text{otherwise} \end{cases} \quad \zeta = \begin{cases} \gamma_n & \text{if } n > 0 \\ \gamma_{-1} & \text{otherwise} \end{cases} \] (2.59)

and

\[ Y_1 = \begin{cases} 0 & \text{if } m = 0 \\ -e^{-2\alpha_0 z'} & \text{otherwise} \end{cases} \quad Y_2 = \begin{cases} e^{2\alpha_0 z'} & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases} \] (2.60)

which has as a solution

\[ \eta = \frac{U^{(0)}_{12} Y_2 - U^{(0)}_{22} Y_1}{V^{(0)}_{11} U^{(0)}_{22} - V^{(0)}_{21} U^{(0)}_{12}} \]

\[ \zeta = \frac{V^{(0)}_{11} Y_2 - V^{(0)}_{21} Y_1}{V^{(0)}_{11} U^{(0)}_{22} - V^{(0)}_{21} U^{(0)}_{12}} \] (2.61)

Now we can use the transformation matrices to determine the reflection and transmission coefficients in any region. To the left of the source they are simply

\[ X^{(-j)} = \begin{bmatrix} \gamma_{-j} \\ \gamma_{-j-1} \end{bmatrix} = V^{(-j)} \begin{bmatrix} \gamma_{-m} \\ 0 \end{bmatrix} = V^{(-j)} X^{-m}. \] (2.62)

There is a useful result when \( n > 0 \):

\[ \Gamma_{-1} = V^{(0)}_{21} \gamma_{-1}. \] (2.63)

Similarly, for regions to the right, we have the expression

\[ X^{(j)} = \begin{bmatrix} \Gamma_{j+1} \\ \gamma_j \end{bmatrix} = U^{(j)} \begin{bmatrix} 0 \\ \gamma_n \end{bmatrix} = U^{(j)} X^{(n)}, \] (2.64)

and for \( m > 0 \),

\[ \Gamma_1 = U^{(0)}_{12} \gamma_1. \] (2.65)

2.4.1 Half-space revisited

A simple example should illustrate this procedure. For the half-space example considered in Section 2.3.2, \( m = 0 \) and \( n = 1 \). Let \( z_0 = 0 \). Therefore \( Y^{(-1)} = 0 \), \( V^{(0)} = I \), while

\[ A_0 = \begin{bmatrix} \mu_0 & \mu_0 \\ \alpha_0 & -\alpha_0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} \mu_1 & \mu_1 \\ \alpha_1 & -\alpha_1 \end{bmatrix} \] (2.66)

\[ X^{(0)} = \begin{bmatrix} \Gamma_1 \\ 0 \end{bmatrix}, \quad X^{(1)} = \begin{bmatrix} 0 \\ \gamma_1 \end{bmatrix} \] (2.67)

and

\[ U^{(0)} = T^{(0)} = \frac{1}{2\mu_0 \alpha_0} \begin{bmatrix} U^{(0)}_{11} & \mu_1 \alpha_0 - \mu_0 \alpha_1 \\ U^{(0)}_{21} & (\mu_1 \alpha_0 + \mu_0 \alpha_1) e^{2\alpha_0 z'} \end{bmatrix} \]

\[ Y = \begin{bmatrix} 0 \\ e^{2\alpha_0 z'} \end{bmatrix} \] (2.68)
Using equations (2.59) and (2.61), we find that

\[
\eta = \Gamma_1 = \frac{U_{12}^{(0)} Y_2}{U_{22}^{(0)}} = \frac{\mu_1 \alpha_0 - \mu_0 \alpha_1}{\mu_1 \alpha_0 + \mu_0 \alpha_1},
\]

\[
\zeta = \Gamma_1 = \frac{Y_2}{U_{22}^{(0)}} = \frac{2\mu_0 \alpha_0}{\mu_1 \alpha_0 + \mu_0 \alpha_1},
\]

which is the same solution obtained in equation (2.29).

### 2.5 Vector potential

Using the air-cored probe physical model for the electric source, equation (1.12) can be used to express the vector potential in any region \( j > 0 \) as

\[
\tilde{A}_j(\rho, z) = -\mu_0 I_0 \int \tilde{G}_j(\rho, z|\rho', z') \tilde{J}(\rho', z') d\rho' dz'
\]

where \( \tilde{G}_j(\rho, z|\rho', z') \) is the axially symmetric Green's function for region \( j \). The current density in the probe model is assumed to be constant \( I_0 = I \Delta \) throughout the coil. Using equations (2.9) and (2.39) to give

\[
0 = \int \frac{1}{2\alpha_0} \left[ \Gamma_{j+1} e^{\alpha_j z + \alpha_0 z'} + \Gamma_j e^{-\alpha_j z + \alpha_0 z'} \right] J_1(\alpha \rho) J_1(\rho') d\alpha.
\]

The vector potential can now be determined by assuming the quasi-static limit, which implies \( \alpha_0 \approx \alpha \), and integrating over the coil dimensions yields

\[
\tilde{A}_j(\rho, z) = -\frac{\mu_0 I_0}{2} \int_0^\infty \frac{\psi(\alpha, r_0, r_1)}{\alpha} J_1(\alpha \rho) \left[ e^{-\alpha z} - e^{-\alpha h} \right] \left[ \Gamma_{j+1} e^{\alpha_j z} + \Gamma_j e^{-\alpha_j z} \right] d\alpha.
\]

This is the desired result for all regions \( j > 0 \). The vector potential in the source region will be computed in the next section.

### 2.6 Coil impedance

Using equations (2.38 and (2.71) the vector potential in the source region, region 0, can be written as

\[
\tilde{A}_0(\rho, z) = -\frac{\mu_0 I_0}{2} \int_{-h}^0 \int_{-r_0}^{r_1} \int_{-h_0}^{r_1} r J_1(\alpha r) J_1(\alpha \rho) \cdot \left[ e^{-\alpha(z-h)} + \Gamma_{-1} e^{-\alpha(z+h)} + \Gamma_1 e^{\alpha(z+h)} \right] d\alpha d\rho dr
\]

After, interchanging the order of integration and integrating over the coil dimensions, the vector potential can be expressed for the region to above the coil, \( z < -h \), as[17]

\[
\tilde{A}_{\text{above}}(\rho, z) = -\frac{\mu_0 I_0}{2} \int_0^\infty \frac{\psi(\alpha, r_0, r_1)}{\alpha} J_1(\alpha \rho) \left[ e^{\alpha(z-h)} - e^{\alpha(h_0-z)} \right] \Gamma_{-1} \left( e^{\alpha(z+h)} - e^{\alpha(z+1)} \right) + \Gamma_1 \left( e^{-\alpha(z-1)} + e^{-\alpha(z+h)} \right) d\alpha.
\]
Similarly, for the region below the coil, \( z > -l_0 \), we find

\[
\hat{\mathbf{A}}_{\text{below}}(\rho, z) = \frac{-\mu_0 l_0}{2} \int_0^\infty \frac{\Psi(\alpha, r_0, r_1)}{\alpha} J_1(\alpha \rho) \left[ e^{\alpha(z-l_0)} - e^{\alpha(z-l_1)} + \Gamma_1 \left( e^{-\alpha(z+l_1)} + e^{-\alpha(z+l_0)} \right) \right] d\alpha. \tag{2.75}
\]

The vector potential \( \hat{\mathbf{A}}(\rho, z) \) for \(-l_0 \geq z \geq -l_1 \) can be found by combining these vector potentials, as we have done for the free-space case, to find

\[
\hat{\mathbf{A}}_{\text{coil}}(\rho, z) = \frac{-\mu_0 l_0}{2} \int_0^\infty \frac{\Psi(\alpha, r_0, r_1)}{\alpha} J_1(\alpha \rho) \left[ 2 - e^{\alpha(z-l_1)} - e^{-\alpha(l_0-z)} + \Gamma_1 \left( e^{-\alpha(z+l_1)} + e^{-\alpha(z+l_0)} \right) \right] d\alpha. \tag{2.76}
\]

The induced voltage can be found by using equation (2.20), integrating over the coil dimensions. The current in the coil is \( I = l_0 \omega / \pi \). Therefore, using the relation \( Z = V/I \), the coil impedance can be expressed as

\[
Z = \frac{\pi \omega \mu_0 n^2}{A^2} \int_0^\infty \frac{\Psi^2(\alpha, r_0, r_1)}{\alpha^2} \left[ 2\alpha(l_1 - l_0) + 2e^{-\alpha(l_1-l_0)} - 2 + \Gamma_1 \left( e^{\alpha l_1} - e^{\alpha l_0} \right)^2 + \Gamma_1 \left( e^{-\alpha l_1} - e^{-\alpha l_0} \right)^2 \right] d\alpha. \tag{2.77}
\]

### 2.7 Further planar examples

In later chapters, Green's functions will be needed for specific examples of conductors with planar interfaces. For each type of conductor the Green's function must be determined for the source region, region 0, when the source region is the free-space above the conductor. The Green's functions must also be determined for the same source, but in a particular region, the host region \( j \), where \( j > 0 \). The scalar Green's function will also be needed to be determined when the source is in the host region itself. The host region is of particular interest, because defects in the conductor will be introduced only in this region. The Green's functions in source regions will be of the form

\[
\tilde{G}_0 = \frac{1}{2\alpha_0} \left[ e^{-\alpha_0|z-z'|} + \Gamma_1 e^{\alpha_0(z+z')} + \Gamma_1 e^{\alpha_0(z+z')} \right], \tag{2.78}
\]

while for the host region with an external source the Green's function will be of the form

\[
\tilde{G}_j = \frac{1}{2\alpha_0} \left[ \Gamma e^{\alpha_0(z+z')} + \Gamma e^{\alpha_0(z+z')} \right]. \tag{2.79}
\]

The corresponding reflection and transmission coefficients need to be determined for each media type.
2.7.1 Layered half-space conductors

Figure 2.6 schematically represents the layered half-space media type. This geometry can be thought of as a half-space with a single non-flawed linear and isotropic layer on top. The layer has conductivity $\sigma_1$, permeability $\mu_1$ and height $l$. The $z$ direction is again defined to be positive going down, with $z = 0$ at the top of the layer. This physical model can be used to study work pieces with 'flat' uniform layers of corrosion or coatings. As can be seen in the Figure, the coatings can have any kind of electromagnetic properties.

Source above layered half-space conductor

Suppose that there is a source in region $0$ of Figure 2.6. The Green's functions are to be determined for regions $0$ and $2$. From section 2.4 we find that $\Gamma_{-1} = \Gamma_3 = 0$ and only the reflection coefficient at the top of the layer, $\Gamma_1$, and the transmission coefficient into the host region $\Gamma_2$ need to be determined. For this example $m = 0$, $n = 2$, $z_0 = 0$, $z_1 = l$ and $V = I$. The transformation matrices are

$$T^{(0)} = \frac{1}{2\mu_0\alpha_0} \begin{bmatrix} \mu_1\alpha_0 + \mu_0\alpha_1 & \mu_1\alpha_0 - \mu_0\alpha_1 \\ (\mu_1\alpha_0 - \mu_0\alpha_0)e^{2\alpha_0z'} & (\mu_1\alpha_0 + \mu_0\alpha_1)e^{2\alpha_0z'} \end{bmatrix}$$

$$T^{(1)} = \frac{e^{(\alpha_1+\alpha_2)t}}{2\mu_1\alpha_1} \begin{bmatrix} \mu_2\alpha_1 + \mu_1\alpha_2 & \mu_2\alpha_1 - \mu_1\alpha_2 \\ \mu_2\alpha_1 - \mu_1\alpha_2 & (\mu_2\alpha_1 + \mu_1\alpha_2)e^{-2(\alpha_1+\alpha_2)} \end{bmatrix}.$$  \hspace{1cm} (2.80)

From equation (2.55) $U^{(0)} = T^{(0)}T^{(1)}$, so

$$U_{12}^{(0)} = \frac{e^{(\alpha_1-\alpha_2)t}}{4\mu_0\mu_1\alpha_0\alpha_1} \left[ (\mu_1\alpha_0 + \mu_0\alpha_1)(\mu_2\alpha_1 - \mu_1\alpha_2)e^{-2\alpha_1l} + (\mu_1\alpha_0 - \mu_0\alpha_1)(\mu_2\alpha_1 + \mu_1\alpha_2) \right]$$

$$U_{22}^{(0)} = \frac{e^{(\alpha_1-\alpha_2)t}}{4\mu_0\mu_1\alpha_0\alpha_1e^{2\alpha_0z'}} \left[ (\mu_1\alpha_0 - \mu_0\alpha_1)(\mu_2\alpha_1 + \mu_1\alpha_2)e^{-2\alpha_1l} + (\mu_1\alpha_0 + \mu_0\alpha_1)(\mu_2\alpha_1 - \mu_1\alpha_2) \right].$$  \hspace{1cm} (2.81)
Using equations (2.59) and (2.61), we find the desired reflection and transmission terms to be

\[
\Gamma_1^{(lyr)} = \eta = \frac{U_{12}^{(0)} e^{2\alpha_0 z'}}{U_{22}^{(0)}}
\]

\[
= \frac{(\mu_1 \alpha_0 - \mu_0 \alpha_1)(\mu_2 \alpha_1 + \mu_2 \alpha_2) + (\mu_1 \alpha_0 + \mu_0 \alpha_1)(\mu_2 \alpha_1 - \mu_1 \alpha_2)e^{-2\alpha_1 l}}{\left(\mu_1 \alpha_0 + \mu_0 \alpha_1\right)(\mu_2 \alpha_1 + \mu_1 \alpha_2) + (\mu_1 \alpha_0 - \mu_0 \alpha_1)(\mu_2 \alpha_1 - \mu_1 \alpha_2)e^{-2\alpha_1 l}}
\]

\[
\Gamma_2^{(lyr)} = \zeta = \frac{4\mu_0 \mu_1 \alpha_0 \alpha_1}{U_{22}^{(0)}}
\]

\[
= \frac{4\mu_0 \mu_1 \alpha_0 \alpha_1}{\left(\mu_1 \alpha_0 + \mu_0 \alpha_1\right)(\mu_2 \alpha_1 + \mu_1 \alpha_2) + (\mu_1 \alpha_0 - \mu_0 \alpha_1)(\mu_2 \alpha_1 - \mu_1 \alpha_2)e^{-2\alpha_1 l}}.
\]  

(2.82)

**Source in the host region**

Suppose that the source is now in the host region, region 2 in Figure 2.6. To be consistent with the notation of the theory, this region should now be region 0, so all of the regions in the Figure need to be re-indexed down by two. Therefore, the Green's function to be determined is of the form of equation (2.78), where \(\Gamma_1 = 0\), because there is no lower surface. For this example \(m = 2, n = 0, z_{-2} = 0, z_{-1} = l\) and \(U = I\). The transformation matrices are

\[
T^{(-1)} = \frac{1}{2\mu_0 \alpha_0} \begin{bmatrix}
    (\mu_1 \alpha_0 + \mu_0 \alpha_1)e^{i(\alpha_{-1} - \alpha_0) - 2\alpha_0 z'} & (\mu_1 \alpha_0 - \mu_0 \alpha_1)e^{i(\alpha_{-1} + \alpha_0)} \\
    (\mu_1 \alpha_0 - \mu_0 \alpha_1)e^{i(\alpha_{-1} + \alpha_0)} & (\mu_1 \alpha_0 + \mu_0 \alpha_1)e^{i(\alpha_{-1} - \alpha_0)}
\end{bmatrix}
\]

\[
T^{(-2)} = \frac{1}{2\mu_1 \alpha_1} \begin{bmatrix}
    (\mu_2 \alpha_1 + \mu_1 \alpha_2) & (\mu_2 \alpha_1 - \mu_1 \alpha_2) \\
    (\mu_2 \alpha_1 - \mu_1 \alpha_2) & (\mu_2 \alpha_1 + \mu_1 \alpha_2)
\end{bmatrix}.
\]  

(2.83)

From equation (2.52) \(V^{(0)} = T^{(-1)} T^{(-2)}\), so

\[
V_{11}^{(0)} = \frac{1}{4\mu_0 \alpha_0 \mu_1 \alpha_1} e^{2\alpha_0 z'} \begin{bmatrix}
    (\mu_1 \alpha_0 + \mu_0 \alpha_1)(\mu_2 \alpha_1 - \mu_1 \alpha_2)e^{i(\alpha_{-1} - \alpha_0)} + \\
    (\mu_1 \alpha_0 - \mu_0 \alpha_1)(\mu_2 \alpha_1 + \mu_1 \alpha_2)e^{i(\alpha_{-1} + \alpha_0)}
\end{bmatrix}
\]

\[
V_{21}^{(0)} = \frac{1}{4\mu_0 \alpha_0 \mu_1 \alpha_1} \begin{bmatrix}
    (\mu_1 \alpha_0 + \mu_0 \alpha_1)(\mu_2 \alpha_1 - \mu_1 \alpha_2)e^{i(\alpha_{-1} - \alpha_0)} + \\
    (\mu_1 \alpha_0 - \mu_0 \alpha_1)(\mu_2 \alpha_1 + \mu_1 \alpha_2)e^{i(\alpha_{-1} + \alpha_0)}
\end{bmatrix}.
\]  

(2.84)

Using equation (3.37), we find the desired reflection term to be

\[
\Gamma_{-1} = \zeta = \frac{V_{21}^{(0)} e^{-2\alpha_0 z'}}{V_{11}^{(0)}}
\]

(2.85)

Switching back to the notation of Figure 2.6 by adding 2 to all the indices in equations (2.84) and (2.85) yields

\[
\Gamma_{-1}^{(lyr)} = \frac{(\mu_1 \alpha_2 + \mu_2 \alpha_1)(\mu_0 \alpha_1 - \mu_1 \alpha_0)e^{i(\alpha_1 - \alpha_2)} + (\mu_1 \alpha_2 - \mu_2 \alpha_1)(\mu_0 \alpha_1 + \mu_1 \alpha_0)e^{i(\alpha_1 + \alpha_2)}}{(\mu_1 \alpha_2 + \mu_2 \alpha_1)(\mu_0 \alpha_1 - \mu_1 \alpha_0)e^{i(\alpha_1 - \alpha_2)} + (\mu_1 \alpha_2 - \mu_2 \alpha_1)(\mu_0 \alpha_1 + \mu_1 \alpha_0)e^{i(\alpha_1 + \alpha_2)}}
\]

(2.86)
2.7.2 Slab Conductors

Figure 2.7 schematically represents the slab media type. This media is assumed linear and isotropic with conductivity $\sigma_1$ and permeability $\mu_1$ extending infinitely in both the $x$ and $y$ coordinate directions, but only a finite direction in the $z$ direction. $z$ is again assumed positive in the downward direction. The top of the slab is positioned at $z = 0$, so the bottom is at $z = s$. The space below the slab is assumed to be free-space; therefore, having the same material properties as region 0.

Source above the slab conductor

For sources in region 0, the scalar Green's function in the source region $\tilde{G}_0$ is just a special case of the layered half-space media type just considered with $\sigma_2 = \sigma_0$ and $\mu_2 = \mu_0$; therefore, only the reflection term off the top surface of the slab $\Gamma_1$ is needed. This coefficient is given in equation (2.82). The Green's function in the host region is of the form of equation (2.79), so the transmission coefficient $\Upsilon_1$ at the top surface of the slab conductor and the reflection term $\Gamma_2$ from the bottom need to be determined. Setting $s = l$ in equation (2.82) then $\Gamma_1 = \Gamma_1^{(lir)}$ and $\Upsilon_2 = \Upsilon_2^{(lir)}$. Using equations (2.64) and (2.81) the reflection and transmission coefficients for the slab region are

$$\Gamma_2 = U_{22}^{(l)} \Upsilon_2$$
$$\Upsilon_1 = U_{12}^{(l)} \Upsilon_2.$$

(2.87)

The coefficients can be simplified by using the assumptions that $\sigma_2 = \sigma_0$ and $\mu_2 = \mu_0$. Therefore,

$$\Gamma_1^{(slab)} = \frac{(\mu_1^2 \sigma_0^2 - \mu_0^2 \sigma_1^2)(1 - e^{-2\alpha_1 l})}{(\mu_1 \sigma_0 + \mu_0 \sigma_1)^2 - (\mu_1 \sigma_0 - \mu_0 \sigma_1)^2 e^{-2\alpha_1 l}}$$
2.7. FURTHER PLANAR EXAMPLES

\[ \gamma_{1}^{(\text{slab})} = \frac{2\mu_{1}\alpha_{0}(\mu_{0}\alpha_{1} - \mu_{1}\alpha_{0})e^{-2\alpha_{1}s}}{(\mu_{1}\alpha_{0} + \mu_{0}\alpha_{1})^2 - (\mu_{1}\alpha_{0} - \mu_{0}\alpha_{1})^2e^{-2\alpha_{1}s}} \]
\[ \gamma_{2}^{(\text{slab})} = \frac{2\mu_{1}\alpha_{0}(\mu_{0}\alpha_{1} + \mu_{1}\alpha_{0})}{(\mu_{1}\alpha_{0} + \mu_{0}\alpha_{1})^2 - (\mu_{1}\alpha_{0} - \mu_{0}\alpha_{1})^2e^{-2\alpha_{1}s}}. \] (2.88)

Sources in the slab conductor

Suppose that the source is now in the slab region, region 1 in Figure 2.6. To be consistent with the notation of the theory, this region should now be region 0, so all of the regions in the figure need to be re-indexed down by one. The Green's function for the source region 0 is of the form of equation (2.78). For this example \( m = 1 \), \( n = 1 \), \( z_{-1} = 0 \) and \( z_{0} = s \). Only the two reflection coefficients in the slab, \( \Gamma_{-1} \) and \( \Gamma_{1} \) are desired. These coefficients can be determined by using equations (2.59), (2.61), (2.63), and (2.65) to show that

\[ \begin{align*}
\Gamma_{-1} &= \frac{V_{21}^{(0)}U_{21}^{(0)}e^{2\alpha_{0}s'} + U_{22}^{(0)}e^{-2\alpha_{0}s'}}{V_{11}^{(0)}U_{11}^{(0)} - V_{21}^{(0)}U_{12}^{(0)}} \\
\Gamma_{1} &= \frac{U_{12}^{(0)}(V_{11}^{(0)}e^{2\alpha_{0}s'} + V_{12}^{(0)}e^{-2\alpha_{0}s'})}{V_{11}^{(0)}U_{22}^{(0)} - V_{21}^{(0)}U_{12}^{(0)}}. 
\end{align*} \] (2.89)

After simplifying, re-indexing up by one and setting \( \sigma_{2} = \sigma_{0} \) and \( \mu_{2} = \mu_{0} \), the final form of the coefficients are

\[ \begin{align*}
\Gamma_{-1}^{(\text{slab})} &= \frac{(\mu_{0}\alpha_{1} - \mu_{1}\alpha_{0})[(\mu_{0}\alpha_{1} + \mu_{1}\alpha_{0}) + (\mu_{0}\alpha_{1} - \mu_{1}\alpha_{0})e^{-2\alpha_{1}(s'+s)}]}{(\mu_{1}\alpha_{0} + \mu_{0}\alpha_{1})^2 - (\mu_{1}\alpha_{0} - \mu_{0}\alpha_{1})^2e^{-2\alpha_{1}s}} \\
\Gamma_{1}^{(\text{slab})} &= \frac{(\mu_{0}\alpha_{1} - \mu_{1}\alpha_{0})[(\mu_{0}\alpha_{1} + \mu_{1}\alpha_{0}) + (\mu_{0}\alpha_{1} - \mu_{1}\alpha_{0})e^{2\alpha_{0}s'}]}{(\mu_{1}\alpha_{0} + \mu_{0}\alpha_{1})^2 - (\mu_{0}\alpha_{1} - \mu_{1}\alpha_{0})^2e^{-2\alpha_{1}s}}. \] (2.90)

2.7.3 Layered slab conductors

Figure 2.8 schematically represents the layered slab physical model. This geometry is like the slab described above with the addition of a linear and isotropic layer of conductivity \( \sigma_{1} \) permeability \( \mu_{1} \) and height \( l \) placed on top. As in the layered half-space media type the layer is always defect free and is closest to the probe. Below the host material, as in the slab case, there is assumed to be free-space, hence \( \sigma_{3} = \sigma_{0} \) and \( \mu_{3} = \mu_{0} \). The interfaces occur at \( z_{-1} = 0 \), \( z_{0} = l \) and \( z_{1} = l + s \).

Source above the conductor

Suppose the source is above the conductor, in region 0 in Figure 2.8, the Green's functions in the source region is of the form equation (2.78), with \( \Gamma_{-1} = 0 \). The Green's function in the host region is of the form of equation (2.79). The reflection coefficient at the top
surface, $\Gamma_1$ and the reflection and transmission coefficients in the host region, $\Gamma_3$ and $T_2$ must be determined. The analysis is just an extension of that done for the layered half-space conductor, except here we have four regions to consider. Referring to Section 2.4 we see that for this case $m = 0$, $n = 3$, $z_0 = 0$, $z_1 = l$, $z_2 = l + s$, $\mu_3 = \mu_0$ and $\sigma_3 = \sigma_0$. Proceeding as before by using equations (2.59) and (2.61) to find

$$U(0) = \frac{U(0)^0 e^{2\alpha_0 z'}}{U(0)}$$

$$r(I_{slab}) = U(2) r^2 2 3 1$$

$$U(2) = T(0) T(1) T(2)$$

Equation (2.64) implies

$$\Gamma_3^{(slab)} = U_3^{(2)} \chi_3$$
$$\gamma_2^{(slab)} = U_2^{(2)} \chi_3$$

where $U^{(2)} = T^{(2)}$.

**Source in host region**

Suppose that the source is now in the slab region, region 1 in Figure 2.6. To be consistent with the notation of the theory, this region should now be region 0, so all of the regions in the Figure need to be re-indexed down by two. The Green's function for the source region is then of the form of equation (2.78). For this example $m = 2$, $n = 1$, $z_{-2} = 0$, $z_{-1} = l$ and $z_0 = l + s$. Only the two reflection coefficients in the slab, $\Gamma_{-1}$ and $\Gamma_1$ are
desired. These coefficients can be determined by using equations (2.59), (2.61), (2.63), and (2.65) to show that

$$\Gamma_{-1} = \frac{U_{12}^{(0)}T_1 = \frac{U_{12}^{(0)}(U_{12}^{(0)}e^{\alpha_2 z'} + U_{22}^{(0)}e^{-2\alpha_2 z'})}{V_{21}^{(0)}U_{22}^{(0)} - V_{21}^{(0)}U_{12}^{(0)}}}$$

$$\Gamma_1 = \frac{V_{21}^{(0)}T_{-1} = \frac{V_{21}^{(0)}(V_{11}^{(0)}e^{\alpha_2 z'} + V_{21}^{(0)}e^{-2\alpha_2 z'})}{V_{21}^{(0)}U_{22}^{(0)} - V_{21}^{(0)}U_{12}^{(0)}}}.$$  (2.93)

After simplifying, re-indexing back up by two and setting $\sigma_3 = \sigma_0$ and $\mu_3 = \mu_0$, the final form of the coefficients are

$$\Gamma_{-1}^{(islab)} = \frac{\Gamma_a + \Gamma_b + (\Gamma_c + \Gamma_d)e^{-2\alpha_2(l+s-z')}}{\Gamma_e + \Gamma_f + \Gamma_g + \Gamma_h}$$

$$\Gamma_1^{(islab)} = \frac{\Gamma_i + \Gamma_j + (\Gamma_k + \Gamma_l)e^{-2\alpha_2(l+s+z')}}{\Gamma_e + \Gamma_f + \Gamma_g + \Gamma_h},$$  (2.94)

where

$$\Gamma_a = (\mu_1\alpha + \alpha_1)(\alpha_2 + \alpha)(\mu_1\alpha_2 - \alpha_1)e^{2\alpha_1 l}$$

$$\Gamma_b = (\alpha_1 - \mu_1\alpha)(\alpha_2 + \alpha)(\mu_1\alpha_2 + \alpha_1)$$  (2.95)

$$\Gamma_c = (\mu_1\alpha + \alpha_1)(\alpha_2 - \alpha)(\mu_1\alpha_2 - \alpha_1)e^{2\alpha_1 l}$$

$$\Gamma_d = (\alpha_1 - \mu_1\alpha)(\alpha_2 - \alpha)(\mu_1\alpha_2 + \alpha_1)$$  (2.96)

$$\Gamma_e = (\mu_1\alpha + \alpha_1)(\alpha_2 + \alpha)(\mu_1\alpha_2 + \alpha_1)e^{2l(\alpha_1 - \alpha_2)}$$

$$\Gamma_f = (\alpha_1 - \mu_1\alpha)(\alpha_2 + \alpha)(\mu_1\alpha_2 - \alpha_1)e^{-2\alpha_1 l}$$  (2.97)

$$\Gamma_g = (\mu_1\alpha + \alpha_1)(\alpha - \alpha_2)(\mu_1\alpha_2 - \alpha_1)e^{2l(\alpha_1 - \alpha_2) - 2\alpha_2 s}$$

$$\Gamma_h = (\mu_1\alpha - \alpha_1)(\alpha - \alpha_2)(\mu_1\alpha_1 + \alpha_1)e^{-2\alpha_2(l+s)}$$

$$\Gamma_i = (\mu_1\alpha + \alpha_1)(\alpha - \alpha_2)(\mu_1\alpha_2 + \alpha_1)e^{2\alpha_2 l}$$

$$\Gamma_j = (\alpha_1 - \mu_1\alpha)(\alpha_2 + \alpha)(\mu_1\alpha_2 - \alpha_1)$$

$$\Gamma_k = \Gamma_c e^{2\alpha_2 l}$$

$$\Gamma_l = \Gamma_d e^{2\alpha_2 l}.$$  (2.98)

2.7.4 Multi-layered slab conductors

Figure 2.9 represents the last of the planar media types to be considered, the multi-layered slab media type. This geometry is the slab described above with the addition of a linear and isotropic layer of conductivity $\sigma_1$, permeability $\mu_1$ and height $l$ placed on top and a linear and isotropic layer of conductivity $\sigma_3$, permeability $\mu_3$ and height $\sigma$ on the bottom. The layers are always defect free. Below the lower layer is assumed to be free-space, hence $\sigma_4 = \sigma_0$ and $\mu_4 = \mu_0$. The interfaces occur at $z_{-1} = 0$, $z_0 = l$, $z_1 = l+s$ and $z_2 = l+s+\sigma$. 
Source above the conductor

Suppose the source is above the conductor, in region 0 in Figure 2.8. The Green's function for the source region is of the form of equation (2.78) with \( \Gamma_{-1} = 0 \) and the Green's function for the host region is of the form of equation (2.79). Only the reflection coefficient at the top surface, \( \Gamma_1 \) and the reflection and transmission coefficients in the host region, \( \Gamma_4 \) and \( \Gamma_3 \), need to be determined. The analysis is just an extension of that done for the layered half-space conductor, except here we have four regions to consider. Referring to Section 2.4 we see that for this case \( m = 0 \), \( n = 4 \), \( z_0 = 0 \), \( z_1 = l \), \( z_2 = l + s \), \( z_3 = l + s + o \), \( \mu_4 = \mu_0 \) and \( \alpha_4 = \alpha_0 \). Proceeding as before by using equations (2.59) and (2.61) to find

\[
\Gamma^{(\text{mslab})}_1 = \frac{U^{(0)}_{12} e^{2\alpha_0 z'}}{U^{(0)}_{22}}, \\
\Gamma^{(\text{mslab})}_4 = \frac{e^{2\alpha_0 z'}}{U^{(0)}_{22}},
\]

where \( U^{(0)} = T^{(0)}T^{(1)}T^{(2)}T^{(3)} \). Equation (2.64) implies

\[
\Gamma^{(\text{mslab})}_4 = U^{(2)}_{12} \Gamma_4, \\
\Gamma^{(\text{mslab})}_3 = U^{(2)}_{22} \Gamma_4,
\]

where \( U^{(2)} = T^{(2)}T^{(3)} \).
2.7. FURTHER PLANAR EXAMPLES

Source in host region

Suppose that the source is now in the host region, region 2 in Figure 2.9. To be consistent with the notation of the theory, this region should now be region 0, so all of the regions in the Figure need to be re-indexed down by two. The Green’s function for the source region will then be of the form of equation (2.78). For this example \( m = 2, n = 2, z_{-2} = 0, z_{-1} = l, z_0 = l + s \) and \( z_1 = l + s + a \). Only the two reflection coefficients in the slab, \( \Gamma_0 \) and \( \Gamma_1 \) are desired. These coefficients can be determined by using equations (2.59), (2.61), (2.63), and (2.65) to show that

\[
\Gamma_{-1} = U_{12}^{(0)} \Gamma_1 = \frac{U_{12}^{(0)} (U_{12}^{(0)} e^{aoz'} + U_{22}^{(0)} e^{-2aoz'})}{V_{12}^{(0)} U_{22}^{(0)} - V_{22}^{(0)} U_{12}^{(0)}}
\]

\[
\Gamma_1 = V_{21}^{(0)} \tilde{\Gamma}_{-1} = \frac{V_{21}^{(0)} (V_{11}^{(0)} e^{aoz'} + V_{21}^{(0)} e^{-2aoz'})}{V_{12}^{(0)} U_{22}^{(0)} - V_{22}^{(0)} U_{12}^{(0)}}
\]

(2.101)

After simplifying, re-indexing back up by two and setting \( \alpha_4 = \sigma_0 \) and \( \mu_4 = \mu_0 \), the final form of the coefficients are

\[
\Gamma_{-1}^{(masslab)} = \frac{\Gamma_a + \Gamma_b + \Gamma_c + \Gamma_d + \Gamma_e + \Gamma_f}{\Gamma_g + \Gamma_h + \Gamma_i + \Gamma_j + \Gamma_k + \Gamma_l}
\]

\[
\Gamma_1^{(masslab)} = \frac{\Gamma_m + \Gamma_n + \Gamma_o + \Gamma_p}{\Gamma_g + \Gamma_h + \Gamma_i + \Gamma_j + \Gamma_k + \Gamma_l}
\]

(2.102)

where

\[
\Gamma_a = 2\mu_3 \alpha_3 (\mu_1 \alpha_2 - \alpha_1) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 + \mu_3 \alpha_0) e^{(l+s)(\alpha_1+\alpha_2)+2l(\alpha_2+\alpha_1)}
\]

\[
\Gamma_b = 2\mu_3 \alpha_3 (\mu_1 \alpha_2 + \alpha_1) (\alpha_1 - \mu_1 \alpha_0) (\alpha_3 + \mu_3 \alpha_0) e^{2ao(l+s)(\alpha_1+\alpha_2)}
\]

\[
\Gamma_c = (\alpha_1 - \mu_1 \alpha_2) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 + \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao+2a_3(l+s)+2a_1l}
\]

\[
\Gamma_d = (\mu_1 \alpha_2 + \alpha_1) (\mu_1 \alpha_0 - \alpha_1) (\alpha_3 + \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao(l+s)+\alpha_1l}
\]

\[
\Gamma_e = (\mu_1 \alpha_2 - \alpha_1) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao(l+s)+2a_1l}
\]

\[
\Gamma_f = (\mu_1 \alpha_2 + \alpha_1) (\alpha_1 - \mu_1 \alpha_0) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao+2a_2l}
\]

\[
\Gamma_g = 2\mu_3 \alpha_3 (\mu_1 \alpha_2 + \alpha_1) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 + \mu_3 \alpha_0) e^{ao+ao_2(l+s)+2a_1l}
\]

\[
\Gamma_h = 2\mu_3 \alpha_3 (\mu_1 \alpha_2 - \alpha_1) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{ao+ao_2(2l+s)+ao_1l}
\]

\[
\Gamma_i = -(\mu_1 \alpha_2 + \alpha_1) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 + \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao+2a_2l+2a_1l}
\]

\[
\Gamma_j = (\mu_1 \alpha_2 - \alpha_1) (\mu_1 \alpha_0 - \alpha_1) (\alpha_3 + \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{ao+ao_2(l+s)}
\]

\[
\Gamma_k = (\alpha_1 - \mu_1 \alpha_2) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao+ao_1l}
\]

\[
\Gamma_l = (\mu_1 \alpha_2 + \alpha_1) (\mu_1 \alpha_0 - \alpha_1) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao+ao_1l}
\]

\[
\Gamma_m = (\mu_1 \alpha_2 + \alpha_1) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_2) e^{2ao+ao_1l}
\]

\[
\Gamma_n = (\alpha_1 - \mu_1 \alpha_2) (\mu_1 \alpha_0 - \alpha_1) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0)
\]

\[
\Gamma_o = (\mu_1 \alpha_2 - \alpha_1) (\mu_1 \alpha_0 + \alpha_1) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao+ao_1l-2ao_2s'}
\]

\[
\Gamma_p = (\mu_1 \alpha_2 + \alpha_1) (\alpha_1 - \mu_1 \alpha_0) (\alpha_3 - \mu_3 \alpha_2) (\alpha_3 + \mu_3 \alpha_0) e^{2ao(l-s')}.
\]

(2.103)
Chapter 3

Basic Theory: Cylindrical Interfaces

The inquiry of truth, which is love making, or wooing of it, the knowledge of truth, which is the presence of it, and the belief of truth, which is the enjoying of it, is the sovereign good of human nature.

Francis Bacon 1561 – 1626

Eddy current tube inspection is a common NDE activity. Bobbin probes are pulled through small thin-walled tubing to inspect the tubes. These probes are the same as the air-cored probes already examined. However, for tube inspection the probe axis is assumed coaxial to the tube or bore-hole being examined. This motivates the study of conductors with cylindrical boundaries. The simplest will be a bore-hole in an infinite media, then a simple tube, and finally two types of layered tubes. Scalar Green's functions will be developed for the general stratified cylindrical case and then they will be applied to analyze each of these particular cylindrical media types in turn.

Dodd and Deeds[17] and Wait[62,63] analyzed special cases of cylindrical conductors, while Dodd, Cheng and Deeds[16] extended the theory for the general case of any number of cylindrical conductors, with the source in any region. In this chapter this theory is modified and presents it in a form that is consistent with the planar media types considered in Chapter 2. The theory is then extended to include the prediction of coil impedances for bobbin coils inside the source region.

The analysis of cylindrical media cannot begin with a simple one-dimensional case, as was done for stratified media with planar boundaries, because the analysis will be inherently three-dimensional. However, the problem can be reduced down to a quasi-one-dimensional problem in a similar way to the planar example in Section 2.4, in this case in terms of the radial direction.
3.1 Cylindrical interface problem: bore-hole conductors

Figure 3.1 schematically represents the bore-hole physical model. This physical model represents an infinite bore-hole of radius $\rho_0$ in a linear and isotropic host material of conductivity $\sigma_1$ and permeability $\mu_1$ of infinite extent. The interior region of the bore-hole is assumed to be free-space, with conductivity $\sigma_0$ and permeability $\mu_0$.

This physical model is to be used for bore-hole inspections with a probe assumed coaxial with the bore-hole. The probe is always assumed to be positioned well away from the top or bottom of the hole, at least 2 skin depths, to avoid large edge effects.

In order to develop the theory the simplest media type which has a cylindrical boundary will be studied first: the bore-hole. In a sense, this is a half-space conductor with a periodic solution. The physical model is shown in Figure 3.1, where the source here is assumed to be a delta-function coil of radius $\rho_0$. The conductor is assumed to have infinite extent in all directions. The scalar Green's functions which describe the field in either the bore-hole, region 0, or in the conductor, region 1 are to be developed. Once the Green's functions for a delta-function coil have been determined, the impedance of any driving coil can be analyzed, if the coil is axially symmetric and concentric with the centre of the bore-hole.

Just as in the planar case, the Green's functions must satisfy

$$ (\nabla^2 + k_0^2)G_0(r|r') = \delta(\rho - \rho')\delta(z - z') $$
$$ (\nabla^2 + k_1^2)G_1(r|r') = 0. \tag{3.1} $$

For this example, a point source $r'$ at the centre of the bore-hole is not assumed, but instead a delta-function coil source of radius $\rho'$. Switching to a cylindrical coordinate system before restating equation (3.1) in terms of this new source. Assuming that the conductor is homogeneous and that the driving current is time harmonic with frequency $\omega$, the current density $J$ and the vector potential $A$ will have only azimuthal components in cylindrical coordinates, hence

$$ J(r) = J(\rho, z)\hat{\phi} $$
The vector potential at \((\rho,z)\), produced by the driving coil with a current density \(\vec{J}\) at \((\rho',z')\), can be expressed as
\[
\vec{A}_j (\rho, z) = \mu_0 \int \int_{\text{coil}} \hat{G}_j (\rho, z, \rho', z') \hat{J} (\rho', z') d\rho' dz',
\]
where \(\hat{G} (\rho, z, \rho', z')\) is the axially symmetric scalar Green's function for the delta-function current at \((\rho', z')\). This is analogous to case the of stratified media with planar boundaries, except the source region must be divided into two parts: one inside the delta-function current loop and one outside. This is shown in Figure 3.1, where region 0\(^-\) is assumed inside and region 0\(^+\) is assumed outside.

Using the cylindrical form of Laplace operator and integrating with respect to the azimuthal variable, (3.1) can be rewritten as
\[
\begin{align*}
\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + k_0^2 \right] \hat{G}_0^- (\rho, z | \rho', z') &= \delta (\rho - \rho') \delta (z - z'), \quad \text{region 0}^-
\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + k_0^2 \right] \hat{G}_0^+ (\rho, z | \rho', z') &= \delta (\rho - \rho') \delta (z - z'), \quad \text{region 0}^+
\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + k_1^2 \right] \hat{G}_1 (\rho, z | \rho', z') &= 0, \quad \text{region 1},
\end{align*}
\]
with interface conditions
\[
\left. \frac{\partial \hat{G}_0^-}{\partial \rho} \right|_{\rho = \rho'} = \left. \frac{\partial \hat{G}_0^+}{\partial \rho} \right|_{\rho = \rho}, \quad \left. \frac{\partial \hat{G}_0^+}{\partial \rho} \right|_{\rho = \rho_0} = \left. \frac{\partial \hat{G}_1}{\partial \rho} \right|_{\rho = \rho_0}.
\]

The solution to the differential equations in (3.4) can be obtained by separation of variables[17,16]. Let
\[
\hat{G}_j (\rho, z | \rho', z') = R_j (\rho | \rho') Z_j (z | z').
\]
Divide (3.4) by \(R_j (\rho | \rho') Z_j (z | z')\) and, after choosing a negative separation constant \(-\alpha^2\), find that
\[
\frac{1}{\rho R_j (\rho | \rho')} \frac{\partial}{\partial \rho} \left( \frac{\partial R_j}{\partial \rho} \right) + \frac{1}{Z_j (z | z')} \frac{\partial^2 Z_j}{\partial z^2} = -\alpha^2.
\]
The \(z\) dependence is found to be
\[
\frac{1}{Z_j (z | z')} \frac{\partial^2 Z_j (z | z')}{\partial z^2} = -\alpha^2,
\]
which has solutions of the form

\[ Z_j(z|z') = A_j \sin(\alpha(z - z')) + B_j \cos(\alpha(z - z')). \]  

(3.9)

The radial dependence can be written

\[ \rho^2 \frac{\partial^2 R_j(\rho|\rho')}{\partial \rho^2} + \rho \frac{\partial R_j(\rho|\rho')}{\partial \rho} - (\rho^2 \alpha_j^2 + 1) R_j(\rho|\rho') = 0, \]

(3.10)

where, as before, \( \alpha_j^2 = \alpha^2 - i \omega \mu_j \sigma_j \). This Bessel differential equation has solutions of the form

\[ R_j(\rho|\rho') = \gamma_{1j} I_1(\alpha_j \rho) + \gamma_{2j} K_1(\alpha_j \rho), \]

(3.11)

where \( I_1 \) and \( K_1 \) are modified Bessel functions of first order and \( \Gamma_j \) and \( \Upsilon_j \) are constants that will be determined to satisfy the interface conditions.

Since these solutions are valid for an arbitrary value \(-\alpha^2\), they will be valid for all values. Therefore, the complete Green’s function in each region \( j \) will be the integral over the separation constant. Since the sine term in \( Z_j \) is an odd function \( A_j = 0 \). The result can be expressed as integrals over positive constants as

\[ \hat{G}_0-(\rho, z|\rho', z') = \int_0^{\infty} \Gamma_{0-} K_1(\alpha_j \rho) \cos(\alpha j(z - z'))d\alpha \]

\[ \hat{G}_0+(\rho, z|\rho', z') = \int_0^{\infty} [\Gamma_{0+} K_1(\alpha_j \rho) + \Upsilon_{0+} I_1(\alpha_j \rho)] \cos(\alpha j(z - z'))d\alpha \]

\[ \hat{G}_1(\rho, z|\rho', z') = \int_0^{\infty} \Upsilon_j I_1(\alpha_j \rho) \cos(\alpha j(z - z'))d\alpha, \]

(3.12)

where \( \Gamma_j \) and \( \Upsilon_j \) are analogous to the reflection and transmission coefficients from Section 2.4.

Unlike the half-space conductor, there are two extra transmission terms and \( \Gamma_{0-} \) and \( \Upsilon_{0+} \) corresponding to regions \( 0^- \) and \( 0^+ \), respectively, and two extra interface conditions. Imposing the interface conditions (3.5) to (3.12) yields

\[ \int_0^{\infty} \Gamma_{0-} I_1(\alpha \rho') \cos(\alpha j(z - z'))d\alpha = \int_0^{\infty} [\Gamma_{0+} K_1(\alpha \rho') + \Upsilon_{0+} I_1(\alpha \rho')] \cos(\alpha j(z - z'))d\alpha \]

\[ \int_0^{\infty} \delta(z - z') + \Upsilon_{0-} I_0(\alpha_0 \rho') \cos(\alpha_0 z - z'))d\alpha = \]

\[ \int_0^{\infty} [\Gamma_{0+} I_0(\alpha_0 \rho') - \Upsilon_{0+} K_0(\alpha_0 \rho')] \cos(\alpha_0 j(z - z'))d\alpha + \frac{1}{\alpha_0^2} \delta(\rho - \rho') \delta(z - z') \]

\[ \mu_0 \int_0^{\infty} [\Gamma_{0+} K_1(\alpha \rho_0) + \Upsilon_{0+} I_1(\alpha \rho_0)] \cos(\alpha_j z - z'))d\alpha = \]

\[ \mu_1 \int_0^{\infty} \Upsilon_1 K_1(\alpha \rho_0) \cos(\alpha j z - z'))d\alpha \]

\[ \alpha_0 \int_0^{\infty} [\Gamma_{0+} I_0(\alpha_1 \rho_0) - \Upsilon_{0+} K_0(\alpha_1 \rho_0)] \cos(\alpha_0 z - z'))d\alpha = \]

\[ -\alpha_1 \int_0^{\infty} \Upsilon_1 K_0(\alpha_1 \rho_0) \cos(\alpha_j z - z'))d\alpha \]

(3.13)
3.2. GENERAL 3D CYLINDRICAL INTERFACE PROBLEM

Now, use the Fourier integral theorem[53],
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} f(\alpha) \left[ \int_{-\infty}^{\infty} \cos\{\alpha(u)\} \cos\{\alpha'(u)\} du \right] d\alpha = f(\alpha'),
\]
(3.14)
by multiplying (3.13) by \(\cos\{\alpha'(z - z')\}\), to find
\[
\begin{align*}
\Gamma_0 - I_1(\alpha_0 \rho') &= \Gamma_1 I_1(\alpha_0 \rho') + \Gamma_0 + K_1(\alpha_0 \rho') \\
\Gamma_0 - I_0(\alpha_0 \rho') &= \Gamma_1 I_0(\alpha_0 \rho') - \Gamma_0 K_0(\alpha_0 \rho') + \frac{1}{\pi} \\
\mu_0 [\Gamma_1 I_1(\alpha_0 \rho_0) + \Gamma_0 + K_1(\alpha_0 \rho_0)] &= \mu_1 \Gamma_1 K_1(\alpha_1 \rho_0) \\
\alpha_0 [\Gamma_1 I_0(\alpha_0 \rho_0) - \Gamma_0 + K_0(\alpha_0 \rho_0)] &= -\alpha_1 \Gamma_1 K_0(\alpha_1 \rho_0).
\end{align*}
\]
(3.15)
Solving for the unknown reflection and transmission coefficients
\[
\begin{align*}
\Gamma_0 &= \frac{\rho' I_1(\alpha_0 \rho') C + K_1(\alpha_0 \rho') D}{\pi D} \\
\Gamma_1 &= \frac{\rho' I_1(\alpha_0 \rho') C}{\pi D} \\
\Gamma_1 &= \frac{\rho' I_1(\alpha_0 \rho') C}{\pi \rho_0 D},
\end{align*}
\]
(3.16)
where
\[
\begin{align*}
C &= \mu_1 \alpha_0 K_0(\alpha_1 \rho_0) K_1(\alpha_0 \rho_0) - \mu_0 \alpha_1 K_0(\alpha_0 \rho_0) K_1(\alpha_1 \rho_0) \\
D &= \mu_1 \alpha_0 I_0(\alpha_0 \rho_0) K_1(\alpha_1 \rho_0) + \mu_0 \alpha_1 I_1(\alpha_0 \rho_0) K_0(\alpha_1 \rho_0)
\end{align*}
\]
(3.17)

3.2 General 3D cylindrical interface problem

Figure 3.2: 3D interface problem for any stratified cylindrical conductor

Now, consider the general case of any multi-layered conductor with cylindrical boundaries. Suppose there are \(m\) layers inside the source region and \(n\) layers outside. Each region \(j\) is isentropic and linear with conductivity \(\sigma_j\) and permeability \(\mu_j\). This situation is illustrated in figure 3.2. All the interfaces between regions are assumed to be cylindrical and coaxial and occur at the radial distances:

\[
0 < \rho_{-m} < \rho_{-m+1} < \cdots < \rho_{-1} < \rho' < \rho_0 < \cdots < \rho_{n-1} < \rho_n.
\]
Therefore, region \(-m\) is an infinite rod of radius \(\rho_{-m}\), while region \(n\) has infinite radial and axial extent.

In each region \(-j\) inside the source region there is a corresponding reflection and transmission coefficient \(\Gamma_{-j-1}\) and \(\Upsilon_{-j}\). In each region \(j\) outside the source region, similarly, there are \(\Gamma_{j+1}\) and \(\Upsilon_{j}\). The innermost and outermost reflection term must be zero, so we have \(\Gamma_{-m-1} = \Gamma_{n+1} = 0\). Following the development of Section 2.4 we want to find the general relationship that holds between adjacent regions. Equation (3.15) can be generalized for any region \(j\) outside the source region as

\[
\mu_j [\Gamma_{j+1} I_1(\alpha_j \rho_j) + \Upsilon_j I_1(\alpha_j \rho_j)] = \mu_{j+1} [\Gamma_{j+2} I_1(\alpha_{j+1} \rho_j) + \Upsilon_{j+1} K_1(\alpha_{j+1} \rho_j)]
\]

\[
\alpha_j [\Gamma_{j+1} I_0(\alpha_j \rho_j) - \Upsilon_j K_0(\alpha_j \rho_j)] = \alpha_{j+1} [\Gamma_{j+2} I_0(\alpha_{j+1} \rho_j) - \Upsilon_{j+1} K_0(\alpha_{j+1} \rho_j)],
\] (3.18)

and write the Green's function for this region as

\[
\mathcal{G}_j (\rho, z|\rho', z') = \int_0^\infty [\Gamma_{j+1} I_1(\alpha_j \rho) + \Upsilon_j K_1(\alpha_j \rho)] \cos(\alpha_j z - \alpha_j z') d\alpha.
\] (3.19)

For any region \(-j\) inside the source region we find the similar relationships

\[
\mu_{-j} [\Upsilon_{-j} I_1(\alpha_{-j} \rho_{-j}) + \Gamma_{-j-1} K_1(\alpha_{-j} \rho_{-j})] = \mu_{-j+1} [\Upsilon_{-j+1} I_1(\alpha_{-j+1} \rho_{-j}) + \Gamma_{-j} K_1(\alpha_{-j+1} \rho_{-j+1})]
\]

\[
\alpha_{-j} [\Upsilon_{-j} I_0(\alpha_{-j} \rho_{-j}) - \Gamma_{-j-1} K_0(\alpha_{-j} \rho_{-j})] = \alpha_{-j+1} [\Upsilon_{-j+1} I_0(\alpha_{-j+1} \rho_{-j}) - \Gamma_{-j} K_0(\alpha_{-j+1} \rho_{-j+1})],
\] (3.20)

and write the Green's function for this region as

\[
\mathcal{G}_{-j} (\rho, z|\rho', z') = \int_0^\infty [\Upsilon_{-j} I_1(\alpha_{-j} \rho) + \Gamma_{-j-1} K_1(\alpha_{-j} \rho)] \cos(\alpha_{-j} z - \alpha_{-j} z') d\alpha.
\] (3.21)

There is a Green's function associated with each part of the source region, region \(0^-\), where \(\rho < \rho'\), and \(0^+\), where \(\rho > \rho'\). They are

\[
\mathcal{G}_{0^-} (\rho, z|\rho', z') = \int_0^\infty [\Upsilon_{0^-} I_1(\alpha_0 \rho) + \Gamma_{-1} K_1(\alpha_0 \rho)] \cos(\alpha_0 z - \alpha_0 z') d\alpha
\]

\[
\mathcal{G}_{0^+} (\rho, z|\rho', z') = \int_0^\infty [\Gamma_{0^+} I_1(\alpha_0 \rho) + \Upsilon_{0^+} K_1(\alpha_0 \rho)] \cos(\alpha_0 z - \alpha_0 z') d\alpha.
\] (3.22)

The relationships in (3.18) and (3.20) can be written in matrix form for \(-m \leq j \leq -1\) or \(1 \leq j \leq n - 1\), as

\[
A_j X^{(j)} = B_j X^{(j+1)},
\] (3.23)

where

\[
A_j = \begin{bmatrix}
\mu_j I_1(\alpha_j \rho_j) & \mu_j K_1(\alpha_j \rho_j) \\
\alpha_j I_0(\alpha_j \rho_j) & -\alpha_j K_0(\alpha_j \rho_j)
\end{bmatrix}
\]

\[
B_j = \begin{bmatrix}
\mu_{j+1} I_1(\alpha_{j+1} \rho_j) & \mu_{j+1} K_1(\alpha_{j+1} \rho_j) \\
\alpha_{j+1} I_0(\alpha_{j+1} \rho_j) & -\alpha_{j+1} K_0(\alpha_{j+1} \rho_j)
\end{bmatrix},
\] (3.24)
3.2. GENERAL 3D CYLINDRICAL INTERFACE PROBLEM

and

\[
X^{(j)} = \begin{cases} 
\begin{bmatrix} \Gamma_j & \Gamma_{j-1} \\
\Gamma_0 & \Gamma_1 \\
\Gamma_{j+1} & \Gamma_j 
\end{bmatrix}^T & \text{if } j < 0 \\
\begin{bmatrix} \Gamma_0 & \Gamma_1 \\
\Gamma_{j+1} & \Gamma_j 
\end{bmatrix}^T & \text{if } j = 0 \\
\begin{bmatrix} \Gamma_j & \Gamma_{j+1} \\
\Gamma_0 & \Gamma_1 
\end{bmatrix}^T & \text{otherwise}
\end{cases}
\] (3.25)

Using equation (3.24) we can define the transformation matrix from region \(-j\) to region \(-j + 1\) as [16]

\[
T(-j) = B_{-j}^{-1} A_{-j} = \frac{1}{D_{-j+1}} \begin{bmatrix} T_{11} & T_{12} \\
T_{21} & T_{22} \end{bmatrix} = \frac{\mu_{-j+1}}{\rho_{-j}} \begin{bmatrix} T_{11} & T_{12} \\
T_{21} & T_{22} \end{bmatrix} 
\] (3.26)

where

\[
T_{11} = \mu_{-j}\alpha_{-j+1} I_1(\alpha_{-j}\rho_{-j})K_0(\alpha_{-j+1}\rho_{-j}) + \mu_{-j+1}\alpha_{-j} I_0(\alpha_{-j}\rho_{-j})K_1(\alpha_{-j+1}\rho_{-j})
\]

\[
T_{12} = \mu_{-j}\alpha_{-j+1} K_0(\alpha_{-j+1}\rho_{-j})K_1(\alpha_{-j}\rho_{-j}) - \mu_{-j+1}\alpha_{-j} K_0(\alpha_{-j}\rho_{-j})K_1(\alpha_{-j+1}\rho_{-j})
\]

\[
T_{21} = \mu_{-j}\alpha_{-j+1} I_0(\alpha_{-j+1}\rho_{-j})I_1(\alpha_{-j}\rho_{-j}) - \mu_{-j+1}\alpha_{-j} I_0(\alpha_{-j}\rho_{-j})I_1(\alpha_{-j+1}\rho_{-j})
\]

\[
T_{22} = \mu_{-j}\alpha_{-j+1} I_0(\alpha_{-j+1}\rho_{-j})K_1(\alpha_{-j}\rho_{-j}) + \mu_{-j+1}\alpha_{-j} I_1(\alpha_{-j+1}\rho_{-j})K_0(\alpha_{-j}\rho_{-j})
\] (3.27)

and

\[
D_{-j} = \mu_{-j}\alpha_{-j} [I_1(\alpha_{-j}\rho_{-j-1})K_0(\alpha_{-j}\rho_{-j-1}) + I_0(\alpha_{-j}\rho_{-j-1})K_1(\alpha_{-j}\rho_{-j-1})].
\] (3.28)

The terms in square brackets in (3.28) can be simplified by the use of a Wronskian relationship between modified Bessel functions[2](eq. 9.6.15); therefore, \(D_{-j} = \mu_{-j}/\rho_{-j-1}\).

The transmission matrix from region \(-m\) to region \(-j\) is just the product of these 2x2 transformation matrices. We can define this transformation as

\[
V(-j) = T(-j-1)T(-j-2) \ldots T(-m).
\]

The transformation from region \(-j\) to itself is just the identity matrix, hence \(V(m) \equiv I\).

The transformation matrix from region \(j + 1\) to region \(j\) is

\[
T^{(j+1)} = A_j^{-1}B_j = \frac{1}{D_j} \begin{bmatrix} T_{11} & T_{12} \\
T_{21} & T_{22} \end{bmatrix} = \frac{\rho_j}{\mu_j} \begin{bmatrix} T_{11} & T_{12} \\
T_{21} & T_{22} \end{bmatrix} 
\] (3.29)

where

\[
T_{11} = \mu_{j+1}\alpha_j I_1(\alpha_{j+1}\rho_j)K_0(\alpha_j\rho_j) + \mu_j\alpha_{j+1} I_0(\alpha_{j+1}\rho_j)K_1(\alpha_j\rho_j)
\]

\[
T_{12} = \mu_{j+1}\alpha_j K_0(\alpha_j\rho_j)K_1(\alpha_{j+1}\rho_j) - \mu_j\alpha_{j+1} K_0(\alpha_{j+1}\rho_j)K_1(\alpha_j\rho_j)
\]

\[
T_{21} = \mu_{j+1}\alpha_j I_0(\alpha_j\rho_j)I_1(\alpha_{j+1}\rho_j) - \mu_j\alpha_{j+1} I_0(\alpha_{j+1}\rho_j)I_1(\alpha_j\rho_j)
\]

\[
T_{22} = \mu_{j+1}\alpha_j I_0(\alpha_j\rho_j)K_1(\alpha_{j+1}\rho_j) + \mu_j\alpha_{j+1} I_1(\alpha_{j+1}\rho_j)K_0(\alpha_{j+1}\rho_j),
\] (3.30)

and

\[
D_j = \mu_j\alpha_j [I_1(\alpha_j\rho_j)K_0(\alpha_j\rho_j) + I_0(\alpha_j\rho_j)K_1(\alpha_j\rho_j)] = \frac{\mu_j}{\rho_j}.
\] (3.31)
The transformation matrix from region \( n \) to region \( j \) can be defined as
\[
U(j) = T(j+1)T(j+2) \ldots T(n),
\]
where \( U(n) = I \).

The relationship between the two parts of the source region, regions \( 0^- \) and \( 0^+ \), can be written as
\[
A'X^{(0)} = A'X' + Y^{(0)}, 
\]
where
\[
A' = \begin{bmatrix}
\mu_0 I_1(\alpha_0 \rho') & \mu_0 K_1(\alpha_0 \rho') \\
\alpha_0 I_0(\alpha_0 \rho') & -\alpha_0 K_0(\alpha_0 \rho')
\end{bmatrix}, 
X' = \begin{bmatrix}
\Gamma_1 \\
\Gamma_{0+}
\end{bmatrix}, 
Y^{(0)} = \begin{bmatrix}
0 \\
1/\pi
\end{bmatrix}. 
\] (3.32)

By using (3.24) recursively along with (3.32), we can express a relationship between \( X^{(-m)} \) and \( X^{(n)} \). This relationship is
\[
U^{(0)}X^{(n)} - V^{(0)}X^{(-m)} = -A_0^{-1}Y^{(0)} = \frac{\rho'}{\pi} \begin{bmatrix}
-K_1(\alpha_0 \rho') \\
I_1(\alpha_0 \rho')
\end{bmatrix}. 
\] (3.33)

Since \( X^{(-m)} \) and \( X^{(n)} \) have only one unknown each, (3.34) can be rewritten as
\[
\begin{bmatrix}
-V_{11}^{(0)} & U_{12}^{(0)} \\
-V_{21}^{(0)} & U_{22}^{(0)}
\end{bmatrix} \begin{bmatrix}
\eta \\
\zeta
\end{bmatrix} = \frac{\rho'}{\pi} \begin{bmatrix}
-K_1(\alpha_0 \rho') \\
I_1(\alpha_0 \rho')
\end{bmatrix}, 
\] (3.35)

where
\[
\eta = \begin{cases} 
\Gamma_{-m} & \text{if } m > 0 \\
\Gamma_{-n} & \text{otherwise}
\end{cases}, 
\zeta = \begin{cases} 
\Gamma_{n} & \text{if } n > 0 \\
\Gamma_{0+} & \text{otherwise}
\end{cases}, 
\] (3.36)

which has as a solution
\[
\eta = \frac{\rho'}{\pi} \frac{U_{22}^{(0)} K_1(\alpha_0 \rho') + U_{12}^{(0)} I_1(\alpha_0 \rho')}{V_{11}^{(0)} U_{22}^{(0)} - V_{21}^{(0)} U_{12}^{(0)}}, 
\]
\[
\zeta = \frac{\rho'}{\pi} \frac{V_{11}^{(0)} I_1(\alpha_0 \rho') + V_{21}^{(0)} K_1(\alpha_0 \rho')}{V_{11}^{(0)} U_{22}^{(0)} - V_{21}^{(0)} U_{12}^{(0)}}. 
\] (3.37)

Now the transformations matrices can be used to determine the reflection and transmission coefficients in any region. Inside the source region they are simply
\[
X^{(-j)} = \begin{bmatrix}
\Gamma_{-j} \\
\Gamma_{-j-1}
\end{bmatrix} = V^{(-j)} \begin{bmatrix}
\Gamma_{-m} \\
0
\end{bmatrix} = V^{(-j)}X^{(-m)} 
\] (3.38)

and for regions outside the source region we have a similar expression
\[
X^{(j)} = \begin{bmatrix}
\Gamma_{j+1} \\
\Gamma_j
\end{bmatrix} = U^{(j-1)} \begin{bmatrix}
0 \\
\Gamma_n
\end{bmatrix} = U^{(j-1)}X^{(n)}, \quad j > 1 
\]
\[
X' = U^{(0)}X^{(n)} 
\] (3.39)
3.3. VECTOR POTENTIAL

3.2.1 Bore-hole revisited

A simple example should illustrate this procedure. For the bore-hole example considered in Section 3.1, we would have \( m = 0 \) and \( n = 1 \), with the source inside the conducting region, region 1. Therefore, \( V^{(0)} = I \), while \( A' \) is given in (3.33) and

\[
A_0 = \begin{bmatrix}
\mu_0 I_1(\alpha_0 \rho_0) & \mu_0 K_1(\alpha_0 \rho_0) \\
\alpha_0 I_0(\alpha_0 \rho_0) & -\alpha_0 K_0(\alpha_0 \rho_0)
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
\mu_1 I_1(\alpha_1 \rho_0) & \mu_1 K_1(\alpha_1 \rho_0) \\
\alpha_1 I_0(\alpha_1 \rho_0) & -\alpha_1 K_0(\alpha_1 \rho_0)
\end{bmatrix},
\]

(3.40)

\[
X^{(0)} = \begin{bmatrix}
\Gamma_{0-} \\
0
\end{bmatrix}, \quad X' = \begin{bmatrix}
\Gamma_1 \\
\Gamma_{0+}
\end{bmatrix}, \quad X^{(1)} = \begin{bmatrix}
0 \\
\Gamma_1
\end{bmatrix}
\]

(3.41)

and

\[
U^{(0)} = \frac{\rho_0}{\rho_0} \begin{bmatrix}
U_{11}^{(0)} \\
U_{21}^{(0)}
\end{bmatrix}, \quad U_{22}^{(0)} = \frac{\mu_1 \alpha_0 K_0(\alpha_0 \rho_0) K_1(\alpha_1 \rho_0) - \mu_0 \alpha_1 K_0(\alpha_1 \rho_0) K_1(\alpha_0 \rho_0)}{\mu_1 \alpha_0 I_0(\alpha_0 \rho_0) K_1(\alpha_1 \rho_0) + \mu_1 \alpha_0 I_1(\alpha_0 \rho_0) K_0(\alpha_1 \rho_0)}
\]

(3.42)

Using equation (3.37), we find that

\[
\eta = \Gamma_{0-} = \frac{\rho' U_{22}^{(0)} K_1(\alpha_0 \rho') + \rho' U_{12}^{(0)} K_1(\alpha_0 \rho')}{\pi U_{22}^{(0)}}
\]

\[
\zeta = \Gamma_1 = \frac{\alpha_0 I_1(\alpha_0 \rho')}{\pi U_{22}^{(0)}}
\]

(3.43)

Finally, we have from (3.39)

\[
\begin{bmatrix}
\Gamma_1 \\
\Gamma_{0+}
\end{bmatrix} = X' = U^{(0)} X^{(1)} = \begin{bmatrix}
\Gamma_1 U_{12}^{(0)} \\
\Gamma_1 U_{22}^{(0)}
\end{bmatrix}
\]

(3.44)

These results are the same obtained in equation (3.16), with \( C = U_{12}^{(0)} \) and \( D = U_{22}^{(0)} \).

3.3 Vector potential

We can use (3.19), (3.37) and (3.39) to write the Green’s function in any region \( j \) outside the source as

\[
\tilde{G}_j (\rho, z | \rho', z') = \int_0^\infty \left[ \Gamma_{j+1} I_1(\alpha_j \rho) + \Gamma_j K_1(\alpha_j \rho) \right] \cos(\alpha \{ z - z' \}) d\alpha
\]

\[
= \int_0^\infty \left[ U_{12}^{(j)} I_1(\alpha_j \rho) + U_{22}^{(j)} K_1(\alpha_j \rho) \right] \Gamma_j \cos(\alpha \{ z - z' \}) d\alpha
\]

\[
= \frac{\rho'}{\pi} \int_0^\infty \left[ U_{12}^{(j)} I_1(\alpha_j \rho) + U_{22}^{(j)} K_1(\alpha_j \rho) \right] \frac{V_1^{(0)} I_1(\alpha_0 \rho') + V_2^{(0)} K_1(\alpha_0 \rho')}{\Delta_0} \cos(\alpha \{ z - z' \}) d\alpha,
\]

(3.45)

where

\[
\Delta_j = V_{11}^{(j)} U_{22}^{(j)} - V_{21}^{(j)} U_{12}^{(j)}
\]

(3.46)
Similarly, we can use (3.21), (3.37) and (3.38) to write the Green's function for region $-j$ inside the source as

$$
\hat{G}_j (\rho, z|\rho', z') = \frac{\rho'}{\pi} \int_0^\infty \left[ V_{11}^{(-j)} I_1 (\alpha_- j \rho) + V_{21}^{(-j)} K_1 (\alpha_- j \rho) \right] \cdot \\
\left[ \frac{U_{22}^{(0)} K_1 (\alpha_0 \rho') + U_{12}^{(0)} I_1 (\alpha_0 \rho')}{\Delta_0} \right] \cos (\alpha \{z - z'\}) \, d\alpha,
$$

(3.47)

Once the scalar Green's function is known for a region, the vector potential can be found by using (3.3). The air-cored physical model shown in Figure 2.2 has a densely and uniformly wound coil, so the current density can be assumed to be a constant $i_0 = I n / A_c$. Using this fact with (3.3) we can now write down the vector potential in any region $j$ as

$$
\hat{A}_j (\rho, z) = i_0 \mu_0 \int_{l_0}^{l_1} \int_{r_0}^{r_1} \hat{G}_j (\rho, z, r, z') r' \, dr' \, dz'.
$$

(3.48)

After changing the order of integration and then integrating over the coil dimensions, we have for any region $j$ outside the source region

$$
\hat{A}_j (\rho, z) = \frac{i_0 \mu_0}{\pi} \int_0^\infty \left[ U_{12}^{(j)} I_1 (\alpha_j \rho) + U_{22}^{(j)} K_1 (\alpha_j \rho) \right] \cdot \\
\left[ \frac{V_{11}^{(0)} \Omega (\alpha, r_0, r_1) + V_{21}^{(0)} \chi (\alpha, r_0, r_1)}{\alpha \Delta_0} \right] \cdot \\
\left[ \sin (\alpha \{z - l_0\}) - \sin (\alpha \{z - l_1\}) \right] \, d\alpha.
$$

(3.49)

and for any region inside the coil

$$
\hat{A}_j (\rho, z) = \frac{i_0 \mu_0}{\pi} \int_0^\infty \left[ V_{11}^{(-j)} I_1 (\alpha_- j \rho) + V_{21}^{(-j)} K_1 (\alpha_- j \rho) \right] \cdot \\
\left[ \frac{U_{22}^{(0)} \chi (\alpha, r_0, r_1) + U_{12}^{(0)} \Omega (\alpha, r_0, r_1)}{\alpha \Delta_0} \right] \cdot \\
\left[ \sin (\alpha \{z - l_0\}) - \sin (\alpha \{z - l_1\}) \right] \, d\alpha,
$$

(3.50)

where

$$
\Omega (\alpha, r_0, r_1) = \int_{r_0}^{r_1} \rho I_1 (\alpha \rho) \, d\rho, \quad \chi (\alpha, r_0, r_1) = \int_{r_0}^{r_1} \rho K_1 (\alpha \rho) \, d\rho.
$$

(3.51)

For the infinite bore-hole example, if the field in the infinite conductor is to be determined, equation (3.49) can be used since $E = i \omega A$. For the bore-hole $U^{(0)} = I$, so

$$
\hat{E}_1 (\rho, z) = \frac{i \omega i_0 \mu_0}{\pi} \int_0^\infty \frac{K_1 (\alpha \rho) \Omega (\alpha, r_0, r_1) \left[ \sin (\alpha \{z - l_0\}) - \sin (\alpha \{z - l_1\}) \right]}{\alpha U_2^{(0)}} \, d\alpha.
$$

(3.52)

where

$$
U_2^{(0)} = \mu_1 \alpha_0 J_0 (\alpha_0 \rho_0) K_1 (\alpha_1 \rho_0) + \mu_0 \alpha_1 I_1 (\alpha_0 \rho_0) K_0 (\alpha_1 \rho_0)
$$

(3.53)
Region 0, containing the coil, needs special treatment. The vector potential in region $0^+$, $A_{0^+}(\rho, z)$, is given by equation (3.49), with $j \equiv 0$, while $A_{0^-}(\rho, z)$ is found using (3.50). We want the vector potential only in the region of the coil itself. To find the vector potential at a point in the coil region, we must, after integration with respect to $z$, add the integral of $A_{0^-}$ for a coil going from $r_0$ to $r$ to the integral of $A_{0^+}$ for a coil going from $r$ to $r_1$. Therefore, replacing $r$ for $r_1$ in (3.49) and replacing $r$ for $r_0$ in (3.50) and adding, we find

$$
\tilde{A}_{\text{coil}}(\rho, z) = \frac{\mu_0 \mu_0}{\pi} \int_0^\infty \left\{ \left[ U_{12}^{(0)} I_1(\alpha_0 \rho) + U_{22}^{(0)} K_1(\alpha_0 \rho) \right] \cdot \left[ V_{11}^{(0)} \Omega(\alpha, r_0, r) + V_{21}^{(0)} \chi(\alpha, r_0, r) \right] + \\
\left[ V_{11}^{(0)} I_1(\alpha_0 \rho) + V_{21}^{(0)} K_1(\alpha_0 \rho) \right] \left[ U_{12}^{(0)} \Omega(\alpha, r, r_1) + U_{22}^{(0)} \chi(\alpha, r, r_1) \right] \cdot \left[ \frac{\sin(\alpha(z - l_0)) - \sin(\alpha(z - l_1))}{\alpha \Delta_0} \right] \right\} d\alpha.
$$

(3.54)

After expanding, adding and subtracting the term

$$
U_{12}^{(0)} V_{21}^{(0)} [I_1(\alpha \rho) \chi(\alpha, r, r_1) + K_1(\alpha \rho) \Omega(\alpha, r_0, r)],
$$

and simplifying, we find

$$
\tilde{A}_{\text{coil}}(\rho, z) = \frac{\mu_0 \mu_0}{\pi} \int_0^\infty \left\{ U_{12}^{(0)} V_{11}^{(0)} I_1(\alpha_0 \rho) \Omega(\alpha, r_0, r_1) + U_{22}^{(0)} V_{21}^{(0)} K_1(\alpha_0 \rho) \chi(\alpha, r_0, r_1) + \\
U_{12}^{(0)} V_{21}^{(0)} [I_1(\alpha_0 \rho) \chi(\alpha, r_0, r_1) + K_1(\alpha_0 \rho) \Omega(\alpha, r_0, r_1)] \cdot \left[ \frac{\sin(\alpha(z - l_0)) - \sin(\alpha(z - l_1))}{\alpha \Delta_0} \right] \right\} d\alpha + \\
\frac{\mu_0 \mu_0}{\pi} \int_0^\infty \frac{1}{\alpha} \left[ K_1(\alpha_0 \rho) \Omega(\alpha, r_0, r) + I_1(\alpha_0 \rho) \chi(\alpha, r, r_1) \right] \cdot \left[ \sin(\alpha(z - l_0)) - \sin(\alpha(z - l_1)) \right] d\alpha.
$$

(3.55)

If there are no conductors present, that is $m = n = 0$, then only the second integral in (3.55) remains, which is just a different formulation to that in equation (2.18), the vector potential for a coil in free-space, hence

$$
\frac{\pi}{\alpha} \Psi^2(\alpha, r_0, r_1) \left[ (l_1 - l_0) + e^{-\alpha(l_1 - l_0)} - 1 \right] d\alpha = \\
\frac{\mu_0 \mu_0}{\pi} \int_0^\infty \frac{1}{\alpha} \left[ K_1(\alpha_0 \rho) \Omega(\alpha, r_0, r) + I_1(\alpha_0 \rho) \chi(\alpha, r, r_1) \right] \cdot \left[ \sin(\alpha(z - l_0)) - \sin(\alpha(z - l_1)) \right] d\alpha.
$$

(3.56)

### 3.4 Coil impedance

As in Section 2.2.2, in order to compute the coil impedance we first compute the induced voltage $V$ in a single loop

$$
V = -i \omega 2 \pi r \tilde{A}_{\text{coil}}(\rho, z).
$$

(3.57)
As before, the total induced voltage is found by equation (2.20). Using equations (2.20), (3.55) and (2.18), with the relation \( Z = V / I \), and integrating over the coil dimensions, the coil impedance is found to be

\[
Z = \frac{2i\omega\mu_0 n^2}{A_c^2} \int_0^\infty \frac{2}{\alpha A_0} \left[ U^{(0)}_{11} V^{(0)}_{11} \Omega^2(\alpha, r_0, r_1) + U^{(0)}_{22} V^{(0)}_{21} \chi^2(\alpha, r_0, r_1) \right] \left[ 1 - \cos(\alpha(l_1 - l_0)) \right] + \frac{\pi}{\alpha} \Psi^2(\alpha, r_0, r_1) \left[ (l_1 - l_0) + \frac{e^{-\alpha(l_1 - l_0)} - 1}{\alpha} \right] d\alpha. \tag{3.58}
\]

For the bore-hole example, therefore, the change in impedance in the coil, between free-space and the bore-hole media types is just the difference between equations (2.22) and (3.58)

\[
\Delta Z = \frac{4i\omega\mu_0 n^2}{A_c^2} \int_0^\infty \frac{U^{(0)}_{11}}{U^{(0)}_{22}} \Omega^2(\alpha, r_0, r_1) [1 - \cos(\alpha(l_1 - l_0))] d\alpha, \tag{3.59}
\]

since \( V = I \).

### 3.5 Further cylindrical examples

In later chapters results will be needed for three additional cylindrical media types in addition to the bore-hole example already considered. All three media types will be tubes, some with layers. For each example it will be necessary to determine the change in coil impedance between the tube conductor and the bore-hole, as well as determine the electric field inside a particular region, the host region in each example. When flaws are introduced into the conductors in Chapter 5 they will occur only in the host region.

#### 3.5.1 Conducting tubes

![Figure 3.3: Tube medium](image)

Figure 3.3 schematically represents the tube media type. This physical model is similar to the bore-hole type except that the host material of conductivity \( \sigma_1 \) and permeability...
3.5. FURTHER CYLINDRICAL EXAMPLES

μ₁ only extends to a finite radial distance ρ₁. The tube is assumed to be surrounded by free-space. The host region is the tube itself, region 1.

Suppose the source is in the interior region of the conducting tube. Following the notation of section 3.2, for the tube, m = 0, n = 2, σ₂ = σ₀, μ₂ = μ₀ and V⁽₀⁾ = I. The change in impedance from free-space is given by equation (3.59) with U⁽₀⁾ = T⁽¹⁾T⁽²⁾. Since T⁽¹⁾₁₂ = U⁽¹⁾₁₂ and T⁽¹⁾₂₂ = U⁽¹⁾₂₂, these coefficients were determined in equation (3.42). Equation (3.29) can be employed to find that

\[
T⁽¹⁾_{11} = \frac{ρ₀}{μ₀} \left[ μ₁ α₀ I₁(α₁ ρ₀) K₀(α₀ ρ₀) + μ₀ α₁ I₀(α₁ ρ₀) K₁(α₀ ρ₀) \right]
\]

\[
T⁽¹⁾_{21} = \frac{ρ₀}{μ₀} \left[ μ₁ α₀ I₀(α₀ ρ₀) I₁(α₁ ρ₀) - μ₀ α₁ I₀(α₀ ρ₀) I₁(α₀ ρ₀) \right]
\]

\[
T⁽²⁾₁₂ = \frac{ρ₁}{μ₁} \left[ μ₀ α₁ K₀(α₁ ρ₁) K₁(α₀ ρ₁) - μ₁ α₀ K₀(α₀ ρ₁) K₁(α₁ ρ₁) \right]
\]

\[
T⁽²⁾₂₂ = \frac{ρ₁}{μ₁} \left[ μ₀ α₁ I₀(α₁ ρ₁) K₁(α₀ ρ₁) + μ₁ α₀ I₁(α₁ ρ₁) K₀(α₀ ρ₀) \right], \tag{3.60}
\]

The field in the host region is then given by

\[
\mathbf{E}_1(ρ, z) = \frac{i ω l₀}{π} \int_0^{∞} \frac{U⁽¹⁾₁₂(I₁(α₁ ρ) + U₂₂⁽¹⁾(α₁ ρ) K₁(α₀ ρ)}{α U₂₂⁽₀⁾} \Omega(α, r₁, ρ₀) \left[ \sin(α(z - l₀)) - \sin(α(z - l₁)) \right] dα. \tag{3.61}
\]

Since U₁₂⁽¹⁾ = T₂₂⁽¹⁾ and U₂₂⁽¹⁾ = T₂₂⁽²⁾ these coefficients are given in equation (3.60).

3.5.2 Layered tube conductors

Figure 3.4: Layered tube medium

Figure 3.4 represents the layered tube media type. This physical model is similar to the tube media type with the addition of a linear and isotropic layer on the inside of the tube. This layer is defect free and has conductivity σ₁, permeability μ₁ and inner radius ρ₀ as shown in the Figure. The host region is the outside part of the tube, region 2.

Suppose that the source is in the interior of the layered tube shown in Figure 3.4. For the layered tube we find m = 0, n = 3, σ₃ = σ₀, μ₃ = μ₀ and V⁽₀⁾ = I. Equation (3.59) can be used to compute the impedance change in the coil by recomputing the
coefficients $U_{12}^{(0)}$ and $U_{22}^{(0)}$ using the fact that $U^{(0)} = T^{(1)}T^{(2)}T^{(3)}$ in a similar way to the tube example above. Similarly, the electric field in the host region can be determined by equation (3.61), except the host region is now in region 2, so

$$
\vec{E}_2 (\rho, z) = \frac{i \omega \mu_0}{\pi} \int_0^\infty \frac{U_{12}^{(2)} I_1 (\alpha_2 \rho) + U_{22}^{(2)} K_1 (\alpha_2 \rho)}{\alpha U_{22}^{(0)}} \Omega (\alpha, r_1, r_0) \left[ \sin (\alpha (z - l_0)) - \sin (\alpha (z - l_1)) \right] d\alpha.
$$

(3.62)

### 3.5.3 Multi-layered tube conductors

![Figure 3.5: Multi-layered tube medium](image)

Figure 3.5 represents the last media type to be studied, the multi-layered tube. The host region is the central region of the tube, region 2.

Suppose that the source is in the interior of the multi-layered tube shown in Figure 3.5. For the layered tube we find $m = 0, n = 4, \sigma_4 = \sigma_0, \mu_4 = \mu_0$ and $V^{(0)} = I$. Equation (3.59) can be used to compute the impedance change in the coil by recomputing the coefficients $U_{12}^{(0)}$ and $U_{22}^{(0)}$ using the fact that $U^{(0)} = T^{(1)}T^{(2)}T^{(3)}T^{(4)}$ in a similar way to the layered tube example above. Similarly, the electric field in the host region can be determined by equation (3.62), with $U^{(3)} = T^{(3)}T^{(4)}$ and this newly computed value of $U^{(0)}$. 
Chapter 4

The Layered Media Forward Problem

If to do were as easy as to know what were good to do, chapels had been churches, and poor men's cottages prince's palaces.

William Shakespeare 1564 – 1616

The theoretical machinery is now in place to develop the complete numerical model for un-flawed layered media. The actual models are discussed and the issues that arise in the numerical approximation to the analytic expressions are addressed. Finally, validation and applications of the models are discussed.

4.1 Numerical Evaluation

The layered media forward problem can be expressed formally as

\[ Z = F(\omega; M, P), \quad (4.1) \]

where \( M \) and \( P \) are sets of known parameters defining the media and probe, respectively. \( Z \) is the probe impedance, which is to be determined. Figure 4.1 shows the forward model schematically and represents a simplified flow chart of the computer programme developed.

4.1.1 Analytic expression of the models

In Chapters 2 and 3 analytic expressions were developed for the layered media forward problem. For planar media the general expression was given in equation (2.77), repeated
Figure 4.1: Layered media forward model

and renumbered here

\[
Z = \frac{-\pi \omega \mu_0 n^2}{A^2} \int_0^\infty \frac{\Psi^2(\alpha, r_0, r_1)}{\alpha^2} \left[ 2\alpha(l_1 - l_0) + 2e^{-\alpha(l_1 - l_0)} - 2\Gamma^{-1}_{-1} \left( e^{\alpha l_1} - e^{\alpha l_0} \right)^2 + \Gamma_1 \left( e^{-\alpha l_1} - e^{-\alpha l_0} \right)^2 \right] d\alpha, \tag{4.2}
\]

while equations (3.58) and (3.59) combine to give the general expression for cylindrical systems,

\[
Z = \frac{2i\omega \mu_0 n^2}{A^2} \int_0^\infty \frac{2}{\alpha \Delta_0} \left[ U_{12}(0)V_{11}(0)\Omega^2(\alpha, r_0, r_1) + U_{22}(0)V_{21}(0)\chi^2(\alpha, r_0, r_1) \right] \left[ 1 - \cos(\alpha(l_1 - l_0)) \right] + \frac{\pi \Psi^2(\alpha, r_0, r_1)}{\alpha} \left( l_1 - l_0 \right) + \frac{e^{-\alpha(l_1 - l_0)} - 1}{\alpha} \right] d\alpha. \tag{4.3}
\]

Both expressions involve improper integrals.

Computationally, it is more accurate to compute the change in impedance in the coil due to some small perturbation in the structure of the medium than to compute absolute impedance \(Z\) in the coil. This is exactly analogous to measuring the probe response, where the signal is usually "zeroed" on some base material and the probe signal measured is then just the change between the base material and the workpiece. Therefore, the models have been implemented to compute the difference in signal from some base media type and if absolute impedance is desired, the absolute impedance signal from the base type is added in. There is no base type for free-space media, so the model always computes absolute impedance. For the half-space and bore-hole conductors, the base type is free space. All other planar media use the half-space as the base type, while all other cylindrical media use the bore-hole base type. The actual analytic expressions are given below.

**Planar media**

The change in coil impedance due the presence of a half-space conductor can be expressed by taking the difference between (4.2) using the free-space reflection coefficients and the
4.1. NUMERICAL EVALUATION

half-space reflection coefficients. For all planar types \( \Gamma_{-1}^{(\text{planar})} \equiv 0 \). In free-space, \( \Gamma_1 \equiv 0 \); therefore,

\[
\Delta Z = \frac{-\pi \mu_0 n^2}{A_c^2} \int_0^\infty \frac{\Psi^2(\alpha, r_0, r_1)}{\alpha^2} \Gamma_1^{(\text{half})} \left( e^{-\alpha l_1} - e^{-\alpha l_0} \right)^2 d\alpha. \tag{4.4}
\]

For the remaining planar media types, the change in coil impedance due the change in the structure of the half-space is computed. Again, using equation (4.2) to take the difference, yielding

\[
\Delta Z = \frac{-\pi \mu_0 n^2}{A_c^2} \int_0^\infty \frac{\Psi^2(\alpha, r_0, r_1)}{\alpha^2} \left( \Gamma_1^{(\text{half})} - \Gamma_1^{(\text{planar})} \right) \left( e^{-\alpha l_1} - e^{-\alpha l_0} \right)^2 d\alpha. \tag{4.5}
\]

Cylindrical media

As in the case of media with planar boundaries, equation (4.3) is a general expression for all the cylindrical media types. For all the cylindrical media types, equation (4.3) can be used to compute the probe impedance \( Z \) (Ohms). Since the coil is inside the conducting regions, \( m = 0 \) and \( n > 0 \). Therefore \( V^{(0)} = I \), so we have \( V_{11}^{(0)} = 1 \) and \( V_{21}^{(0)} = 0 \) and equation (4.3) reduces to

\[
Z = \frac{4i\omega\mu_0 n^2}{A_c^2} \int_0^\infty \frac{\Omega^2(\alpha, r_0, r_1)}{\alpha} \frac{U_{12}^{(0)}}{U_{22}^{(0)}} \left[ 1 - \cos(\alpha\{l_1 - l_0\}) \right] + \frac{-2\pi \mu_0 n^2}{A_c^2} \int_0^\infty \frac{\Psi^2(\alpha, r_0, r_1)}{\alpha} \left[ (l_1 - l_0) + \frac{e^{-\alpha(l_1-l_0)} - 1}{\alpha} \right] d\alpha. \tag{4.6}
\]

Equation (4.6) represents the general mathematical model for the layered media forward problem for cylindrical media types. As with the reflection coefficients, the two coefficients \( U_{12} \) and \( U_{22} \) must be determined for each physical model studied. Using equation (4.6) twice, once with \( U_{12}^{(\text{free})} = 0 \) and \( U_{22}^{(\text{free})} = 1 \) and once with the reflection coefficients for the bore-hole conductor and subtracting, yields

\[
\Delta Z = \frac{4i\omega\mu_0 n^2}{A_c^2} \int_0^\infty \frac{\Omega^2(\alpha, r_0, r_1)}{\alpha} \frac{U_{12}^{(\text{bore})}}{U_{22}^{(\text{bore})}} \left[ 1 - \cos(\alpha\{l_1 - l_0\}) \right] d\alpha. \tag{4.7}
\]

The integral expression in equation (4.7) is the mathematical model used for the layered media forward problem for bore-hole conductors. For the remaining cylindrical media types a similar expression,

\[
\Delta Z = \frac{4i\omega\mu_0 n^2}{A_c^2} \int_0^\infty \frac{\Omega^2(\alpha, r_0, r_1)}{\alpha} \left[ \frac{U_{12}^{(\text{bore})}}{U_{22}^{(\text{bore})}} - \frac{U_{12}^{(\text{cyln})}}{U_{22}^{(\text{cyln})}} \right] \left[ 1 - \cos(\alpha\{l_1 - l_0\}) \right] d\alpha. \tag{4.8}
\]
4.1.2 Numerical evaluation of the improper integrals

For both planar and cylindrical media, the function $F$ in equation (4.1) represents an improper integral, which needs to be approximated numerically. Since the integral is well-defined, a straightforward transformation can be implemented to transform the infinite interval to a finite one. The mapping used is\[44\]

$$
\int_a^b f(x)dx = \int_{\frac{b}{a}}^{\frac{b}{a}} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt, \quad ab > 0. \tag{4.9}
$$

This transformation cannot be applied to the entire intervals of integration in equations (4.2), (4.7) or (4.8) since the lower limit is $a = 0$; therefore, the interval of integration $[0, \infty)$ is divided into two intervals: one finite, $[0, c]$, the other infinite, $[c, \infty)$. The mapping (4.9) is then applied to the second interval for a fixed value of $c$, resulting in two finite integrals to be numerically approximated. These integrals are then approximated by using standard Romberg integration techniques.

The solution of equation (4.1) is independent of the particular value chosen for $c$ in (4.9); however, the rate of convergence of the Romberg scheme of the transformed integral can be quite slow if small values are used. It was found that values of $c$ above $1/(3\delta)$ gave good rates of convergence of the adaptive numerical integration scheme.

4.1.3 The planar Fourier integrand

The numerical approximation of the Fourier integral for planar media types involves the evaluation of the integrand of the form shown in equation (4.2). The computation of this integrand is straightforward, except for the term

$$
\Psi(\alpha, r_0, r_1) = \int_{r_0}^{r_1} \rho J_1(\alpha \rho) \, d\rho. \tag{4.10}
$$

This term can be evaluated by using Struve functions, but this approach does not offer any computational advantage. Instead, $\Psi(\alpha, r_0, r_1)$ is approximated using polynomial interpolation.

The polynomial approximation is based on the following analysis\[15\]. By introducing a change of variable, we have

$$
\Psi(\alpha, r_0, r_1) = \frac{1}{\alpha^2} \int_{r_0}^{r_1} x J_1(x) \, dx. \tag{4.11}
$$

Then integration by parts yields

$$
\int_{r_0}^{r_1} x J_1(x) \, dx = \alpha r_0 J_0(\alpha r_0) - \alpha r_1 J_0(\alpha r_1) + \int_{r_0}^{r_1} J_0(x) \, dx. \tag{4.12}
$$

Since

$$
\int_{r_0}^{r_1} J_0(x) \, dx = \int_0^{r_1} J_0(x) \, dx - \int_0^{r_0} J_0(x) \, dx, \tag{4.13}
$$
only the integral of $J_0$ need be approximated. On the interval $0 \leq x < 3$ and on the
interval $3 \leq x \leq 8$ polynomial interpolation in $x$ was used, while for large arguments,
a polynomial in $1/x$ was used. This approach proved to be considerably faster than
using the Struve functions and still gave accurate results. Efficiency is important when
evaluating $\Psi$ only because it appears again in the full 3D forward problem and needs to
be evaluated many times.

4.1.4 The cylindrical improper integrand

The evaluation of the integrand for cylindrical media types involves only the evaluation
of exponentials and modified Bessel functions. The only complication here is that the
arguments to the modified Bessel functions are of the form $\alpha_i \rho_j$ which is complex. No
standard library functions existed for this approximation and the efficiency of this compu-
tation was not important so they were evaluated by using their integral representation[3]:

\[
I_0(z) = \int_0^\pi e^{z \cos \theta} \, d\theta
\]
\[
I_1(z) = \int_0^\pi e^{z \cos \theta} \cos \theta \, d\theta
\]
\[
K_0(z) = \int_0^{\infty} e^{-z \cosh \theta} \, d\theta
\]
\[
K_1(z) = \int_0^{\infty} e^{-z \cosh \theta} \cosh \theta \, d\theta
\]

Similarly, the term $\Omega$, defined as

\[
\Omega(\alpha, r_0, r_1) = \int_{r_0}^{r_1} \rho I_1(\alpha \rho) \, d\rho,
\]

is also approximated numerically using its definition.

4.2 Validation Exercises

<table>
<thead>
<tr>
<th>Probe Number</th>
<th>Inner Radius</th>
<th>Outer Radius</th>
<th>Top of Coil</th>
<th>Bottom of Coil</th>
<th>Number of Turns</th>
<th>Self-Inductance</th>
<th>Predicted Self-Indct.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>.8</td>
<td>1.8</td>
<td>1.5</td>
<td>.5</td>
<td>106</td>
<td>20 mH</td>
<td>22.52 mH</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>182</td>
<td>160 mH</td>
<td>168.8 mH</td>
</tr>
<tr>
<td>8</td>
<td>4.75</td>
<td>7.75</td>
<td>4</td>
<td>1</td>
<td>296</td>
<td>1200 mH</td>
<td>1138 mH</td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td>13</td>
<td>4</td>
<td>1</td>
<td>351</td>
<td>4000 mH</td>
<td>3989 mH</td>
</tr>
</tbody>
</table>

Table 4.1: Probe parameters

A set of air-cored eddy current probes was manufactured with the parameters given in
Table 4.1[32]. The layered media forward model can predict the free-space impedance
of the probe. The impedance in free-space of idealized coils is purely inductive, so this is often referred to as the self-inductance of the probe. The predicted self-inductance for all the probes are shown in the last column of the Table and are in good agreement with the measured values shown in the adjacent column. The air-cored coil in presented in Section 7.4.5 also had its self-inductance measured and was found to be 221.8±4 mH[14]. The theoretical value computed by Burke was found to be 226.0 mH, while the forward model predicted \( L_0 = 225.98 \text{ mH} \). So the model is in good agreements with experimental and published results.

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Measured Direction</th>
<th>Predicted Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.35 kHz</td>
<td>136°</td>
<td>141.7°</td>
</tr>
<tr>
<td>4.00 kHz</td>
<td>126°</td>
<td>126.8°</td>
</tr>
<tr>
<td>12.0 kHz</td>
<td>114°</td>
<td>113.7°</td>
</tr>
</tbody>
</table>

Table 4.2: Measured versus predicted lift-off direction angles

Another simple experimental measurement is to determine the direction in the impedance plane in which the probe signal moves as the probe is lifted off the workpiece. This lift-off direction can be predicted by simply evaluating the partial derivative of equation (4.4) with respect to the lift-off parameter. The lift-off angle: the anti-clockwise angle between the positive x-axis and the lift-off direction vector, is a function not only of the probe parameters, but also of the material properties of the workpiece.

The lift-off direction was carefully measured in the laboratory[33] at various frequencies using a thick plate of austenitic steel, treated as a half-space conductor. The plate had a relative permeability of 1 and measured conductivity of \( 1.3 \times 10^6 \text{ S/m} \). The measured and predicted results at 3 different frequencies are presented in Table 4.2. Again the results are in good agreement.

### 4.3 Applications of layered media forward problems

The layered media forward models are straightforward giving predicted results for stratified conductors in the absence of flaws. This does not on the surface appear to very useful for real world type of NDE problems. Therefore; it is important that their application be pointed out. These models can be usefully applied in the following general areas:

- Provide simple experimental calibration checks
- Aid in probe design
• Provide a tool for choosing probes and inspection frequencies

Equipment calibration

When setting up equipment for measurements, either in the laboratory or on site, it is useful to take a series of simple measurements where the signals are well understood. The simplest measurements in eddy current NDE are measuring the probe response in the absence of defects. In these cases, parameters describing the workpiece and the operating conditions should all be well defined. Checking the measured signals at several frequencies against the layered model results should provide a quick check on equipment calibration and material properties. If there is a wide variation, then either the equipment is faulty, the parameters describing the probe or material properties are not correct, or the application of the physical model is inappropriate. The latter case would apply if a half-space model were being used for a plate where the skin depth is large compared to the plate thickness. Calibrating the equipment to these models should allow the inspection to be carried out with more confidence.

4.3.1 Probe design

The layered forward models are also useful in probe design. NDE probes are often designed to provide 50 Ohms of resistance at the designed operating frequency. The free-space model provides a way of predicting the operating frequency of a particular probe, or used iteratively, to help design new probes. Another important consideration in NDE probe design is the coupling between the workpiece and the probe. Coupling is a measure of how well the probe excites a field in the workpiece. The degree of coupling, for a particular probe, can be measured using Förster diagrams like the one shown in Figure 4.2.

A Förster diagram is the locus of points traversed in the normalized impedance plane as the operating frequency is varied from low frequencies to high frequencies with the probe on a workpiece. Normalized impedances are defined to be

\[ ||Z|| = \frac{Z}{\omega L_0}, \]

where \( L_0 \) is the free-space inductance of the probe. At the low frequency limit the impedance is purely inductive, approaching the free-space value, so at the normalized low frequency limit is (0,1) in the impedance plane if the material is non-ferromagnetic. In the presence of a ferromagnetic material, the inductance of the coil will be increased at low frequencies; therefore, the normalized low frequency limit will be greater than unity on the reactance axis. As the frequency is increased the normalized impedance values trace a curve through the normalized impedance plane. The high frequency limit is also purely inductive. Of course the probe will resonate before this high frequency limit is
reached, but the point where the curve should theoretically intersect the reactance axis can be easily measured. This intersection point provides the measure of coupling between probe and workpiece: the lower the intersection, the higher the coupling. Values in the range of 0.4 to 0.6 are considered good, values above 0.7 are considered poor.

Figure 4.2 shows the predicted Förster diagrams for 4 different eddy current probes described in Table 4.1. From the Figure, it is easy to see that coil #9 has the best coupling, while coil #6 has the worst. Since the larger coils have better coupling but poorer discrimination capabilities, coupling is really best used to compare similar size coils. The addition of a ferrite core in the windings also increase the coupling[12], but they have not been modelled. The coupling is a coil property, as can be seen in Figure 4.3. All curves are predicted using coil #6. The first, third and fourth curve in the Figure are the predicted responses to a conducting half-space of austenitic steel, an austenitic steel half-space with a 1mm layer corrosion and a similar layered half-space with a slightly ferrous conducting layer, respectively, where all three curves were generated using the same lift-off. Starting at the top left in the Figure, the first symbol represents the predicted signal at 10kHz, while each subsequent symbols represent a doubling of the inspection frequency. Even with the wide variation in materials the high frequency limit is the same. The second curve shows the predicted response when the probe is lifted 0.1mm off the surface. As can be seen in the Figure, coupling is a strong function of lift-off; decreasing as lift-off increases.

Figure 4.4 shows the predicted Förster diagrams for 2mm, 3mm and 4mm slabs of austenitic steel compared to the half-space curve from Figure 4.3. The 3mm slab has a 1mm ferrous layer on top, while the 4mm slab has a 1mm ferrous layer on the top and
4.3. APPLICATIONS OF LAYERED MEDIA FORWARD PROBLEMS

Figure 4.3: Förster diagrams for half-space and layered half-space conductors

bottom. The results in the Figure show that the coupling is independent of material properties. Figure 4.5 shows that the coupling measure extends to bore-hole or tube inspection. This Figure shows the predicted responses using coil #8 for a 16mm bore-hole, 3 16mm diameter tubes, all of austenitic steel. The tubes have 2mm, 3mm and 4mm thick walls, with the 3mm having a 1mm ferrous layer on the inside, while the 4mm has a 1mm ferrous layer on the inside and outside. Again the coupling measure is independent of material properties. For cylindrical geometries, the inside radius plays the role of lift-off in planar geometries. The coupling will decrease rapidly with increasing inner radii.

4.3.2 Inspection frequencies

The operating frequency use for NDE inspection is very important. The higher the frequency, the finer the flaw discrimination; however, the higher the frequency the shallower the penetration. So the operating frequency decisions always involve trade offs. As can be seen in all of the Förster diagrams in the previous section the eddy current signal differentiates different materials and/or geometries only over a certain range of frequencies. If the NDE task is to measure uniform coatings, uniform layers of corrosion, plate thicknesses, etc., a simple Förster diagram is useful in helping to determine the inspection frequency which will give the best discrimination.

There is an eddy current NDE technique that avoids the problem of choosing the operating frequency: transient eddy current inspection. Using a transient, time decaying driving current has the effect of inspecting at all frequencies. Therefore, the signal would
provides a measure of the total variation between two Förster diagrams. Harrison[25] has successfully applied this technique.

At all frequencies however, a major source of noise in the signal is due to variations in the probe lift-off. This is due to surface irregularities and probe positioning errors. So inspection equipment is often set up to look in quadrature to this signal. The models can quickly provide phase information, so that the phase separation between variations in the inspection parameter with probe lift-off variations can be predicted. Therefore, the frequency which gives the optimum phase separation, hence the best signal to noise, can be quickly determined. Figure 4.6 can be used to illustrate this point.

Figure 4.6 presents predicted probe responses from coil #6, plotted against measured values in the impedance plane at three different frequencies. The test-piece here was a block of austenitic steel which had been attacked with a corrosive agent to form a "uniform" layer of IGA[33]. The depth was measured destructively on one of the blocks surfaces. The actual parameters used for the forward problem in the Figure were determined by the inverse problem discussed in Chapter 8. In each image of an eddy current instrument in Figure 4.6, represents a different test frequency. The measured data and its scatter is presented as solid lines with □ symbols at each of the three frequencies. The solid arrows indicate the measured lift-off angles. The ∇ symbol indicates ΔZ predicted by the layered forward model. The broken arrow shows the predicted lift-off angle. The data scatter due to lift-off effects can be clearly seen in these figures. The data at fixed frequencies is nearly collinear, with the exception of the 1.2 MHz data which does seem to show an outlier.

The plot in the lower right-hand side of Figure 4.6 shows the amount of separation
between the three variables lift-off $l_2 - l_1$, conductivity of the layer $\sigma_1$ and depth of layer $l$. For a lack of a better name, they are referred to as lift-off, conductivity and depth angles. These angles represent the direction that the signal $\Delta Z$ change on an impedance plane with respect to each variable and analytically just represent the phase of the partial derivatives of the analytical solution of $\Delta Z$ evaluated at each of the test conditions, e.g. $\arg\left(\frac{\partial \Delta Z}{\partial l_2} - \frac{\partial \Delta Z}{\partial l_1}\right)$. These curves show that at low frequencies changes in any of the three variables will give signal variations in nearly the same phase. As the frequency and as the depth increase however the depth signal becomes distinct. In fact one can choose operation conditions where $\arg\left(\frac{\partial \Delta Z}{\partial l_1}\right)$ is normal to lift-off. These curves also imply that the layer conductivity is not accurately defined by the data, since the response due to conductivity variation has the same phase as the lift-off "noise".
Figure 4.6: Separation of signals due to test parameters
Chapter 5

3D Analysis: The Volume Integral Approach

*People who like this sort of thing will find this the sort of thing they like.*

Abraham Lincoln 1809 – 1865

5.1 Introduction

In Chapters 2 and 3 the change in impedance in an air-cored probe in the presence of several different media types was analyzed; however, all the media were restricted so that each of its sub-regions were isotropic and homogeneous. Now one of the regions, the *host* region, is allowed to have an anomaly.

These anomalies can be cracks, inclusions, local changes in the host conductivity. Three major assumptions are made about the flawed region:

- the host region is non ferromagnetic,
- the flaw itself is made up of uniform cells, *elements*, each of which has a constant conductivity.
- the flaw does not extend out of the host region.

Cracks or voids in the host region can be modelled as regions with no conductivity. Corrosion or other inclusions can be modelled as regions with different conductivities in the same manner. Figure 5.1 shows one flaw arrangement, along with the indexing conventions used. If the boundaries of the flaw itself are not planar, the flaw can be
approximated by a series of elements, where the conductivity of each element is just the volume fraction of the conductivity over that element, that is

$$\sigma_{klm} = \frac{1}{V_{klm}} \int_{V_{klm}} \sigma(r) dr.$$  \hfill (5.1)

For media types with cylindrical boundaries, the flaw elements no longer have a simple shape, but are assumed to be as shown in Figure 5.2. The conductivity of each element is constant, as before, and is just a per cent host region conductivity.

Media types with cylindrical boundaries are handled in the same way, except here the elements are no longer rectangular boxes, but are 'bent' around the axis to form sectors in the $\rho s$-plane. This arrangement is shown in Figure 5.2. Now each element has four flat faces and two curved faces. Again, the conductivity is assumed constant in each element.

The volume integral technique has been successfully applied to eddy current NDE problems by Sabbagh[53], Bowler[12] and McKirdy[37]. This technique involves subdividing the flaw into smaller elements and then, using an approach based on Green's functions and the superposition principle outlined in Chapter 1, constructing a global solution to the forward problem by solving a series of simpler problems of how one element interacts with another. The volume integral approach has a major advantage over finite element schemes for eddy current NDE in that the interaction between the probe and media boundaries can be built into the Green's functions; therefore, only the flaw volume needs to subdivided.

For eddy current NDE three things are needed to apply the volume integral approach:
5.2. **DYADIC GREEN'S FUNCTIONS**

![Diagram of flaw in hosts with cylindrical boundaries]

**Figure 5.2: Flaw in hosts with cylindrical boundaries**

1. Assemble the matrix by applying the method of moments

2. Assemble the forcing function, the incident field

3. Solving the system of equations for the unknown dipole density

The assembling of the matrix is discussed in the next section and in Chapter 6. The incident field is discussed later in this chapter for three different probes: air-cored, differential and ACPD. The actual solution to the system of equations is discussed in Chapter 7.

For air-cored probes, once the dipole density distribution is known, the impedance in the coil can be determined. For differential probes, the differential voltage in the pick-up coils can be determined. If an ACPD probe is used, the potential drop can also be determined by from the dipole distribution. The following sections describe the application of the volume integral method to eddy current NDE.

### 5.2 Dyadic Green's functions

The free-space scalar Green's function for a point source is [59]

\[
G(r|r') = \frac{e^{ik|r-r'|}}{4\pi|r-r'|},
\]  

(5.2)
This scalar function describes the propagation of electromagnetic waves through free-space from an electrical point source. We want to use a Green's function approach that allows for a three dimensional source at a point \( r' \) and provides the three dimensional field vector at a point \( r \). It will be convenient to use dyadic Green's functions. The dyadic Green's function \( \mathbf{G}^{(ee)}(r|r') \) is referred to as the electric-electric dyadic Green's function\(^{[5,8]} \) and will be described in the next section. Dyads, like tensors when studying elasticity, are introduced to prevent the explosion of notation, we will use dyads\(^{[59]} \) to express the coupling of the components of the electric fields involved\(^{[46]} \).

### 5.2.1 Dyads

The word *dyad* means a group of two or a pair of quantities\(^{[59]} \). It is formed by two vectors by the following equation

\[
\mathbf{D} = \mathbf{A}\mathbf{B}.
\]

Define two scalar operations as

\[
\begin{align*}
\mathbf{C} \cdot \mathbf{D} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} \\
\mathbf{D} \cdot \mathbf{C} &= \mathbf{A}(\mathbf{B} \cdot \mathbf{C}),
\end{align*}
\]

where the scalar operations return vectors.

We can define the unitary dyads \( \mathbf{I} \) and \( \mathbf{J} \) as

\[
\mathbf{I} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}, \quad \mathbf{J} = \hat{x}\hat{y} + \hat{y}\hat{x} - \hat{z}\hat{z},
\]

and \( \mathbf{I} \) has the property that

\[
\mathbf{A} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{A} = \mathbf{A}
\]

and

\[
\nabla \cdot (\mathbf{I}f) = \nabla f.
\]

### 5.2.2 Free-space dyadic Green's function

Let an \( x \)-directed electric source be described by

\[
\mathbf{J}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}')\hat{z}.
\]

Let \( \mathbf{G}^{(0)}(\mathbf{r}|\mathbf{r}') \) denote the electric field produced by this source. Then\(^{[59]} \)

\[
\mathbf{G}^{(0)}(\mathbf{r}|\mathbf{r}') = \left[ 1 + \frac{1}{k^2} \nabla \nabla \cdot \right] \mathbf{G}(\mathbf{r}|\mathbf{r}')\hat{z}
\]

which is a solution of the equation derived from Maxwell's equations

\[
\left[ \nabla \times \nabla \times -k^2 \right] \mathbf{G}^{(0)}(\mathbf{r}|\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\hat{z}.
\]
5.2. DYADIC GREEN'S FUNCTIONS

$G^{(o)}_{y}$ and $G^{(o)}_{z}$ can be defined in a similar way for $y$ and $z$ directed sources, respectively. We combine all of these vector quantities into a the dyad $\tilde{G}^{(o)}$ by the definition

$$\tilde{G}^{(o)}(r|r') = G^{(o)}_{x}(r|r')\hat{x} + G^{(o)}_{y}(r|r')\hat{y} + G^{(o)}_{z}(r|r')\hat{z}. \quad (5.11)$$

Equation (5.11) is the free-space dyadic Green's function. So the dyadic Green's function element $G^{(o)}_{ij}(r|r')$ is the $i$th component of the electric field, at the observation point $r$, due to a point unit dipole source at $r'$ oriented in the $j$th direction. Using dyadic definition in (5.9), equation (5.11) can be generalized to give[59]

$$\tilde{G}^{(o)}(r|r') = \left[ \mathbf{I} + \frac{1}{k^2} \nabla \nabla \right] G(r|r'). \quad (5.12)$$

5.2.3 Scalar decomposition

In Chapter 2 scalar Green's functions were derived for various stratified planar conductors. From equation (2.38), for the source in region 0, the general form of the scalar Green's function in region 0 was found to be

$$\tilde{G}(r|r') = \frac{1}{2\alpha} \left[ e^{-\alpha_0|z-z'|} + \Gamma^{(\text{media})}_{-1} e^{-\alpha_0(z+z')} + \Gamma^{(\text{media})}_{1} e^{\alpha_0(z+z')} \right], \quad (5.13)$$

where $\Gamma^{(\text{media})}_{-1}$ and $\Gamma^{(\text{media})}_{1}$ are the media specific reflection coefficients. The electric-electric dyadic Green's function will be formed from this Green's function by using scalar decomposition and Hertz potentials[52,20].

In analyzing the electromagnetic field in isotropic planar stratified regions, it can be convenient to represent the field using Hertz potentials. This approach has the advantage of reducing the vector field problem to two independent scalar problems. In this formulation, the dyadic Green's functions used to determine the electric field are reduced to two scalar functions.

The transverse electric and magnetic fields in the source region can be written quite generally as the sum of a transverse gradient and a transverse curl. Thus

$$E_t(r) = \nabla_t \frac{\partial \Pi^{(m)}(r)}{\partial z} + i\omega \mu_0 \nabla_t \times \hat{z} \Pi^{(e)}(r)$$

$$H_t(r) = \sigma_0 \nabla_t \times \Pi^{(m)}(r) + \nabla_t \frac{\partial \Pi^{(e)}(r)}{\partial z}, \quad (5.14)$$

where $\hat{z}$ is the unit normal to the interface and the subscript $t$ refers to components transverse to the $z$-direction, for example $\nabla_t = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}$. The transverse magnetic scalar potential is $\Pi^{(m)}(r)$, while $\Pi^{(e)}(r)$ is the transverse electric scalar potential.

In the source region, the Hertz potentials satisfy[52]

$$\left[ \nabla^2 + k_0^2 \right] \nabla_t \Pi^{(p)}(r) = j^{(p)}(r), \quad (5.15)$$
where \( p \in \{ m, e \} \) and the inhomogeneous terms representing the source are

\[
j^{(m)}(r) = \frac{1}{\sigma_0} \dot{z} \cdot \nabla \times \nabla \times J(r)
\]

\[
j^{(e)}(r) = \dot{z} \cdot \nabla \times J(r).
\]  

(5.16)

The transverse magnetic and transverse electric field do not couple, hence one can solve for them independently. They can be solved by introducing a scalar Green's function such that

\[
G(r'|r) = -\nabla_2^2 U^{(p)}(r|r').
\]

(5.17)

Substituting this into equation (5.15), we see that we are left with the scalar wave equation, which has solutions[59]

\[
\Pi^{(m)}(r) = \int U^{(m)}(r|r') j^{(m)}(r') \, dr'
\]

\[
\Pi^{(e)}(r) = \int U^{(e)}(r|r') j^{(e)}(r') \, dr'.
\]

(5.18)

Suppose, that there are upper and lower boundaries to the source region, region 0 and let \( c \) be the distance between the interfaces, assuming the upper interface is at \( z = 0 \), the lower at \( z = c \) and that the adjacent regions are non-conducting. Then

\[
G^{(m)}(z|z') = \left( \frac{\alpha^2}{2\alpha_0} \right) \left[ e^{-\alpha_0|z-z'|} + \Gamma_1^{(m)} e^{-\alpha_0(z+z')} + \Gamma_1^{(m)} e^{-\alpha_0(z+z') - 2\alpha} \right]
\]

\[
G^{(e)}(z|z') = \left( \frac{\alpha^2}{2\alpha_0} \right) \left[ e^{-\alpha_0|z-z'|} + \Gamma_1^{(e)} e^{-\alpha_0(z+z')} + \Gamma_1^{(e)} e^{-\alpha_0(z+z')} \right]
\]

(5.19)

where \( \alpha^2 = \alpha_0^2 - k^2_0 \) and \( \Gamma_1^{(p)} \) and \( \Gamma_1^{(p)} \) are the electric and magnetic reflection terms off the upper and lower interfaces, respectively. This situation corresponds to the slab media type and the electric reflection coefficients are given in equation (2.90) after re-indexing. The magnetic reflection coefficients are determined in the same way as the electric ones were, by applying the interface conditions. In the absence of one or both of these surfaces the corresponding electric and magnetic reflection term is zero.

The magnetic scalar potential must satisfy

\[
\xi_j G^{(m)}_j = \xi_{j+1} G^{(m)}_{j+1}
\]

\[
\frac{\partial G^{(m)}_j}{\partial z} = \frac{\partial G^{(m)}_{j+1}}{\partial z},
\]

(5.20)

at \( z = z_{j+1} \), for \( j = -1, 0 \). The term \( \xi_j \) is the complex permittivity defined to be \( \xi_j = \epsilon_j - i\sigma_j/\omega \). Hence, the magnetic scalar potential has the same form as the electric scalar potential and has the same boundary conditions, except \( \xi_j \) must be substituted for \( \mu_j \) through out. Since \( \xi_{-1} \approx 0 \) and \( \xi_1 \approx 0 \) equation (2.90), with \( \xi_j \) substituted for \( \mu_j \) implies

\[
\Gamma^{(m)}_{-1} \approx 1 - e^{-2\alpha_0(z'+c)}
\]

\[
\Gamma^{(m)}_1 \approx (1 - e^{2\alpha_0 z'}) e^{-2\alpha_0 c}
\]

(5.21)
In order to separate the $z + z'$ dependence from the $z - z'$ dependence, define $\tilde{U}(z|z')$ as

$$\alpha^2 \tilde{U}(z|z') = \frac{1}{2\alpha_0} \left[ e^{-\alpha_0|z-z'|} + e^{-\alpha_0(z+z')} + e^{\alpha_0(z+z'-2c)} + e^{-\alpha_0(z-z'+2c)} + e^{\alpha_0(z-z'-2c)} \right].$$

Equations (5.21) and (5.19) imply that $\tilde{U}^{(m)}(z|z') \approx \tilde{U}(z|z')$. The electric reflection coefficients depend on $z'$ only in exponential terms of the form $\exp(\pm \alpha_0 z')$, hence the electric scalar potential can be rewritten in a similar form to (5.22) as

$$\alpha^2 \tilde{U}^{(e)}(z|z') = \frac{1}{2\alpha_0} \left[ e^{-\alpha_0|z-z'|} + \Lambda^{(r)}_\alpha e^{-\alpha_0(z+z')} + \Lambda^{(r)}_\beta e^{\alpha_0(z+z'-2c)} + \Lambda^{(r)}_\gamma e^{-\alpha_0(z-z'+2c)} + \Lambda^{(r)}_\delta e^{\alpha_0(z-z'-2c)} \right],$$

upon introduction of four new media specific coefficients $\Lambda^{(r)}_\alpha, \Lambda^{(r)}_\beta, \Lambda^{(r)}_\gamma$ and $\Lambda^{(r)}_\delta$. These coefficients will be derived for each planar media type in Chapter 6.

Now, define a scalar potential correction term $V$ as

$$V(z|z') = \tilde{U}^{(e)}(z|z') - \tilde{U}(z|z'),$$

therefore

$$\alpha^2 V(z|z') = \frac{1}{2\alpha_0} \left[ (\Lambda^{(r)}_\alpha - 1) e^{-\alpha_0(z+z')} + (\Lambda^{(r)}_\beta - 1) e^{\alpha_0(z+z'-2c)} - (\Lambda^{(r)}_\gamma - 1) e^{-\alpha_0(z-z'+2c)} - (\Lambda^{(r)}_\delta - 1) e^{\alpha_0(z-z'-2c)} \right].$$

Equation 5.25 describes the correction scalar potential in Fourier space, the analysis of this term is given in detail in Chapter 6. Again, in the absence of a lower interface, all but one of the terms in (5.25) disappears, since the electric and magnetic reflection terms are zero. For this case $\Lambda^{(r)}_\alpha = \Gamma^{(e)}_1$ and all the other terms vanish.

The dyadic Green's function $\vec{G}^{(ee)}(r|r')$ is identified by substituting (5.18) into (5.14) and repeatedly transferring the curls from with the definition of the source terms $j^{(p)}$ to the Green's functions by integration by parts. This gives[52,8]

$$-i\omega \epsilon_0 \vec{G}^{(ee)}(r|r') = (\nabla \times \nabla \times \hat{z})(\nabla' \times \nabla' \times \hat{z})U^{(e)}(r|r') + k^2(\nabla \times \hat{z})(\nabla' \times \hat{z})U^{(m)}(r|r') - \hat{z} \hat{z} \delta(z - z').$$

Equation (5.24) and the fact that $\tilde{U} = \tilde{U}^{(m)}$ implies

$$-i\omega \epsilon_0 \vec{G}^{(ee)}(r|r') = (\nabla \times \nabla \times \hat{z})(\nabla' \times \nabla' \times \hat{z})U(r|r') + k^2(\nabla \times \hat{z})(\nabla' \times \hat{z})U(r|r') - \hat{z} \hat{z} \delta(z - z') + k^2(\nabla \times \hat{z})(\nabla' \times \hat{z})V(r|r').$$
Now apply the dyadic identity\[20\]
\[
\nabla_i^2 \left[ \mu_0 \frac{\partial}{\partial t} \mathbf{I} - \nabla \nabla \right] = -\frac{1}{\varepsilon_0 \frac{\partial}{\partial t}} \left[ \hat{\varepsilon} \hat{\varepsilon} \nabla_i^2 + \nabla_i \nabla_i \right] \left[ \nabla^2 - \frac{1}{k_2 \frac{\partial}{\partial t}} \right] + \mu_0 \frac{\partial}{\partial t} \left( \nabla \times \hat{\varepsilon} \right) \left( \nabla \times \hat{\varepsilon} \right) + \\
\frac{1}{\varepsilon_0 \frac{\partial}{\partial t}} \left[ \nabla \times \nabla \times \hat{\varepsilon} \right] \left[ \nabla \times \nabla \times \hat{\varepsilon} \right]
\]
(5.28)

and the fact that \( \nabla = -\nabla' \), when applied to scalar potentials with \( r - r' \) dependence, to equation (5.27) to find\[111\]

\[
\left( \Phi(r)(z) (r)(z') = \frac{e^{1/2}}{\varepsilon_0 \frac{\partial}{\partial t}} G(rlr') + \left[ \frac{e^{1/2}}{\varepsilon_0 \frac{\partial}{\partial t}} \right] G(rlr' - 2z'z) + \\
\frac{e^{1/2}}{\varepsilon_0 \frac{\partial}{\partial t}} G(rlr' + 2z'z) + \frac{e^{1/2}}{\varepsilon_0 \frac{\partial}{\partial t}} \left[ \left( \nabla \times \hat{\varepsilon} \right) \left( \nabla \times \hat{\varepsilon} \right) \right] V(rlr')
\]
(5.29)

Grouping terms in equation (5.29) with the same \( z \pm z' \) dependence, we can write

\[
\tilde{G}^{(ee)}(r|r') = \tilde{G}^{(r)}(r|r') + \tilde{G}^{(T)}(r|r')
\]
(5.30)

where \( \tilde{G}^{(r)}(r|r') \) has \( z - z' \) dependence and \( \tilde{G}^{(T)}(r|r') \) has \( z + z' \) dependence. These dyads are defined

\[
\tilde{G}^{(T)}(r|r') = \tilde{G}^{(0)}(r|r') + \tilde{G}^{(0)}(r|r + 2c\hat{z}) + \\
\tilde{G}^{(0)}(r|r' - 2c\hat{z}) + \tilde{V}^{(r)}(r|r' + 2c\hat{z}) + \tilde{V}^{(T)}(r|r' - 2c\hat{z})
\]

\[
\tilde{G}^{(r)}(r|r') = \left[ \tilde{I} - \frac{1}{k_2^2} \nabla \nabla \right] G(rlr' - 2z'\hat{z}) + \left[ \tilde{I} - \frac{1}{k_2^2} \nabla \nabla \right] G(rlr' + 2z'\hat{z}) + \\
\tilde{V}^{(r)}(r|r' - 2z'\hat{z}) + \tilde{V}^{(T)}(r|r' - 2z'\hat{z})
\]
(5.31)

where

\[
\tilde{G}^{(0)}(r|r') = \left[ \tilde{I} + \frac{1}{k_2^2} \nabla \nabla \right] G(rlr')
\]

\[
\tilde{V}^{(\Xi)}_{\eta}(r|r') = \frac{1}{k_2^2} \left( \nabla \times \hat{\varepsilon} \right) \left( \nabla \times \hat{\varepsilon} \right) \tilde{V}^{(\Xi)}_{\eta}(r|r').
\]
(5.32)

The correction dyadic terms are defined by

\[
\tilde{V}^{(\Xi)}_{\eta}(z|z') = \frac{1}{2\alpha_0} \left( \Lambda^{(\Xi)}_{\eta} - 1 \right) e^{-\alpha_0(z+z')},
\]
(5.33)

where \( \Xi \in \{ T, \Gamma \} \) and \( \eta \in \{ r, \beta \} \), hence

\[
\tilde{V}(z|z') = \tilde{V}^{(T)}(z|z') + \tilde{V}^{(r)}(z|z') + \tilde{V}^{(\Gamma)}(z|z') + \tilde{V}^{(\beta)}(z|z').
\]
(5.34)
5.3. INTEGRAL FORMULATION

The dyadic terms in equation (5.30) \( \mathcal{G}^{(0)}(r|r') \) is the convolutional dyadic Green's function and \( \mathcal{G}^{(c)}(r|r') \) is the correlational dyadic Green's function. In the absence of a lower reflecting surface, the convolutional dyadic Green's function reduces to just the free-space dyad Green's function discussed in Section 5.2.2, while just the terms with \( G(r|r'-2z'\hat{z}) \) in the correlational dyad.

5.3 Integral Formulation

In general, Maxwell's equations for any isotropic conductor containing some anomaly, represented by a dipole distribution \( P(r) = [\sigma(r) - \sigma_0]E(r) \), may be written as

\[
\nabla \times E = i\omega \mu_0 H(r),
\nabla \times H(r) = \sigma_0 E(r) + P(r).
\]

The volume integral formulation is developed from the solution of Maxwell's equations expressed in terms of the electric-electric dyadic Green's function as

\[
E(r) = E^{(i)}(r) + i\omega \mu_0 \int_{\text{flaw}} \mathcal{G}^{(ee)}(r|r') \cdot P(r')dr',
\]

where \( E^{(i)}(r) \) is the incident field and the integral represents the electric field scattered by the defect or flaw.

Normalize the equation by introducing a flaw function[53]

\[
u(r) = \frac{\sigma(r) - \sigma_0}{\sigma_0}
\]

and multiplying equation (5.36) by \( \sigma_0 v(r) \) and rewriting to yield an integral equation for \( P(r) \),

\[
P^{(i)}(r) = P(r) - k^2 v(r) \int_{\text{flaw}} \mathcal{G}^{(ee)}(r|r') \cdot P(r')dr',
\]

where

\[
P^{(i)}(r) = \sigma_0 v(r) E^{(i)}(r).
\]

By solving equation (5.38) the effective source distribution of the flaw for a given excitation \( P(r) \) is found.

The appropriate dyad may be chosen for a given problem to satisfy specific boundary conditions: a half-space dyadic Green's function for a source in a half-space, or a dyadic Green's function satisfying continuity conditions on a cylindrical boundary for calculating the induced source in a cylindrical structure. Equation (5.38) is a linear for the unknown vector \( P(r) \) and the moment method can be applied to provide approximate solutions. Before proceeding, we present a brief review of the method of moments.
5.3.1 Definition of moment method

Given an operator expression

\[ L(f) = g, \quad (5.40) \]

how does one go about finding \( f \) for a known operator \( L \) and forcing function \( g \)? If the operator is linear, and this will be the case of interest here, the method of moments can be used to solve this general non-homogeneous equation. The method is briefly outlined here, following the development in the classic work by Harrington[24].

Let \( f \) in equation (5.40) be approximated by expanding it in a finite series of functions \( f_1, f_2, f_3, \ldots, f_n \) in the domain of the linear operator \( L \), as

\[ f \approx \sum_{i=1}^{n} \alpha_i f_i, \quad (5.41) \]

where \( \alpha_i \) are scalar constants. The \( f_i \) are called expansion functions or basis functions. Because \( L \) is linear, \( L \) can operate on equation (5.41) to yield

\[ \sum_{i=0}^{n} \alpha_i L(f_i) \approx g. \quad (5.42) \]

Now define a set of weighting functions, or testing functions, \( w_1, w_2, w_3, \ldots w_n \) in the range of \( L \), and test equation (5.42) with each \( w_j \) as follows

\[ \int \sum_{i=1}^{n} \alpha_i L(f_i) w_j = \sum_{i=1}^{n} \alpha_i \langle L(f_i), w_j \rangle = \langle g, w_j \rangle = \int g w_j, \quad (5.43) \]

for \( j = 1, 2, 3, \ldots, n \). This set of equations can be written in matrix form as

\[ [l_{nn}] [\alpha_n] \approx [g_n], \quad (5.44) \]

where

\[ [l_{nn}] = \begin{bmatrix} \langle w_1, L(f_1) \rangle & \langle w_1, L(f_2) \rangle & \cdots & \langle w_1, L(f_n) \rangle \\ \langle w_2, L(f_1) \rangle & \langle w_2, L(f_2) \rangle & \cdots & \langle w_2, L(f_n) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle w_n, L(f_1) \rangle & \langle w_n, L(f_2) \rangle & \cdots & \langle w_n, L(f_n) \rangle \end{bmatrix} \]

\[ [\alpha_n] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad [g_n] = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \vdots \\ \langle w_n, g \rangle \end{bmatrix}. \quad (5.45) \]

Equation (5.44) is a linear system of equations that can now be solved using any standard technique. Once the unknowns \( \alpha_i \) have been solved for, the \( \alpha_i \) can be substituted into equation (5.41) to yield an approximation to the solution \( f \).
5.3.2 Moment method and electromagnetic problems

The integral expression in equation (5.38) is discretized by applying the moment method. The flaw volume can be subdivided into a regular grid of \( n_x \times n_y \times n_z \) cells, each of size \( \delta_x \times \delta_y \times \delta_z \) and then the flaw current dipole density and the flaw function can be expanded using pulse functions defined over the grid as basis functions[12]. Thus

\[
P_{KLM} = \int P(r)p_K \left( \frac{x}{\delta_x} \right) p_L \left( \frac{y}{\delta_y} \right) p_M \left( \frac{z}{\delta_z} \right) \, dr,
\]

and

\[
u_{KLM} = \int v(r)p_K \left( \frac{x}{\delta_x} \right) p_L \left( \frac{y}{\delta_y} \right) p_M \left( \frac{z}{\delta_z} \right) \, dr,
\]

where the pulse basis functions \( p_j(s) \) are defined by

\[
p_j(s) = \begin{cases} 
1 & \text{if } j \leq s < j + 1 \\
0 & \text{otherwise}
\end{cases}
\]

The dipole density distribution and flaw function can then approximated by the piece-wise linear functions

\[
P(r) \approx \sum_{K=0}^{n_x-1} \sum_{L=0}^{n_y-1} \sum_{M=0}^{n_z-1} P_{KLM} p_K p_L p_M(z_K, y_L, z_M),
\]

and

\[
v(r) \approx \sum_{K=0}^{n_x-1} \sum_{L=0}^{n_y-1} \sum_{M=0}^{n_z-1} v_{KLM} p_K p_L p_M(z_K, y_L, z_M),
\]

where \( p_{rel}(r) = p_x(x/\delta_x)p_y(y/\delta_y)p_z(z/\delta_z) \).

To complete the discretization testing is carried out by multiplying the integral equation by the testing functions and integrating over the field coordinates. The same basis functions adopted for expanding the unknown could be used for testing, this is known as Galerkin’s method. \( p_{KLM}(r) \) as defined here, is the first order member of a class of 3D spline functions commonly used for defining a discrete approximation. The zero order member is a 3D delta function and higher orders are generated by successive convolutions with pulse functions. It is not necessary to use the same testing and expansion functions. Recent work[43] has shown there is no clear advantage to Galerkin’s method. Therefore, advantages using different test and expansion functions whose convolution give the same order scheme as Galerkin’s using pulse functions were explored.

The second order basis functions, \( \beta_{KLM} \) could be used to expand, where

\[
\beta_{KLM}(r) = \beta_K \left( \frac{x}{\delta_x} \right) \beta_L \left( \frac{y}{\delta_y} \right) \beta_M \left( \frac{z}{\delta_z} \right)
\]

and \( \beta_j(u), (j = 0, 1, 2, 3...) \) is the convolution of 1D pulse functions given by

\[
\beta_j(u) = \begin{cases} 
1 - |u - j| & \text{if } j - 1 \leq u < j + 1 \\
0 & \text{otherwise}
\end{cases}
\]
we have
\[ P_{klm} \star P_{KLM} = \delta_{klm} \beta_{KLM} = \beta_{KLM}, \]  
(5.54)
where \( \delta_{klm} = \delta(k - x/\delta_x)\delta(l - y/\delta_y)\delta(m - z/\delta_z) \). Testing with 3D delta functions is equivalent to computing the value of the integrand at the origin of the 3D delta function, which is why this approach is often referred to as point matching. The incident field is radially symmetric, that is \( P^{(i)}(x, y, z_m) = P^{(i)}(\rho, z_m) \). A point matching scheme can exploit this fact by evaluating \( P^{(i)}(\rho, z) \) at a finite number of points and then approximate the value at \( \rho_{kl} = \sqrt{x_k^2 + y_l^2} \) by interpolation and the resolving the radial component into its \( x \)- and \( y \)- components. For this reason, point matching was used, therefore, to \( P^{(i)}(r) \) we have
\[ P^{(i)}_{KLM} = P^{(i)}(x_K, y_L, z_M). \]  
(5.55)

The discretization of the integral equation in (5.38) is completed by taking moments of the field by multiplying the integral, the linear operator, by \( \delta_{klm} \) and then integrating over each cell, yielding
\[
P^{(i)}_{KLM} = P_{KLM} + v_{KLM} \sum_{k=0}^{n_x-1} \sum_{l=0}^{n_y-1} \sum_{m=0}^{n_z-1} \left[ \int_{V_{KLM}} G^{(ee)}(r_{klm}|r') \delta_{klm} \beta_{KLM}(r') dr' \right] P_{klm}
\]
\[ = P_{KLM} + v_{KLM} \sum_{k=0}^{n_x-1} \sum_{l=0}^{n_y-1} \sum_{m=0}^{n_z-1} \left[ \int_{V_{KLM}} G^{(ee)}(r_{klm}|r') \beta_{KLM}(r') dr' \right] P_{klm}. \]  
(5.56)

The elements of the matrix needed for calculating the effective dipole density at a flaw via a volume element scheme can be defined by
\[
\bar{G}_{klm,KLM}^{(ee)} = k^2 \int_{V_{KLM}} G^{(ee)}(r_{klm}|r') \beta_{KLM}(r') dr'
\]  
(5.57)
In order to avoid assembling the six-dimensional matrix of equation (5.57) the dyad inside the integral is divided into two parts: the convolutional dyad \( \bar{G}^{(T)}(r|r') \) with \( z - z' \) dependence and the correlational dyad \( \bar{G}^{(T)}(r|r') \) with \( z + z' \) dependence. Both dyads have \( x - x' \) and \( y - y' \) dependence. The matrix elements depend on only the difference or the sum of its indices. Therefore, using this partitioning, equation (5.57) can be rewritten
\[
\bar{G}_{klm,KLM}^{(ee)} = \bar{G}_{k-K,l-L,m-M}^{(T)} + \bar{G}_{k-K,l-L,m+M}^{(T)} \]  
(5.58)
where
\[
\bar{G}_{k-K,l-L,m-M}^{(T)} = k^2 \int_{V_{KLM}} G^{(T)}(r_{klm}|r') \beta_{KLM}(r') dr'
\]
\[
\bar{G}_{k-K,l-L,m+M}^{(T)} = k^2 \int_{V_{KLM}} G^{(T)}(r_{klm}|r') \beta_{KLM}(r') dr'
\]  
(5.59)
Now using equation (5.58) any dyad from the matrix \( \bar{G}_{klm,KLM}^{(ee)} \) can be determined from the three index dyads of equation (5.59).
Numerical evaluation of the matrix requires special consideration since the singularity is of high order. Although the nature and treatment of the high order singularity has been discussed extensively in the literature, an explicit prescription for the numerical evaluation of the matrix elements for a system analyzed in Cartesian coordinates is not generally available. In Chapter 6, a systematic way of approximating both the singular and the non-singular integrals over rectangular volumetric cells is given. This development is one of the main contributions of this work.

5.4 Air-cored probes

It is necessary to determine the incident field $J^{(i)}(r)$ in each of the flaw elements assuming that the host region has no anomaly. For the electric source we will assume that an axially symmetric air-cored coil is used to induce the field. Using point matching implies

$$P_{klm}^{(i)} = \sigma_j E^{(i)}(x_k, y_l, z_m)$$

$$= i\omega \sigma_j A^{(i)}(x_k, y_l, z_m)$$

Therefore, we need only determine the field in the centre of the element.

5.4.1 Incident field for planar media

Since the probe is assumed to be an axially symmetric coil, we can use the results from Section 2.5 to write down the vector potential. In equation (2.72), it was found that

$$A_j (\rho, z) = \frac{\mu_0 \alpha_0}{2} \int_0^\infty \frac{\Psi(\alpha, r_0, r_1)}{\alpha} J_1(\alpha \rho) \left[ e^{-\alpha_0} - e^{-\alpha_1} \right] \left[ \Gamma_j^{(\text{media})} e^{\alpha_j z} + \Gamma_j^{(\text{media})} e^{-\alpha_j z} \right] d\alpha,$$  \hspace{1cm} (5.61)

where $\Gamma_j^{(\text{media})}$ and $\Gamma_j^{(\text{media})}$ are the media specific reflection and transmission coefficients for region $j$, respectively.

We can then write down the general expression for the incident field in region $j > 0$ for planar media types as

$$P_{KLM}^{(i)} = \frac{i\omega \mu_0 \alpha_0}{2} V_{KLM} \int_0^\infty \frac{\Psi(\alpha, r_0, r_1)}{\alpha} J_1(\alpha \rho_{KL}) \left[ e^{-\alpha_0} - e^{-\alpha_1} \right] \left[ \Gamma_j^{(\text{media})} e^{\alpha_j z_M} + \Gamma_j^{(\text{media})} e^{-\alpha_j z_M} \right] d\alpha,$$  \hspace{1cm} (5.62)

where $\rho_{KL} = \sqrt{x_K^2 + y_L^2}$, $x_K = K \delta_x$, $y_L = L \delta_L$ and $z_M = M \delta_z$.

For the half-space conductors, the host region is region 1 and there is no lower region, hence $\Gamma_2 = 0$. The transmission coefficient, determined in equation (2.69), is

$$\Gamma_1^{(\text{half})} = \frac{2\alpha_0}{\alpha_1 + \alpha_0},$$  \hspace{1cm} (5.63)
where it is assumed here that \( \mu_1 \) from (2.69) is the permeability of free-space. The expression for the incident dipole density, the forcing function, for half-space media types is, since \( \alpha \approx \alpha_0 \),

\[
P_{KLM}^{(i)} = \frac{i \omega \sigma_1 \mu_0 \omega}{2} v_{KLM} \int_{0}^{\infty} \frac{\Psi(\alpha, r_0, r_1)}{(\alpha_1 + \alpha_0)} J_1(\alpha \rho_{KL}) \left[ e^{-\alpha_0 b} - e^{-\alpha_1 h} \right] e^{\alpha_1 z} \, d\alpha
\]

The reflection and transmission coefficients for the remaining planar media types were determined in Sections 2.7.1 - 2.7.3.

### 5.4.2 Incident fields for cylindrical media

We now want to determine the field in the centre of the flaw elements for cylindrical media physical models again assuming an axially symmetric air-cored probe. Therefore, the results of Section 3.3 can be used to compute the vector potential, hence the incident dipole density in each element in the absence of a flaw. All of the cylindrical media types to be modelled assume the conducting region is on the outside of the coil; therefore, \( m = 0 \), \( n > 0 \) and \( V = I \) for all these media types. For any region \( j > 0 \) outside the coil, equation (3.49) can be used to give the vector potential in this region as

\[
\mathbf{A}_j(\rho, z) = \frac{\mu_0 \omega}{\pi} \int_{0}^{\infty} \frac{\Omega(\alpha, r_0, r_1)}{\alpha} \left[ \frac{U_{12}^{(j)} I_1(\alpha \rho) + U_{22}^{(j)} K_1(\alpha \rho)}{U_{22}^{(0)}} \right] \sin(\alpha \{ z - l_0 \}) - \sin(\alpha \{ z - l_1 \}) \, d\alpha,
\]

where \( j \) is the host region and the coefficient matrix \( U^{(j)} \) corresponds to the particular media type, the type of layered structure of the conductor.

Using (5.60) and (5.65), we can write down the general form for the incident dipole density at each of the flaw elements in region \( j \) as

\[
P_{KLM}^{(i)} = \frac{i \omega \sigma_1 \mu_0 \omega}{\pi} v_{KLM} \int_{0}^{\infty} \frac{\Omega(\alpha, r_0, r_1)}{\alpha} \left[ \frac{U_{12}^{(j)} I_1(\alpha \rho_M) + U_{22}^{(j)} K_1(\alpha \rho_M)}{U_{22}^{(0)}} \right] \sin(\alpha \{ z_L - l_0 \}) - \sin(\alpha \{ z_L - l_1 \}) \, d\alpha,
\]

where \( z_L = L \delta_s \) and \( \rho_M = \delta_s \). The coefficient matrix \( U^{(j)} \) must now be determined for each cylindrical media type.

To illustrate this field computation, we will look at the bore-hole media type. The host region \( j \) is the infinite conductor, region 1. Assume the host region has a permeability of free-space, so \( \mu_1 = \mu_0 \). From equation (3.42)

\[
U_{22}^{(0)} = \rho_0 [\alpha_0 I_0(\alpha_0 \rho_0) K_1(\alpha_1 \rho_0) + \alpha_1 I_1(\alpha_0 \rho_0) K_0(\alpha_1 \rho_0)]
\]

and from the fact that \( U^{(n)} = I \), \( U_{12}^{(i)} = 0 \), \( U_{22}^{(i)} = 1 \). Therefore,

\[
P_{KLM}^{(i)} = \frac{i \omega \sigma_1 \mu_0 \omega}{\pi} v_{KLM} \int_{0}^{\infty} \frac{\Omega(\alpha, r_0, r_1)}{\alpha U_{22}^{(0)}} K_1(\alpha_1 \rho_M) [\sin(\alpha \{ z_L - l_0 \}) - \sin(\alpha \{ z_L - l_1 \})] \, d\alpha.
\]
Equation (5.66) can be used in a similar manner to compute the incident field in the
host region for the remaining cylindrical media types. The coefficient matrices $U^{(0)}$ and
$U^{(host)}$ were determined in Sections 3.5.1 - 3.5.3.

5.4.3 Flaw impedance

The change in impedance in the driving coil is the measured quantity that was to be
modelled. Once the linear system of equations has been approximated, the dipole density
distribution can be used to estimate this quantity. Adopting the probe current as the
phase reference, the probe impedance $\Delta Z$ due to the flaw, expressed in terms of the
electric field $E^{(s)}(r)$ scattered by the flaw, is given by

$$\Delta Z = -\frac{1}{I^2} \int_{\text{coil}} E^{(s)}(r) \cdot J_{\text{coil}}(r) \, dr.$$  (5.69)

Equation (5.69) can be used directly to compute the probe response, but this would entail
the intermediate step of calculating the scattered field at the coil, before integrating over
the coil region to get the impedance. Instead we appeal to the reciprocity theorem
relating the scattered field at the primary source, the coil, to the incident field at the
secondary source, the current dipole density at the flaw. Therefore,

$$\Delta Z = -\frac{1}{I^2} \int_{\text{flaw}} E^{(i)}(r) \cdot P(r) \, dr,$$  (5.70)

which has as a discrete analog form of

$$\Delta Z \approx -\delta_x \delta_y \delta_z \sum_{k=0}^{n_x-1} \sum_{l=0}^{n_y-1} \sum_{m=0}^{n_z-1} \frac{P^{(i)}_{klm} \cdot P_{klm}}{\sigma_j \nu_{klm}},$$  (5.71)

where the flaw is assumed to be in region $j$. So the change in probe impedance can be
approximated by a simple multi-dimensional scalar product.

5.5 Differential probes

Figure 5.3 schematically represents differential probe which is often used in NDE. Al-
ternating current is introduced into the large coil, which drives eddy currents in the
workpiece. This induces magnetic flux passing through the two smaller pick-up coils
which generates a potential in each coil. The output of the probe is the difference in
these two potentials. This probe will give no signal in the absence of a flaw, since the
magnetic flux passing through each pick-up coil will be the same, since the pick-up coils
are assumed to be symmetrically located, relative to the centre of the larger driving coil.

The two pick-up coils can be used to measure the difference in emf as the probe is
scanned over a workpiece. This voltage difference is the probe signal we wish to model.
The incident field created by the outside coil is exactly the same as those considered in Section 5.4. We need only analyze the signal from differential coils.

The flaw lies in the incident field produced by an excitation coil and that, as a result, there is an induced current dipole density \( \mathbf{P} \) at the flaw. This dipole density distribution may be regarded as the source of the scattered electric field, \( \mathbf{E}^{(s)}(r) \). If \( \mathbf{J}_i \) is the current density of pick-up coil \( i \), where \( i = 1, 2 \), then the induced emf \( V_i \) in coil \( i \) is given by

\[
I_i V_i = -\int_{\text{coil}_i} \mathbf{E}^{(s)}(r) \cdot \mathbf{J}_i(r) \, dr \quad i = 1, 2
\]

(5.72)

This may also be expressed in a more compact notation as

\[
I_i V_i = -\langle \mathbf{E}^{(s)} | \mathbf{J}_i \rangle \quad i = 1, 2
\]

(5.73)

Using the reciprocity principle we also have

\[
I_i V_i = -\langle \mathbf{E}_i | \mathbf{P} \rangle \quad i = 1, 2
\]

(5.74)

where \( \mathbf{E}_i \) is the field due to the source \( \mathbf{J}_i \). Hence the differential probe response is given by

\[
\Delta V = V_1 - V_2 = -\langle \mathbf{e}_1 - \mathbf{e}_2 | \mathbf{P} \rangle = -\int_{\text{flaw}} \Delta \mathbf{e}(r) \cdot \mathbf{P}(r) \, dr, \quad i = 1, 2
\]

(5.75)

where \( \mathbf{e}_i = \mathbf{E}_i/I_i \) and \( \Delta \mathbf{e} = \mathbf{e}_1 - \mathbf{e}_2 \).
5.6 ACPD PROBES

If these fields $e_i$ are expanded with delta functions at the centres of the flaw elements, as was done for the incident field, each field can be expressed using equation (5.64) after dividing through by $I\sigma_1$ and using the pick-up coil dimensions to find for each pick-up coil

$$e_{KLM} = \frac{\hat{\omega} \mu_0}{2I} v_{KLM} \int_0^\infty \frac{\alpha_0 \Psi(\alpha, r_{d0}, r_{dl})}{\alpha(\alpha_1 + \alpha_0)} J_1(\alpha r_{KL}) \left[ e^{-\alpha d_0} - e^{-\alpha d_1} \right] e^{\alpha z_M} d\alpha, \quad (5.76)$$

where $x_K = \pm x_d + K\delta_1, y_L = \pm y_d + L\delta_2$ and $z_M = M\delta_3$. Now using the dipole density distribution solution vector $P_{klm}$ an approximation to the probe response can be written as

$$\Delta V \approx \sum_{k=0}^{n_x-1} \sum_{l=0}^{n_y-1} \sum_{m=0}^{n_z-1} \Delta e_{klm} \cdot P_{klm}. \quad (5.77)$$

5.6 ACPD probes

Alternating current potential drop (ACPD) is another electromagnetic technique used for sizing of flaws using eddy currents. The technique is based on detecting the change in potential (voltage) between two probes a fixed distance apart, when a time harmonic current of frequency $\omega$ is injected at two point sources sufficiently far from the defect.

![Figure 5.4: ACPD Probe](image)

Figure 5.4: ACPD Probe

The physical model for the ACPD probe is shown schematically in Figure 5.4. We will use this probe only with the half-space media type. The incident field is created from a time harmonic current being injected away from the flaw. In the absence of the flaw, the incident field approaches a uniform field in the flaw region.
Therefore, we can assume we have a uniform magnetic field, \( \hat{y}H_0 \) in air above the half-space conductor, with an electric field

\[
E^{(i)}(z) = \hat{x}E_0 e^{-ik_1z},
\]

(5.78)
in the conductor, where \( k_1^2 = i\omega\mu_0\sigma_1 \). Applying the induction law allows us to relate the electric field to the magnetic field and noting that the magnetic field is continuous at \( z = 0 \), we have

\[
E_0 = \frac{\omega\mu_0 H_0}{k_1}.
\]

(5.79)

Multiplying through by \( \sigma_1 \nu(r) \), gives the incident electric field at the flaw. Expand, using delta functions, implies that

\[
P_{klm}^{(i)} = ik_1 \nu_{klm} H_0 e^{-ik_1z_k}.
\]

(5.80)

### 5.6.1 Basic theory: alternating current potential drop

As indicated in the Figure 7.18, the potential drop will be measured at point \( x_j \) as the difference in potential at the points \( (x_j - x_d, 0) \) and \( (x_j + x_d, 0) \). We need only determine the surface electric field created from the dipole density distribution in the flaw region. We can assume symmetry about the slot, so the field need be computed on only one side of the flaw. The field will be sampled as

\[
E_j = \hat{x} \cdot E^{(s)}(x_j, 0, 0),
\]

(5.81)

where \( E^{(s)} \) is the scattered field at the surface and

\[
x_j = \frac{c}{2} + jd_x, \quad j = 1, 2, \ldots n
\]

(5.82)

and \( c = n_x \delta_x \), where \( n_x \) is the number of flaw cells across the slot and \( \delta_x \) is their dimension in the \( x \) direction.

For any \( j > 0 \), the ACPD signal is

\[
V(x_j) = V(x_j + x_d) - V(x_j - x_d),
\]

(5.83)

The ACPD signal near a defect can be broken into at most three parts: the contribution across the crack opening and the line integral of the \( x \)-component of the field on each side of the crack. Therefore, the signal \( V(x_j) \) can be approximated as

\[
V(x_j) \approx \frac{1}{\sigma_{host}} \sum_{k=0}^{n_x-1} \hat{x} \cdot P_{k00} \delta_x - \sum_l E_l d_x - \sum_m E_m d_x + V_0^{(i)},
\]

(5.84)

where \( V_0^{(i)} \) is the unperturbed uniform field due to the injected current \( V_0^{(i)} \equiv -2x_d\hat{x} \cdot E_0^{(i)} \).

(5.84) can be rewritten as

\[
V(x_j) \approx \frac{c}{\sigma_{host}} \sum_{k=0}^{n_x-1} \hat{x} P_{k00} - \sum_l \left( E_0^{(i)} - E_l \right) d_x - \sum_m \left( E_0^{(i)} - E_m \right) d_x.
\]

(5.85)
The summations over \( l \) and \( m \) in (5.85) represent numerical approximations to the line integral

\[
V(x_j) = -\int_{x_j-s_d}^{x_j+s_d} E_z(x,0,0)dx.
\]  

(5.86)

Figure 5.5: \( x \) component of the dipole distribution on a slot

Figure 5.5 shows the \( x \)-component of the dipole distribution for a unit input onto a .25mm \( \times \) 20mm \( \times \) 1mm slot machined into austenitic steel[67]. In the next section we will discuss how to use the dipole distribution \( \mathbf{P} \) to compute the \( x \)-component of the surface electric field transverse to the centre of the slot.

Figure 5.6: Surface electric field across slot centre line

Figure 5.6 shows the field computed from the dipole distribution shown in Figure 5.5. At the slot edge, \( c/2 \), the total field is zero, that is \( E^{(0)}(c/2) - E_0 = 0 \). The profile is
symmetric in $x$. The field is constant across the central region of the slot and in this case is approximately 4000 V/m.

### 5.6.2 Scattered electric surface field

In Section 5.2, we described the Green's functions needed to describe the electric field in the host region for an electric source in the same region. Therefore, these Green's functions can be employed here to describe the electric field at the surface of the host region, $E(x_j,0,0) = E_j$. We can write

$$E_j = E_0^{(i)} - iw \mu_0 \int_{\text{flaw}} \hat{z} \cdot \hat{G}(x_j,0,0|x',y',z') \cdot P(x',y',z') \, dr'.$$

(5.87)

After the discretization of the flaw we end up with the simple summation

$$E_j \approx E_0^{(i)} - \sum_{k=0}^{n_x-1} \sum_{l=0}^{n_y-1} \sum_{m=0}^{n_z-1} \tilde{G}_j(x_j,0,0|x'_k,y'_l,z'_m) \cdot P_{klm}$$

(5.88)

Since the field is known at $x = c/2$ and the field varies smoothly to an asymptotic limit, the field is sampled at $n$ points and then a cubic spline fit is made so that the field can then be interpolated at any value $x_j$. The field at $x = 8x_d$ is assumed to have reach the asymptotic limit $E_0^{(i)}$. An adaptive integration scheme is then used to approximate equation (5.86) at any point $x_j$. 
Chapter 6

3D Matrix Elements Analysis

*It seems... to be one of those simple cases which are so extremely difficult.*

*That sounds a little paradoxical. But it is profoundly true. Singularity is almost invariably a clue.*

Sir Arthur Conan Doyle 1859 – 1930

After applying the method of moments, the general dyadic expression can be written down by combining equations (5.56) – (5.58) to yield

\[
P^{(i)}_{KLM} = P_{KLM} + v_{KLM} \sum_{k=0}^{n_x-1} \sum_{l=0}^{n_y-1} \sum_{m=0}^{n_z-1} \left[ G^{(T)}_{k-K,l-L,m-M} + G^{(F)}_{k-K,l-L,m+M} \right] P_{kim}, \quad (6.1)
\]

where the dyadic terms \( G^{(T)} \) are the convolutional matrix elements, while \( G^{(F)} \) are the correlational matrix elements. The integral expressions for the matrix elements cannot be evaluated analytically, but instead must be approximated numerically.

Bowler, Sabbagh and Sabbagh[12] evaluated these dyadic expressions in Fourier space, using the limiting case of the \( xy \)-plane as the exclusion volume. The dyadic terms were then evaluated at a series of discrete points in Fourier space and then an inverse discrete fast Fourier transform (IDFFT) was used to approximate the result in physical space. This approach required a large amount of memory for storing the intermediate results and since it involved three-dimensional IDFFT’s was very slow. When the element aspect ratio, the ratio of the largest side to the smallest side, increased, even more points needed to be evaluated in order to have a high enough spatial resolution and unfortunately the precision decreased.

McKirdy[37] employed physical space integration in implementing his volume integral scheme. His approach involved transforming the volume integrals involving the singularity in (6.1) into equivalent surface integrals in such a way as to avoid the singularity.
These integrals were then approximated numerically. This approach does not require a lot of memory, like the Fourier approach, and provides a high degree of precision; however, the surface integrals require large amounts of CPU time to compute.

In order to overcome these deficiencies of the Fourier techniques and the inefficiency of the numerical approximation using surface integrals the singularities in equation (6.1) were taken head on. In this new approach, the dyadic expressions are approximated in physical space by a series of approximations that isolate the aspects of the singular integrals that depend on the specific geometry being considered from the intrinsic nature of the singular integrals. The resulting technique involves a series of polynomial evaluations to approximate these integrals, with coefficients that have been predetermined for all element sizes. Polynomial evaluation is especially fast; consequently, the CPU requirements have been dramatically reduced, but without sacrificing precision. This physical space integration technique is described in the next section. The treatment of the correlational and convolutional dyadic matrix elements is presented in the following section.

The treatment of the correction dyad, introduced in Chapter 5, is given in detail. The presentation here evaluates some of the correction dyad in Fourier space before a final two-dimensional integration in physical space. This approach is different from that used by Bowler[11] for half-space conductors, where the correction term was treated entirely in physical space. There is no computational gains to be made here; however, this hybrid approach to the correction dyad does extend to other stratified media, unlike the physical space technique.

6.1 Free-space matrix elements for planar media

Only the free-space dyadic Green's function will be treated in this section, because the treatment of the matrix elements that arise from reflections from the interfaces in the surrounding media will be able to be treated as a free-space term with a correction term.

For all planar media types we will use the free-space matrix elements dyad $\hat{G}_{klm|KLM}^{(0)}$, which can be written as

$$
\hat{G}_{klm|KLM}^{(0)} = k^2 \int_{V_{KLM}} \hat{G}^{(0)}(r_{klm}|r') \beta_{KLM}(r') \, dr',
$$

(6.2)

where the basis functions are convolutions of 3D pulse functions[12] given in equation (5.52) and

$$
\hat{G}^{(0)}(r|r') = \left[ I + \frac{1}{k_{host}^2} \nabla \nabla \right] G(r|r'),
$$

(6.3)

where $G(r|r')$ is the dynamic scalar Green's function. Some of these matrix element terms, like $\hat{G}_{klm|klm}^{(0)}$, involve a $1/|r|^3$ singularity. These terms have been treated in a special way, which allows for efficient and precise approximation. This is the subject of the next section.
6.1. FREE-SPACE MATRIX ELEMENTS FOR PLANAR MEDIA

6.1.1 Evaluation of singular matrix elements

Figure 6.1: First octant of cell centred at the singularity. In the example shown $\delta_x$ is the smallest cell dimension.

The choice of basis functions mean that the integration volumes are in the form of rectangular parallelepipeds. For the regularization that follows, a finite exclusion volume around the singularity is introduced[66,35,69,62]. For a given volume $V_{klm}$, there are clear numerical advantages in making the exclusion volume $V_0$ as large as possible and ensuring that the non-excluded region $V_1 = V_{klm} - V_0$ has a simple geometry. For this reason, a cubic exclusion volume centred on the singularity has been used. The sides of the cube have the dimension $2a$, where $a = \min\{\delta_x, \delta_y, \delta_z\}$.

As pointed out by Lee[35], the integral in (6.2) does not exist in the usual sense for volumes containing the origin. However, by appealing to generalized function theory the limit can be well-defined if $\beta_{klm}(r)$ is Hölder continuous[34]. A function $f$ is said to be Hölder continuous if there exists three constants $c, B$ and $\alpha$ such that

$$|f(r) - f(r_0)| \leq BR^\alpha, \quad R = |r - r'|$$

(6.4)

for all points $r$ for which $R \leq c$. Since $\beta_{klm}(r)$ is piecewise linear and continuous, it is Hölder continuous and the singular integral in (6.1) can be regularized. Define a dyad $I_{rest}$ such that $I_{k-L,m-M} = G_{klm|KLM}^{(0)}$. Applying the regularization from Lee[35]

$$I_{klm}^{(pq)} = \int_{V_{klm}} \beta_{klm}(r') \left[ \delta_{pq} k^2 + \partial_p \partial_q' \right] G(r') \, dr'$$

$$= A_{klm}^{(pq)} + B_{klm}^{(pq)} + C_{klm}^{(pq)} + D_{klm}^{(pq)}$$

(6.5)

where

$$A_{klm}^{(pq)} = \int_{V_{klm} - V_0} \beta_{klm}(r') \left[ \delta_{pq} k^2 + \partial_p \partial_q' \right] G(r') \, dr'$$
\[ B_{klm}^{(pq)} = \int_{V_0} [\beta_{klm}(r') - \beta_{klm}(0)] \partial_{r'} \partial_q G(r') dr' \]
\[ + \beta_{klm}(0) \int_{V_0} \partial_{r'} \partial_q [G(r') - G_0(r')] dr' \]
\[ C_{klm}^{(pq)} = \beta_{klm}(0) \left[ \frac{1}{\partial_{r'} \partial_q} \int_{V_0} G_0(r-r') dr' \right]_{r=0} \]
\[ D_{klm}^{(pq)} = k^2 \delta_{pq} \int_{V_0} G(r') \beta_{klm}(r') dr' \]  
(6.6)

where \( G_0(r) \) is the static scalar Green's function and \( G(r) \) the dynamic scalar Green's function defined as
\[ G_0(r) = \frac{1}{4\pi|r|} \]
\[ G(r) = \frac{e^{ik|r|}}{4\pi|r|}. \]  
(6.7)

The partial derivatives of the dynamic scalar Green's function are
\[ \partial_p \partial_p G(r) = k^2 \left[ \frac{p^2}{|r|^2} - \frac{i}{k|r|} \left( 1 + \frac{i}{k|r|} \right) \left( \frac{3p^2}{|r|^2} - 1 \right) \right] G(r), \]
\[ \partial_p \partial_q G(r) = k^2 \frac{pq}{|r|^2} \left[ 1 + \frac{3i}{k|r|} \left( 1 + \frac{i}{k|r|} \right) \right] G(r), \]  
(6.8)

where \( p, q \in \{x, y, z\} \).

![Figure 6.2: Support of 1D basis functions](image)

The convolution of 3D pulse functions
\[ \beta_{klm}(r) = \beta_k(\frac{x}{\delta_x}) \beta_l(\frac{y}{\delta_y}) \beta_m(\frac{y}{\delta_z}) \]  
(6.9)

is non-zero over a volume equal to that of 8 cells. Because \( \beta_{klm}(r) \) extends beyond the \( k, l, m \)-cell to first, second and third nearest neighbours, it is not just the self-matrix element calculation, where the integral extends into the exclusion zone. Since
the singularity is assumed to be at the origin and \( \beta_1(x/\delta_x) \), shown in Figure 6.2, has support of 2, only the matrix elements involving only \( \beta_{-1} \), \( \beta_0 \) and \( \beta_1 \) in equation (6.2) involve the singularity. Thus the only terms in (6.6) that involve the exclusion volume are those where \( k,l,m \in \{ -1,0,1 \} \); however, due to symmetry only elements where \( k,l,m \in \{ 0,1 \} \) give unique results.

**Evaluation of \( A_{klm}^{(pq)} \)**

With the origin of the coordinate system at the singularity, all of the integrations can be done in the first octant where \( x, y \) and \( z \) are positive. The odd-even symmetry properties of the potential and its derivatives are exploited to find the total integral. The first octant of the volume \( V_{oo0} (k,l,m = 0) \) is shown in Figure 6.1. \( O_0 \), a cube of side \( a \), is the first octant of the exclusion volume and in this example \( a = \delta_x \) since \( \delta_x \) happens to be the smallest cell dimension. In general evaluation of \( A_{klm}^{(pq)} \) involves integration over first octant volume \( O_\Sigma = O_{klm} - O_0 \). Since \( O_\Sigma \) does not contain the singularity, \( A_{klm}^{(pq)} \) can be approximated by using an adaptive quadrature for, at most, two rectangular regions. An example is shown in Figure 6.1 where the two regions are denoted by \( O_\alpha \) and \( O_\gamma \); however, one or both could be zero, if an elemental cell was respectively a cuboid or a cube. These integrations are special cases of the non-singular matrix element computations and will be discussed in Section 6.1.2.

**Evaluation of \( C_{klm}^{(pq)} \)**

Introducing the notation \( \delta_{klm} = 1 \) if \( k = l = m = 0 \) and \( \delta_{klm} = 0 \) otherwise, \( C_{klm}^{(pq)} \) for a cube is given by [35,69]

\[
C_{klm}^{(pq)} = \beta_{klm}(0) \left[ \partial_p \partial_q \int_{V_0} G_0(r - r') \, dr' \right]_{r=0} = \frac{1}{4\pi} \int_{S_0} \frac{n_p \hat{e}_r}{|r|^2} \, dS = \frac{2a}{4\pi} \int_{S_0} \frac{\hat{e}_p \hat{e}_q}{|r|^3} \, dS = -\frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \theta \, d\theta - 1 \, d\theta = -\frac{1}{3} \delta_{klm} \delta_{pq},
\]

(6.10)

where \( \delta_{pq} = 1 \) if \( p = q \) and \( \delta_{pq} = 0 \) otherwise.

**Evaluation of \( D_{klm}^{(pq)} \)**

Let the \( xy \)-plane, \( xz \)-plane and the \( yz \)-plane divide \( V_{oo0} \) into eight octants. Consider the octant \( O_0 \), where the coordinates \( x, y \) and \( z \) are positive, for \( k = l = m = 0 \),

\[
\beta_{oo0}(r) = 1 - \frac{x}{\delta_x} - \frac{y}{\delta_y} - \frac{z}{\delta_z} + \frac{xy}{\delta_x \delta_y}
\]
Due to the strictly even or odd nature of the integrands, we need only consider the first octant in all the following integrations. All of the terms $D_{klm}^{(pq)}$ can be expressed as linear combinations of integrations over this volume. Let

$$I_{000} = k^2 \int_{O_0} G(r') \left[ 1 - \frac{x'}{\delta_x} - \frac{y'}{\delta_y} - \frac{z'}{\delta_z} + \frac{x'y'}{\delta_x\delta_y} + \frac{x'z'}{\delta_x\delta_z} + \frac{y'z'}{\delta_y\delta_z} \right] dr'$$

(6.12)

By expanding $e^{ikR}$ as a Taylor's series,

$$e^{ikR} = 1 + (ikR) - \frac{1}{2}(kR)^2 - \frac{i}{6}(kR)^3 + \ldots$$

(6.13)

and integrating term by term, $I_{000}$ may be approximated by a polynomial in $\lambda = iak$. The integrals, being smooth functions, can be evaluated numerically. Thus $D_{000}^{(pq)}$ can be approximated as a summation of polynomials, the approximation being of the form

$$D_{000}^{(pq)} = 8\delta_{pq}I_{000} \approx \sum_{s=0}^{3} d_s p_s.$$  

(6.14)

Here we have taken note of the fact that $G(r)$ and $\beta_{000}(r)$ are both even with respect to $x, y$ and $z$, hence for the 8 octants we get $D_{000}^{(pq)} = 8\delta_{pq}I_{000}$. The polynomials are defined as

$$k^2 \int_{O_0} G(r)dr \approx p_0 = \sum_{s=2}^{5} d_{0s}\lambda^s$$

$$\frac{k^2}{a} \int_{O_0} xG(r)dr \approx p_1 = \sum_{s=2}^{5} d_{1s}\lambda^s$$

$$\frac{k^2}{a^2} \int_{O_0} yG(r)dr \approx p_2 = \sum_{s=2}^{5} d_{2s}\lambda^s$$

$$\frac{k^2}{a^3} \int_{O_0} xyG(r)dr \approx p_3 = \sum_{s=2}^{5} d_{3s}\lambda^s$$

(6.15)

The coefficients $d_{is}$ are shown in the top part of Table 6.1. The computation of the coefficients for the polynomial $p_0$ is presented in detail to illustrate the technique used. The coefficients for the remaining polynomials were computed in a similar way.

First, after switching to polar coordinates, integration with respect to the radial dimension can be done analytically. Due to symmetry the integral can be evaluated by
6.1. FREE-SPACE MATRIX ELEMENTS FOR PLANAR MEDIA

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Table 6.1: Coefficients for \( D_{kim} \)

breaking the integral into integrals over two volumes in the first octant. The resulting integrals are shown in (6.16).

\[
\frac{1}{4\pi} \int_{O_0} G(r') \, dr' = 2 \int_{\tan^{-1} \sec \theta}^{\pi} \int_{0}^{\pi} \frac{(1 - irk) \sin \phi e^{ikr}}{4k^2 \pi} d\phi d\theta
\]

\[
+2 \int_{0}^{\tan^{-1} \sec \theta} \int_{0}^{\pi} \frac{(1 - irk) \sin \phi e^{ikr}}{4k^2 \pi} d\phi d\theta \quad (6.16)
\]

Now, approximate the exponential function by expanding it in its power series, keeping the first 5 terms and perform the integration with respect to \( \phi \). The result is shown in (6.17).

\[
\frac{1}{4\pi} \int_{O_0} G(r') \, dr' = 2 \int_{0}^{\pi} \frac{\lambda^4 S_0^2(\theta)}{256 \pi \cos^4 \theta} + \frac{\lambda^4 \ln |s_0(\theta)|}{8 \pi \cos^2 \theta} + \frac{\lambda^4 \ln |s_0(\theta)|}{64 \pi \cos^4 \theta}
\]

\[
- \frac{\lambda^5 s_2(\theta)}{12 \pi \cos^3 \theta s_1(\theta)} d\theta
\]

\[
+2 \int_{0}^{\pi} \frac{\lambda^5 s_2(\theta)}{180 \pi \cos^5 s_1(\theta)} - \frac{\lambda^5 s_2(\theta)}{360 \pi \cos^5 \theta s_1^3(\theta)} - \frac{\lambda^4}{256 \pi \cos^4 \theta s_0^3(\theta)} d\theta
\]

\[
+2 \int_{0}^{\pi} \frac{\lambda^2}{8 \pi s_2(\theta)} - \frac{\lambda^3}{24 \pi s_2^2(\theta)} - \frac{\lambda^4}{96 \pi s_2^3(\theta)} - \frac{\lambda^5}{480 \pi s_2^4(\theta)} d\theta
\]

\[
+ \frac{\lambda^5}{960} + \frac{\lambda^4}{192} + \frac{\lambda^3}{48} + \frac{\lambda^2}{16} \quad (6.17)
\]

where

\[
s_0(\theta) = \tan \left( \frac{\tan^{-1} (\sec \theta)}{2} \right)
\]

\[
s_1(\theta) = \sin[\tan^{-1} (\sec \theta)]
\]

\[
s_2(\theta) = \cos[\tan^{-1} (\sec \theta)] \quad (6.18)
\]
Chapter 6. 3D Matrix Elements Analysis

At this stage the remaining integrals cannot, in general, be evaluated analytically; however, these integrals are now independent of the problem size and therefore need be evaluated only once. The integrands are all smooth functions on the interval $[0, \frac{\pi}{2}]$ and are therefore well suited for numerical integration to high precision. Once these integrals have been approximated the coefficients of $\lambda$ have been completely determined.

The coefficients $d_s, s = 0, 1, 2, 3$, in (6.14) are written in terms of a set of constants given by

$$
c_0 = 1 \quad c_1 = \frac{a}{\delta_x} + \frac{a}{\delta_y} + \frac{a}{\delta_z}
$$

$$
c_2 = \frac{a^2}{\delta_x \delta_y} + \frac{a^2}{\delta_x \delta_z} + \frac{a^2}{\delta_y \delta_z} \quad c_3 = \frac{a^3}{\delta_x \delta_y \delta_z}.
$$

(6.19)

In the case of $D_{000}^{(pq)}$ the coefficients are given in the second column in the bottom part of Table 6.1.

The remaining terms $D_{k1m}^{(pq)}$ for $k, l, m \in \{0, 1\}$, are all expressible as a combinations of the $p$'s. To illustrate this we consider $D_{100}^{(pq)}$ and define

$$
I_{100} = k^2 \int \beta_{100}(r')Gdr'
$$

$$
= k^2 \int G \left[ \frac{x'}{\delta_x} - \frac{x'y'}{\delta_x \delta_y} - \frac{x'z'}{\delta_x \delta_z} + \frac{x'y'z'}{\delta_x \delta_y \delta_z} \right] dr'
$$

$$
\approx a \frac{p_1}{\delta_x} - \left( \frac{a^2}{\delta_x \delta_y} + \frac{a^2}{\delta_x \delta_z} \right) p_2 + \frac{a^3}{\delta_x \delta_y \delta_z} p_3.
$$

(6.20)

Since we are displaced from the $yz$-plane we have symmetry only in two planes therefore $D_{100}^{(pq)} = 4\delta_{pq} I_{100}$. Hence the results given in the third column of the bottom part of Table 6.1. Similarly with $D_{010}^{(pq)}, D_{001}^{(pq)}, D_{110}^{(pq)}, D_{011}^{(pq)}, D_{101}^{(pq)}$ and $D_{111}^{(pq)}$.

Evaluation of $B_{k1m}^{(pq)}$

We begin this section by considering $B_{k1m}^{(pp)}$ with $k = l = m = 0$. Results for other values of $k, l$ and $m$ can be expressed in terms of the integrals that arise for the 000 case. First notice that the terms in brackets for expression of $B_{k1m}^{(pq)}$ in (6.6) is

$$
[\beta_{000}(r') - \beta_{000}(0)] = -\frac{|x'|}{\delta_x} \frac{|y'|}{\delta_y} \frac{|z'|}{\delta_z}
$$

$$
+ \frac{|x'y'|}{\delta_x \delta_y} + \frac{|x'z'|}{\delta_x \delta_z} + \frac{|y'z'|}{\delta_y \delta_z}
$$

$$
- \frac{|x'y'z'|}{\delta_x \delta_y \delta_z}.
$$

(6.21)

Again in considering the first octant, we can drop the absolute values and express $B_{000}^{(pp)}$ as a linear combination of integrals involving terms in (6.21). Approximating the integrals
by power series expansions, we find that

\[ \frac{1}{a} \int_{O_x} x \frac{\partial^2 G}{\partial x^2} \, dx \approx q_0 = \sum_{s=0}^{5} b_{ts}^{(xx)} \lambda^s \]

\[ \frac{1}{a} \int_{O_x} y \frac{\partial^2 G}{\partial x^2} \, dx \approx q_1 = \sum_{s=0}^{5} b_{ts}^{(xx)} \lambda^s \]

\[ \frac{1}{a^2} \int_{O_x} xy \frac{\partial^2 G}{\partial x^2} \, dx \approx q_2 = \sum_{s=0}^{5} b_{ts}^{(xx)} \lambda^s \]

\[ \frac{1}{a^2} \int_{O_x} yz \frac{\partial^2 G}{\partial x^2} \, dx \approx q_3 = \sum_{s=0}^{5} b_{ts}^{(xx)} \lambda^s \]

\[ \frac{1}{a^3} \int_{O_x} yz \frac{\partial^2 G}{\partial x^2} \, dx \approx q_4 = \sum_{s=0}^{5} b_{ts}^{(xx)} \lambda^s \]

\[ \int_{O_x} \frac{\partial^2 (G - G_0)}{\partial x^2} \, dx \approx q_5 = \sum_{s=0}^{5} b_{ts}^{(xx)} \lambda^s \]  

(6.22)

where the coefficients \( b_{ts}^{(xx)} \) are given in the top part of Table 6.2.

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<td>0.012740</td>
<td>-0.004417</td>
<td>0.001421</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2: Coefficients for \( b_{ts}^{(xx)} \)

As noted earlier, the second order partial derivative with respect to \( x, y \) or \( z \) is an even function as is \( \beta_{000} \) therefore the results above are the same in the remaining 7 octants.

We can now express the final result for \( B_{kml}^{(xx)} \). In general we have

\[ B_{kml}^{(xx)} \approx \sum_{s=0}^{5} b_{ts}^{(xx)} q_s \]  

(6.23)
where the coefficients $b_s$, $s = 0, 1, \ldots, 5$, are given in the bottom part of Table 6.2. Some new coefficients, introduced in the Table, are given by

\[
\begin{align*}
   c_0^x &= \frac{a}{\delta_x} & c_1^x &= \frac{a}{\delta_x} + \frac{a}{\delta_z} \\
   c_2^x &= \frac{a^2}{\delta_x \delta_y} + \frac{a^2}{\delta_x \delta_z} & c_3^x &= \frac{a^2}{\delta_y \delta_z}.
\end{align*}
\]  
(6.24)

Next we consider terms involving derivatives with respect to $x$ and $y$. We need to evaluate the two integrals in the expression for $B_{kilm}^{(xy)}$ in (6.6). The mixed derivatives are odd functions in $x$ and $y$, so the first integral is zero for all terms except $B_{1100}^{(xy)}$ and $B_{1111}^{(xy)}$. The second integral is zero for all $k, l$ and $m$, since $\beta_{kilm}(0) = 0$ unless $k = l = m = 0$ where the integral is zero by anti-symmetry. So we need only concern ourselves with the terms

\[
\frac{1}{a^2} \int_0^1 xy \frac{\partial^2 G}{\partial x \partial y} \, dx \approx u_0 = \sum_{s=0}^5 b_0^{(xy)} \lambda^s
\]
\[
\frac{1}{a^3} \int_0^1 xyz \frac{\partial^2 G}{\partial x \partial y} \, dx \approx u_1 = \sum_{s=0}^5 b_1^{(xy)} \lambda^s
\]  
(6.25)

Then

\[
B_{kilm}^{(xy)} = \sum_{s=0}^1 b_s^{(xy)} u_s
\]  
(6.26)

and the coefficients are given in Table 6.3.

<table>
<thead>
<tr>
<th>$u_{ts}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t \backslash s$</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$b_s^{(xy)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s \mid B_{0000}^{(xy)}</td>
</tr>
<tr>
<td>$b_0^{(xy)}$</td>
</tr>
<tr>
<td>$b_1^{(xy)}$</td>
</tr>
</tbody>
</table>

Table 6.3: Coefficients for $B_{kilm}^{(xy)}$

6.1.2 Adaptive integration of non-singular integrands

Integrating the non-singular terms in (6.1) and indeed the term $A_{kilm}^{(xy)}$ of (6.6) involves a volumetric numerical integration of an infinitely smooth function which should be easy to approximate numerically; however, some care must be taken due to the rapidly decaying nature of the integrand. Volumetric cells with large aspect ratios, that is the
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ratio of the largest cell dimension to that of the smallest, will have rapidly changing integrands in cells adjacent to the singular ones. While for volumetric cells with smaller aspect ratios, the integrands in cells adjacent to the singular volumes vary much more slowly. This behaviour makes a fixed order quadrature scheme impractical, therefore an adaptive scheme was used.

**Adaptive integration of non-singular terms**

The integral in (6.1) is redefined in terms of its iterated integral so that volumetric integration is now cast as a series of one dimensional integrals. Let us define

\[ F^{(pq)}(x, y, z) = \beta_{klm}(r) \left[ \delta_{pq} k^2 + \partial_r \partial_q \right] G(r), \]  

(6.27)

for all \( V_{klm} \) where \( F \) is analytic. (6.1) can be written as

\[ I(\mathbf{p}) = F^{(pq)}(x, y, z) \, dV. \]  

(6.28)

Now define \( F_1^{(pq)}(x, y) \) as

\[ F_1^{(pq)}(x, y) = \int_{(m-1)\delta_x}^{(m+1)\delta_x} F^{(pq)}(x, y, z) \, dz \]  

(6.29)

and \( F_2^{(pq)}(x) \) as

\[ F_2^{(pq)}(x) = \int_{(l-1)\delta_y}^{(l+1)\delta_y} F_1^{(pq)}(x, y) \, dy \]  

(6.30)

Combining equations (6.28), (6.29) and (6.30) gives

\[ I_{klm}^{(pq)} = \int_{(k-1)\delta_z}^{(k+1)\delta_z} F_2^{(pq)}(x) \, dx. \]  

(6.31)

Now each of the integrals in equations (6.29) - (6.31) are one dimensional and can be approximated adaptively to any desired accuracy. Romberg integration is used for each of the one dimensional integrals, using 3 successive refinements of the extended trapezoidal rule to remove all error terms in the error series up to but not including \( O(n^6) \) [44]. The modified trapezoidal rule is useful here since it is a closed formula (evaluating the integral at the end-points) and the basis function forces all the integrands to zero on the boundary of \( V_{klm} \); therefore providing a free evaluation of the trapezoidal rule.

**Adaptive integration to evaluate \( A_{klm}^{(pq)} \)**

Evaluation of the term \( A_{klm}^{(pq)} \) of (6.6) involves the volume \( V_{klm} = \{-a, a\} \times \{-a, a\} \times [a, a] \), where \( a = \min \{ \delta_x, \delta_y, \delta_z \} \), so all the numerical integrations are adjacent to the exclusion volume. Since all the terms in the integrand in the expression of (6.6) are either odd or
even we can do all the integrations in the first octant and use symmetry to compute the final result. Using \( F^{(pq)}(x, y, z) \) defined above, we can write

\[
A_{k\ell m}^{(pq)} = W(p, q)(2 - k)(2 - l)(2 - m) \cdot (I_{k\ell m}^{(pq)} + J_{k\ell m}^{(pq)} + K_{k\ell m}^{(pq)})
\]

(6.32)

where

\[
I_{k\ell m}^{(pq)} = \int_{a}^{b} \int_{a}^{b} \int_{a}^{b} F^{(pq)}(x, y, z) \, dz \, dy \, dx
\]

(6.33)

\[
J_{k\ell m}^{(pq)} = \int_{0}^{\delta_{x}} \int_{0}^{\delta_{y}} \int_{0}^{\delta_{z}} F^{(pq)}(x, y, z) \, dz \, dy \, dx
\]

(6.34)

\[
K_{k\ell m}^{(pq)} = \int_{0}^{\delta_{x}} \int_{0}^{\delta_{y}} \int_{0}^{\delta_{z}} F^{(pq)}(x, y, z) \, dz \, dy \, dx
\]

(6.35)

and

\[
W(p, q) = \begin{cases} 
1 & \text{if } p = q \\
kl & \text{if } p = x, q = y \\
km & \text{if } p = x, q = z \\
lm & \text{if } p = y, q = z.
\end{cases}
\]

(6.36)

We have now expressed the volumetric integral as the sum of 3 integrals, each of which is a series of one dimensional integrations. Notice that at least one, but possibly two or all three of which may be zero, depending on the aspect ratio of the volumetric cell. Again Romberg integration was employed with three refinements; however, an open formula was used for the integrations. This avoided evaluations on the axial planes, keeping the function evaluations as far away from the singularity as possible.

### 6.1.3 Continuity between methods

The value of the integral in (6.1) is independent of the exclusion volume used. For the polynomial approximations a cubic exclusion volume was used with a side length of \( 2a \), where \( a = \min\{\delta_{x}, \delta_{y}, \delta_{z}\} \). The integral over the remaining volume is integrated using the adaptive integration of Section 6.1.2. The independence of the result with respect to the exclusion volume can be used to help verify the results by varying \( a \) for a fixed cell volume. This is not a sufficient test; however, but continuity between methods is a strong check. Further comparisons have shown very good agreement to results using Fourier techniques.

This simple numerical experiment was performed for a cubic volume \( V_{000} \). For cubic cells, the largest possible exclusion volume is \( V_{000} \). This implies that \( A_{000}^{(xx)} = 0 \), and from (6.6),

\[
I_{000}^{(xx)} = B_{000}^{(xx)} + C_{000}^{(xx)} + D_{000}^{(xx)}.
\]

(6.37)

However, as the size of the exclusion cube is decreased, \(|A_{000}^{(xx)}|\) increases, while \(|B_{000}^{(xx)}| + |D_{000}^{(xx)}|\) decreases. \( C_{000}^{(xx)} \equiv \frac{1}{3} \) and is independent of the exclusion volume, being determined strictly by the nature of the singularity.
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This behaviour is shown in Figure 6.3. The sum of these terms should be independent of the exclusion volume size. $A^{(xx)}_{000}$ was computed to a 0.01% tolerance, while the terms $B^{(xx)}_{000}$ and $D^{(xx)}_{000}$ are computed to an accuracy of $O(3^{-6})$. The sum of the exclusion volume dependent terms is also shown in Figure 6.3 and it is indeed constant. This provides a powerful check on the accuracy of the technique.

6.2 Matrix elements for planar media

We will now develop the convolutional and correlational dyadic Green’s function, moments can then be taken to find the matrix elements. The reflection dyadic Green’s function was defined in equation (5.31) and is repeated and renumber here

$$G^{(r)}(r|r') = \left[ \nabla - \frac{1}{k^2} \nabla' \right] G(r|r' - 2z'\hat{z}) + \left[ \nabla - \frac{1}{k^2} \nabla' \right] G(r|r' - 2(z' + c)\hat{z}) + \nabla^r (r|r' - 2z'\hat{z}) + \nabla^r (r|r' - 2(z' + c)\hat{z}). \tag{6.38}$$

The first two dyads will be interpreted as image terms, the last two dyads are the correction terms. The convolutional dyadic Green’s function was also given in equation (5.31) and was found to be

$$G^{(c)}(r|r') = \tilde{G}^{(c)}(r|r') + \tilde{G}^{(c)}(r|r + 2c\hat{z}) + \tilde{G}^{(c)}(r|r' - 2c\hat{z}) + \tilde{G}^{(c)}(r|r' - 2(z' + c)\hat{z}). \tag{6.39}$$

where the first term is the free-space dyadic Green’s function, which after applying the moment method, was analyzed in the last section. The next two dyadic terms can be
interpreted as image dyads, but are evaluated as free-space terms, while the last two are correction terms.

### 6.2.1 Dyadic image terms

In Fourier space, the magnetic scalar potential in the source region when there are upper and lower interfaces was found in equation (5.22) to be

\[
\alpha^2 \tilde{U}(z|z') = \frac{1}{2\alpha_0} \left[ e^{-\alpha_0|z-z'|} + e^{-\alpha_0(z+z')} + e^{\alpha_0(z-z')} + e^{-\alpha_0(z+z') - 2c} + e^{\alpha_0(z-z') - 2c} \right],
\]

(6.40)

where the exponential terms can all be thought of as sources or potential terms. The first term is the physical source, the next two correspond to the simple image terms, that is the observed field at \( r \) due to the mirror image of the source in free-space at \( r' \). This is only true if the wave is perfectly reflecting, which is the case in equation (6.40).

\[ z'' = 2z_1 - z' \]

\[ z''' = 2z_0 - z' \]

This is illustrated in Figure 6.4 for the quasi-one dimensional case for the two simple images. In the Figure the field at \( z \) due to a source at \( z' \) is equivalent to the field due to the two sources in free-space at \( z'' \) and \( z''' \). Let \( z_1 = 0 \) and \( z_0 = c \) in the figure, where \( c > 0 \). The path length for the two waves drawn in the Figure are \( z + z' \) and \( 2c - z - z' \), which is real scalar multiples in the simple image terms in equation (6.40). Notice that for large values of \( c \) that the reflection from the bottom surface will be very weak and can be safely ignored. This is the case for thick plates and in the limit, represents the half-space conductor.

Since these simple image terms have a free-space interpretation, there is a way to evaluate them which utilizes the machinery of the previous section. Define the dyadic operator \( \mathcal{S}[\tilde{G}] \) such that

\[
\mathcal{S}[\tilde{G}] = \begin{bmatrix}
G_{xx} & G_{xy} & -G_{xz} \\
G_{yx} & G_{yy} & -G_{yz} \\
G_{zx} & G_{zy} & -G_{zz}
\end{bmatrix},
\]

(6.41)
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then

\[
\begin{bmatrix}
\vec{\nabla} \cdot \mathbf{E}
\end{bmatrix} G(\mathbf{r}|\mathbf{r}' - 2z' \hat{\mathbf{z}}) = \mathcal{S} \left[ \tilde{G}^{(0)}(\mathbf{r}|\mathbf{r}' - 2z' \hat{\mathbf{z}}) \right]
\]

\[
\begin{bmatrix}
\vec{\nabla} \cdot \mathbf{E}
\end{bmatrix} G(\mathbf{r}|\mathbf{r}' - 2z' \hat{\mathbf{z}}) = \mathcal{S} \left[ \tilde{G}^{(0)}(\mathbf{r}|\mathbf{r}' - 2z' \hat{\mathbf{z}}) \right].
\] (6.42)

Therefore, after applying moments, these terms can be evaluated as the free-space terms from the last section and then changing the sign of the third column of the dyad. The image terms corresponds to the idealized case when the incident wave is perfectly reflected, that is the reflection coefficient is unity. The correction dyads will correct for the actual imperfect reflection and for all odd number of multiply reflected waves.

With the presence of two interfaces, there will be multiple reflections between the interfaces. Figure 6.5 illustrates the path length of the two simple image and the first two multiple reflections: the double image terms. The double image terms refer to the equivalent field due to source points in free-space at distances |z - z' + 2c| and |z' - z + 2c|, where c = z_0 - z_{-1}, and have a z - z' dependence, instead of the z + z' dependence observed for the simple image terms. Therefore, these terms can be treated exactly like free-space terms, that is assume there are no interfaces and the source point is at a distance z - z' + 2c and z' - z + 2c. The correction dyads will be used to take into account the imperfect reflections that occur. The double image term dyads are then just \( \tilde{G}^{(0)}(\mathbf{r}|\mathbf{r} + 2c \hat{\mathbf{z}}) \) and \( \tilde{G}^{(0)}(\mathbf{r}|\mathbf{r}' - 2c \hat{\mathbf{z}}) \) in the expression of the convolutional dyad in equation (6.39).

The free-space term and these four image terms will dominate the dyadic Green's function \( \tilde{G}^{(cs)}(\mathbf{r}|\mathbf{r}') \) and should be computed precisely.
6.2.2 Dyadic correction terms

The correction dyadic terms were defined in Fourier space by equation (5.33). These dyadic expressions can all be written in terms of four scalar potentials

\[
\tilde{V}_\eta^{(\Xi)}(x|z') = \frac{1}{2\alpha_0} \left( \Lambda_\eta^{(\Xi)} - 1 \right) e^{-\alpha_0(z+z')},
\]

where \( \Xi \in \{\Gamma, \Upsilon\} \) and \( \eta \in \{\tau, \beta\} \). This Fourier integral has can be written equivalently as

\[
V_\eta^{(\Xi)}(\rho, z|\rho', z') = \frac{1}{2\alpha_0} \int_0^\infty \frac{1}{2\alpha_0} e^{\alpha_j(x+z')} \left( \Lambda_\eta^{(\Xi)} - 1 \right) J_0(\alpha \rho) \alpha \, d\alpha,
\]

where the positive sign is used for \( \Xi = \Gamma \) and the negative sign when \( \Xi = \Upsilon \).

Using equation (5.32), the correlational correction matrix elements can be formed by applying the differential operator \((\nabla \times \hat{z})(\nabla' \times \hat{z})\) and the method of moments to equation (6.44). This is illustrated in detail by considering the scalar potential \( V_\tau^{(\Xi)} \).

After applying the differential operator and the method of moments to this potential, we have

\[
\tilde{V}_\tau^{(\Xi)}|_{klm} = -\int_{x_k-x_{k'}} \int_{y_l-y_{l'}} \beta_k \left( \frac{x}{\delta_x} \right) \beta_l \left( \frac{y}{\delta_y} \right) \int_{z_m-z_{m'}} (\nabla \times \hat{z})(\nabla' \times \hat{z}) \cdot
\]

\[
\int_0^\infty (\Lambda_\tau^{(\Xi)} - 1) \left[ e^{\alpha(x+z')} \frac{J_0(\rho \alpha)}{4\pi \alpha} \right] \beta_z \left( \frac{z}{\delta_z} \right) d\alpha \, dz \, dy \, dx.
\]

After taking the differential operator inside the integral, the correlational correction dyad can be written

\[
\tilde{V}_\tau^{(\Xi)}|_{klm} = \begin{bmatrix}
-V^{(\Xi)}_{pp} & V^{(\Xi)}_{px} & 0 \\
V^{(\Xi)}_{px} & -V^{(\Xi)}_{xx} & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where

\[
V^{(\Xi)}_{pp} = \frac{1}{2\pi \rho} \int_{x_k-x_{k'}} \int_{y_l-y_{l'}} \beta_k \left( \frac{x}{\delta_x} \right) \beta_l \left( \frac{y}{\delta_y} \right) \int_{z_m-z_{m'}} \cdot
\]

\[
\int_0^\infty \frac{k^2}{2\alpha_1} (\Gamma^{(\Xi)_{-1}} - 1) e^{-\alpha(x+z')} \alpha \left[ \rho \alpha J_0(\rho \alpha) - J_1(\rho \alpha) \right] d\alpha +
\]

\[
\frac{q^2}{\rho^2} \int_0^\infty (\Gamma^{(\Xi)_{-1}} - 1) e^{-\alpha(x+z')} \alpha \left[ \rho \alpha J_0(\rho \alpha) - 2J_1(\rho \alpha) \right] d\alpha \, dz \, dy \, dx
\]

\[
V^{(\Xi)}_{pq} = \frac{pq}{2\pi \rho^3} \int_{x_k-x_{k'}} \int_{y_l-y_{l'}} \beta_k \left( \frac{x}{\delta_x} \right) \beta_l \left( \frac{y}{\delta_y} \right) \int_{z_m-z_{m'}} \cdot
\]

\[
\int_0^\infty (\Gamma^{(\Xi)_{-1}} - 1) e^{-\alpha(x+z')} \alpha \left[ \rho \alpha J_0(\rho \alpha) - 2J_1(\rho \alpha) \right] d\alpha \, dz \, dy \, dx,
\]

where \( p \in \{x, y\} \) and \( p \neq q \).
This expression can be simplified by first recognizing that the integrals need only be evaluated when \( z' = 0 \) and defining two axially symmetric functions, which we will call Surrey functions for a lack of a better name, as

\[
S^{(\Gamma)}_T(\rho, m) = \int_{z_{m-\delta_z}}^{z_{m+\delta_z}} \int_0^\infty \frac{k^2}{2\alpha_1} (\Lambda^{(\Gamma)} - 1) e^{-\alpha_1 \delta_z} \rho \alpha J_0(\rho \alpha) \beta_1 \left( \frac{z}{\delta_z} \right) \, d\alpha \, dz
\]

\[
T^{(\Gamma)}_T(\rho, m) = \int_{z_{m-\delta_z}}^{z_{m+\delta_z}} \int_0^\infty \frac{k^2}{2\alpha_1} (\Lambda^{(\Gamma)} - 1) e^{-\alpha_1 \delta_z} \alpha J_1(\rho \alpha) \beta_1 \left( \frac{z}{\delta_z} \right) \, d\alpha \, dz. \quad (6.48)
\]

Integrating (6.48) with respect to \( z \) we get the final expression

\[
S^{(\Gamma)}_T(\rho, m) = \int_0^\infty \frac{k^2}{2\alpha_1} (\Lambda^{(\Gamma)} - 1) \rho \alpha^2 e^{-\alpha_1 (m-1) \delta_z} \left[ \frac{1 + e^{-\alpha_1 \delta_z} (e^{-\alpha_1 \delta_z} - 2)}{\alpha_1^2 \delta_z^2} \right] J_0(\rho \alpha) \, d\alpha
\]

\[
T^{(\Gamma)}_T(\rho, m) = \int_0^\infty \frac{k^2}{2\alpha_1} (\Lambda^{(\Gamma)} - 1) \alpha e^{-\alpha_1 (m-1) \delta_z} \left[ \frac{1 + e^{-\alpha_1 \delta_z} (e^{-\alpha_1 \delta_z} - 2)}{\alpha_1^2 \delta_z^2} \right] J_1(\rho \alpha) \, d\alpha. \quad (6.49)
\]

Using equation (6.49), the dyadic elements given in (6.47) can be rewritten as

\[
V^{(\Gamma)}_{pp} = \frac{1}{2\pi} \int_{z_{k-\delta_z}}^{z_{k+\delta_z}} \int_{z_{l-\delta_z}}^{z_{l+\delta_z}} \left\{ \left( 1 - \frac{p^2}{\rho^2} \right) S^{(\Gamma)}_T(\rho, m) + \left( 1 - \frac{2p^2}{\rho^2} \right) T^{(\Gamma)}_T(\rho, m) \right\} \delta_\rho \beta_k \left( \frac{x}{\delta_\rho} \right) \beta_l \left( \frac{y}{\delta_\rho} \right) \, dy \, dx
\]

\[
V^{(\Gamma)}_{pq} = -\frac{1}{2\pi} \int_{z_{k-\delta_z}}^{z_{k+\delta_z}} \int_{z_{l-\delta_z}}^{z_{l+\delta_z}} pq \left\{ S^{(\Gamma)}_T(\rho, m) - 2T^{(\Gamma)}_T(\rho, m) \right\} \delta_\rho \beta_k \left( \frac{x}{\delta_\rho} \right) \beta_l \left( \frac{y}{\delta_\rho} \right) \, dy \, dx. \quad (6.50)
\]

Now, only a two-dimensional numerical integration is needed to evaluate this correction dyadic matrix, with a simple interpolation of the Surrey functions.

The remaining three correction dyadic matrices can be treated in the same way, where generally

\[
\tilde{V}_\eta^{(\Xi)}_{ijkl} = \begin{bmatrix}
-V^{(\Xi)}_{\eta vy} & V^{(\Xi)}_{\eta vx} & 0 \\
0 & 0 & 0 \\
-V^{(\Xi)}_{\eta yx} & -V^{(\Xi)}_{\eta yy} & 0
\end{bmatrix}
\]

(6.51)

where

\[
V^{(\Xi)}_{\eta pp} = \frac{1}{2\pi} \int_{z_{k-\delta_z}}^{z_{k+\delta_z}} \int_{z_{l-\delta_z}}^{z_{l+\delta_z}} \left\{ \left( 1 - \frac{p^2}{\rho^2} \right) 1S^{(\Xi)}_T(\rho, m) + \left( 1 - \frac{2p^2}{\rho^2} \right) T^{(\Xi)}_T(\rho, m) \right\} \delta_\rho \beta_k \left( \frac{x}{\delta_\rho} \right) \beta_l \left( \frac{y}{\delta_\rho} \right) \, dy \, dx
\]

\[
V^{(\Xi)}_{\eta pq} = -\frac{1}{2\pi} \int_{z_{k-\delta_z}}^{z_{k+\delta_z}} \int_{z_{l-\delta_z}}^{z_{l+\delta_z}} pq \left\{ S^{(\Xi)}_T(\rho, m) - 2T^{(\Xi)}_T(\rho, m) \right\} \delta_\rho \beta_k \left( \frac{x}{\delta_\rho} \right) \beta_l \left( \frac{y}{\delta_\rho} \right) \, dy \, dx, \quad (6.52)
\]

where \( \Xi \in \{ \Gamma, \Upsilon \} \) and \( \eta \in \{ \tau, \beta \} \). New Surrey functions will be defined for each media type.
6.2.3 Half-space media

For the half-space, there is no lower interface, hence there is only one simple image term, no double image terms and only one correction term with $\vec{V}_0(T) = 0$, $\vec{V}_0(T) = 0$ and $\vec{V}_0(T) = 0$. From equations (6.38) and (6.42) the correlational dyad is

$$G^{(0)}(r|l) = G_0^{(0)}(r|l) + V_0(r)(r|l)$$

and from equation (6.39)

$$G^{(T)}(r|l) = G^{(0)}(r|l).$$

The correction matrix elements are formed by using equation (6.50) with the Surrey functions defined by equation (6.49) with $\Lambda_0 = 1$.

6.2.4 Layered half-space matrix elements

For the layered half-space media type, where again there is no lower surface. Hence there is only one simple image term and no double image terms. The image term is treated slightly differently than the half-space media type, because of the presence of the medium between the host region, the source region and the probe region. Letting the $\alpha_1 \to \alpha_2$ in (2.82), we see that $\Gamma_{-1}^{(lr)} \to \Gamma_{-1}^{(hal)} e^{2\alpha_1 l}$. To compute the image term in this case, we make another idealization is made by assuming that the layer and the source region have the same physical properties, hence $\alpha_1 = \alpha_2$. From equations (6.38), (6.42) and (6.39) the correlational and convolutional dyads are then

$$G^{(T)}(r|l) = S \left[ G^{(0)}(r|l) - \{z' + 2l\}z' \right] + \vec{V}_0^{(T)}(r|l) - \{z' + 2l\}z'$$

$$G^{(T)}(r|l) = G^{(0)}(r|l).$$

The image term, for this situation, is the image assuming that the layer on top of the half-space has the same material properties as the host region. Indeed, when studying a subsurface flaw, the layer on top is exactly the same material as the host. The image terms are computed as before, except the image terms are all offset by a factor of $2l$, twice the layer thickness, in the $z$ direction. This offset is caused by the extra travel needed for the waves to be propagated through the layer, be reflected, and then propagate back to the host region.

The correction term not only corrects for the imperfect reflection, but also corrects for variations in material properties, if necessary. The larger the difference in the material properties of the two regions, the larger this correction will be. Using the term $\Lambda_0^{(T)} = e^{2\alpha_1 l} \Gamma_{-1}^{(lr)}$, two new Surrey functions $S_0^{(T)}$ and $T_0^{(T)}$ are defined by following the same procedure outlined in Section 6.2.3 the final definitions are

$$S_0^{(T)}(\rho, m) = \int_0^{\infty} (\Lambda_0^{(T)} - 1) \rho e^{-\alpha(m-1)\delta_2 - 2\alpha l} \left[ \frac{1 + e^{-\alpha\delta_2}(e^{-\alpha\delta_2} - 2)}{\alpha^2\delta_2^2} \right] J_0(\rho l) d\alpha$$
\[ T_{\tau}^{(\Gamma)}(\rho, m) = \int_{0}^{\infty} (\Lambda_{\tau}^{(\Gamma)} - 1) \alpha e^{-\alpha(m-1)\delta_s - 2a_1} \frac{[1 + e^{\alpha\delta_s}(e^{-\alpha\delta_s} - 2)]}{\alpha^2 \delta_s^2} J_1(\rho \alpha) d\alpha \] (6.56)

Then using equation (6.50) with \( S_\tau^{(\Gamma)} \) and \( T_{\tau}^{(\Gamma)} \) defined in equation (6.56), the correction matrix elements \( \tilde{V}_{\tau}^{(\Gamma)} |_{kim} \) can be determined.

### 6.2.5 Slab matrix elements

We now wish to compute the matrix elements for a slab or plate of finite thickness. Unlike the half-space and layered half-space examples already encountered, we now have both \( z+z' \) and \( z-z' \) dependence. Using the slab reflection coefficients in equation (2.90) in the scalar Green’s function defined in equation (2.38) and rearranging terms, the coefficients in the definition of \( \tilde{V}(z|z') \) of equation (5.25) are found to be

\[
\Lambda_{\tau}^{(\Gamma)} = \Lambda_{\beta}^{(\Gamma)} = \frac{(\alpha_1^2 - \alpha^2)}{(\alpha_1 + \alpha)^2 - (\alpha_1 - \alpha)^2 e^{-2a_1 z}} \\
\Lambda_{\tau}^{(0)} = \Lambda_{\beta}^{(0)} = \frac{(\alpha_1 - \alpha)^2 e^{-2a_1 z}}{(\alpha_1 + \alpha)^2 - (\alpha_1 - \alpha)^2 e^{-2a_1 z}}. 
\] (6.57)

Therefore, from equations (6.38), (6.42) and (6.39) the correlational and convolutional dyads are then

\[
\tilde{G}^{(\Gamma)}(r|r') = S \left[ \tilde{G}^{(0)}(r|r' - z' \hat{z}) \right] + S \left[ \tilde{G}^{(0)}(r|r' - 2\{z' + s\} \hat{z}) \right] + \\
\tilde{V}_{\tau}^{(\Gamma)}(r|r' - z' \hat{z}) + \tilde{V}_{\tau}^{(\Gamma)}(r|r' - 2\{z' + s\} \hat{z}) \\
\tilde{G}^{(\Gamma)}(r|r') = \tilde{G}^{(0)}(r|r') + \tilde{G}^{(0)}(r|r' + 2s \hat{z}) + \\
\tilde{G}^{(0)}(r|r' - 2s \hat{z}) \tilde{V}_{\tau}^{(\Gamma)}(r|r' + 2s \hat{z}) + \tilde{V}_{\tau}^{(\Gamma)}(r|r' - 2s \hat{z}). 
\] (6.58)

Using the coefficients found in equation (6.57), four slab Surrey functions are defined as

\[
S_{\tau}^{(\Gamma)}(\rho, m) = \int_{0}^{\infty} \frac{k^2}{2\alpha_1} (\Lambda_{\tau}^{(\Gamma)} - 1) \rho \alpha^2 \left[ e^{-\alpha_1(m-1)\delta_s} + e^{\alpha_1(m+1)\delta_s - 2a_1} \right] J_0(\rho \alpha) d\alpha \\
T_{\tau}^{(\Gamma)}(\rho, m) = \int_{0}^{\infty} \frac{k^2}{2\alpha_1} (\Lambda_{\tau}^{(\Gamma)} - 1) \alpha \left[ e^{-\alpha_1(m-1)\delta_s} + e^{\alpha_1(m+1)\delta_s - 2a_1} \right] J_1(\rho \alpha) d\alpha \\
S_{\tau}^{(\Gamma)}(\rho, m) = \int_{0}^{\infty} \frac{k^2}{2\alpha_1} (\Lambda_{\tau}^{(\Gamma)} - 1) \rho \alpha^2 \left[ e^{-\alpha_1(m-1)\delta_s} + e^{\alpha_1(m+1)\delta_s - 2a_1} \right] J_0(\rho \alpha) d\alpha \\
S_{\tau}^{(\Gamma)}(\rho, m) = \int_{0}^{\infty} \frac{k^2}{2\alpha_1} (\Lambda_{\tau}^{(\Gamma)} - 1) \rho \alpha^2 \left[ e^{-\alpha_1(m-1)\delta_s} + e^{\alpha_1(m+1)\delta_s} \right] J_0(\rho \alpha) d\alpha
\]
CHAPTER 6. 3D MATRIX ELEMENTS ANALYSIS

\[ T_\alpha^{(T)}(\rho, m) = \int_0^\infty \frac{k^2}{2\alpha_1} (\Lambda_\alpha^{(T)} - 1) \alpha \left[ e^{-\alpha_1(m-1)\delta_2} + e^{\alpha_1((m+1)\delta_2)} \right] \left[ 1 + e^{-\alpha_1\delta_2} (e^{-\alpha_1\delta_2} - 2) \right] J_1(\rho\alpha) d\alpha. \]  

(6.59)

Using equation (6.50) with the Surrey functions defined in equation (6.59), the correction matrix elements \( \vec{V}^{(T)}_{\alpha \beta \gamma \delta} \) and \( \vec{V}^{(T)}_{\alpha \beta \gamma \delta} \) can be determined.

6.2.6 Layered slab matrix elements

Since the layered slab media type has a layer between the probe region 0 and the host region 2, allowances similar to those made for the layered half-space must be made. Therefore, from equations (6.38), (6.42) and (6.39) the correlational and convolutional dyads are

\[
\bar{\mathbf{G}}^{(T)}(r|r') = S \left[ \bar{\mathbf{G}}^{(0)}(r|r' - \{z' - 2l\}z) \right] + S \left[ \bar{\mathbf{G}}^{(0)}(r|r' - 2\{z' + l + s\}z) \right] + \bar{\mathbf{V}}^{(T)}(r|r' - \{z' - 2l\}z) + \bar{\mathbf{V}}^{(T)}(r|z' - 2\{z' + l + s\}z) \]

\[
\bar{\mathbf{G}}^{(T)}(r|r') = \bar{\mathbf{G}}^{(0)}(r|r') + \bar{\mathbf{G}}^{(0)}(r|r' + 2\{l + s\}z) + \bar{\mathbf{G}}^{(0)}(r|z' - 2\{l + s\}z) \bar{V}^{(T)}(r|z' + 2s) + \bar{V}^{(T)}(r|z' - 2s). \]  

(6.60)

The reflection coefficients for the layered slab model were determined in Section 2.7.3. Multiplying the reflection coefficients from equation (2.94) by the exponential functions in equation (2.38) and then recombining terms, an expression of the form of equation (5.25) is found, after adjusting for the layer, to be

\[
\Lambda_\alpha^{(T)} = \frac{(\Gamma_a + \Gamma_b) e^{2\alpha_1 I}}{\Gamma_e + \Gamma_f + \Gamma_g + \Gamma_h} 
\Lambda_\alpha^{(T)} = \frac{\Gamma_i + \Gamma_j}{\Gamma_e + \Gamma_f + \Gamma_g + \Gamma_h} 
\Lambda_\alpha^{(T)} = \frac{(\Gamma_e + \Gamma_d) e^{2\alpha_1 I}}{\Gamma_e + \Gamma_f + \Gamma_g + \Gamma_h} 
\Lambda_\alpha^{(T)} = \frac{\Gamma_k + \Gamma_l}{\Gamma_e + \Gamma_f + \Gamma_g + \Gamma_h}, \]  

(6.61)

where \( \Gamma_\alpha \), for \( \alpha = a, b, \ldots l \), are defined in equation (2.94). Using the coefficients found in equation (6.61), eight slab Surrey functions are defined as

\[
S_\eta^{(T)}(\rho, m) = \int_0^\infty \frac{k^2}{2\alpha_2} (\Lambda_\eta^{(T)} - 1) \rho \alpha^2 e^{-2\alpha_2 I} \left[ e^{-\alpha_2(m-1)\delta_2} + e^{\alpha_2((m+1)\delta_2-2)} \right] \left[ 1 + e^{-\alpha_2\delta_2} (e^{-\alpha_2\delta_2} - 2) \right] J_0(\rho\alpha) d\alpha \]  

\[
S_\eta^{(T)}(\rho, m) = \int_0^\infty \frac{k^2}{2\alpha_2} (\Lambda_\eta^{(T)} - 1) \rho \alpha^2 e^{-2\alpha_2 I} \left[ e^{-\alpha_2(m-1)\delta_2} + e^{\alpha_2((m+1)\delta_2-2)} \right] \left[ 1 + e^{-\alpha_2\delta_2} (e^{-\alpha_2\delta_2} - 2) \right] J_0(\rho\alpha) d\alpha \]  

(6.61)
6.2. MATRIX ELEMENTS FOR PLANAR MEDIA

\[ T^{(\Gamma)}_{\eta}(\rho, m) = \int_0^{\infty} \frac{k^2}{2\alpha_2} (\Lambda^{(\Gamma)}_{\eta} - 1) \alpha e^{-2\alpha z} \left[ e^{-\alpha_2 (m-1)\delta_3} + e^{\alpha_2 (m+1)\delta_3 - 2s} \right] \left[ 1 + e^{-\alpha_2 \delta_3} \right] \frac{J_1(\rho \alpha)}{\alpha^2 \delta_3} d\alpha \]

\[ S^{(\Gamma)}_{\eta}(\rho, m) = \int_0^{\infty} \frac{k^2}{2\alpha_2} (\Lambda^{(\Gamma)}_{\eta} - 1) \rho \alpha^2 e^{-2\alpha z} \left[ e^{-\alpha_2 (m-1)\delta_3} + e^{\alpha_2 (m+1)\delta_3} \right] \left[ 1 + e^{-\alpha_2 \delta_3} \right] \frac{J_0(\rho \alpha)}{\alpha^2 \delta_3} d\alpha \]

\[ T^{(\Gamma)}_{\eta}(\rho, m) = \int_0^{\infty} \frac{k^2}{2\alpha_2} (\Lambda^{(\Gamma)}_{\eta} - 1) \alpha e^{-2\alpha z} \left[ e^{-\alpha_2 (m-1)\delta_3} + e^{\alpha_2 (m+1)\delta_3} \right] \left[ 1 + e^{-\alpha_2 \delta_3} \right] \frac{J_1(\rho \alpha)}{\alpha^2 \delta_3} d\alpha, \quad (6.62) \]

where \( \eta \in \{ \tau, \beta \} \).

Using equation (6.50) with the Surrey functions defined in equation (6.62), the correction matrix elements \( \tilde{V}^{(\Gamma)}_{\tau} |_{klm} \), \( \tilde{V}^{(\Gamma)}_{\tau} |_{klm} \), \( \tilde{V}^{(\Gamma)}_{\beta} |_{klm} \) and \( \tilde{V}^{(\Gamma)}_{\beta} |_{klm} \) can be determined.

6.2.7 Multi-layered slab matrix elements

Since the multi-layered slab media type has a layer between the probe region 0 and the host region 2, as well as a layer below the host region, allowances similar to those made for the layered half-space and layered slab must be made. Therefore, from equations (6.38), (6.42) and (6.39) the correlational and convolutional dyads are

\[ \tilde{G}^{(\Gamma)}(r|r') = S \left[ \tilde{G}^{(0)}(r|r' - \{z' - 2l\} \hat{z}) \right] + S \left[ \tilde{G}^{(0)}(r|r' - 2\{z' + l + s + o\} \hat{z}) \right] + \tilde{V}^{(\Gamma)}_{\tau}(r|r' - \{z' - 2l\} \hat{z}) + \tilde{V}^{(\Gamma)}_{\tau}(r|r' - 2\{z' + l + s + o\} \hat{z}) \]

\[ \tilde{G}^{(\Gamma)}(r|r') = \tilde{G}^{(0)}(r|r') + \tilde{G}^{(0)}(r|r' + 2\{l + s\} \hat{z}) \tilde{G}^{(0)}(r|r' - 2\{s + o\} \hat{z}) + \tilde{V}^{(\Gamma)}_{\tau}(r|r' + 2s \hat{z}) + \tilde{V}^{(\Gamma)}_{\tau}(r|r' - 2s \hat{z}). \quad (6.63) \]

The reflection coefficients for the multi-layered slab model were determined in Section 2.7.4. Multiplying the reflection coefficients from equation (2.102) by the exponential functions in equation (2.38) and then recombining terms, an expression of the form of equation (5.25) is found, after adjusting for the top and bottom layers, to be

\[ \Lambda^{(\Gamma)}_{\tau} = \frac{(\Gamma_a + \Gamma_b + \Gamma_c + \Gamma_d) e^{2\alpha z}}{\Gamma_g + \Gamma_h + \Gamma_i + \Gamma_j + \Gamma_k + \Gamma_l} \]

\[ \Lambda^{(\Gamma)}_{\beta} = \frac{(\Gamma_e + \Gamma_f) e^{-2\alpha z}}{\Gamma_g + \Gamma_h + \Gamma_i + \Gamma_j + \Gamma_k + \Gamma_l} \]

\[ \Lambda^{(0)}_{\tau} = \frac{(\Gamma_m + \Gamma_n) e^{2\alpha z}}{\Gamma_g + \Gamma_h + \Gamma_i + \Gamma_j + \Gamma_k + \Gamma_l} \]
\[ \Lambda_{\beta}^{(0)} = \frac{(\Gamma_{\alpha} + \Gamma_{\beta}) e^{-2\omega_{\alpha}}}{\Gamma_{\gamma} + \Gamma_{\theta} + \Gamma_{i} + \Gamma_{j} + \Gamma_{k} + \Gamma_{l}}, \] (6.64)

where \( \Gamma_{\alpha} \), for \( \alpha = a, b, \ldots p \), are defined in equation (2.102).

Using the coefficients found in equation (6.64), eight slab Surrey functions are defined as

\[ S_{\eta}^{(\Gamma)}(\rho, m) = \int_{0}^{\infty} \frac{k^2}{2\alpha_2} (\Lambda_{\eta}^{(\Gamma)} - 1) \rho \alpha^2 e^{2\alpha_2 \kappa_{\eta}} \left[ e^{-\alpha_2 (m-1) \delta_{s} + e^{\alpha_2 ((m+1) \delta_{s} - 2)}} \right] \left[ 1 + e^{-\alpha_2 \delta_{s} (e^{-\alpha_2 \delta_{s} - 2})} \right] J_{0}(\rho \alpha) \, d\alpha, \] (6.65)

\[ T_{\eta}^{(\Gamma)}(\rho, m) = \int_{0}^{\infty} \frac{k^2}{2\alpha_2} (\Lambda_{\eta}^{(\Gamma)} - 1) \alpha \rho e^{2\alpha_2 \kappa_{\eta}} \left[ e^{-\alpha_2 (m-1) \delta_{s} + e^{\alpha_2 ((m+1) \delta_{s} - 2)}} \right] \left[ 1 + e^{-\alpha_2 \delta_{s} (e^{-\alpha_2 \delta_{s} - 2})} \right] J_{1}(\rho \alpha) \, d\alpha, \]

where \( \eta \in \{ \tau, \beta \} \), \( \kappa_{\tau} = -1 \) and \( \kappa_{\beta} = 0 \).

Using equation (6.50) with the Surrey functions defined in equation (6.65), the correction matrix elements \( V_{\tau}^{(\Gamma')} |_{klm} \), \( V_{\tau}^{(\Gamma')} |_{klm} \), \( V_{\beta}^{(\Gamma')} |_{klm} \) and \( V_{\beta}^{(\Gamma')} |_{klm} \) can be determined.

### 6.3 Matrix elements for cylindrical media types

For a point source outside a conducting circular cylinder of radius \( \rho_1 \) and conductivity \( \sigma_1 \), the scalar Green’s function which describes the potential is given by[20,13]

\[ G(\mathbf{r}|\mathbf{r'}) = \frac{i}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \left( \int_{-\infty}^{\infty} \alpha \left[ J_n(\alpha \rho') - \frac{b(n)}{d(n)} H_n^{(1)}(\alpha \rho') \right] H_n^{(1)}(\alpha \rho) e^{-\alpha_1 |z-z'|} \alpha_1 \right. \, d\alpha, \] (6.66)

where \( H_n^{(1)} \) are Bessel functions of the third kind (Hankel functions), \( \alpha_1 = \sqrt{k^2 - \alpha^2} \), \( k^2 = -i\omega \mu_0 \sigma_1 \) and

\[ b(n) = J_n(\rho_1) + i J_n(\rho_1), \quad d(n) = H_n^{(1)}(\rho_1) + i H_n^{(1)}(\rho_1). \] (6.67)
In order to describe the field inside the cylinder the Helmholtz operator must be applied to this scalar function to derive the dyadic Green's function.

Evaluation of equation (6.66) would be computationally intensive. It involves the infinite sum of an infinite integral. Therefore, some simplifying assumptions need to be made. As \( \rho_1 \to \infty \) in equation (6.66), \( b(n)/d(n) \to 0 \), \( J_0(\alpha \rho') \to 1 \) and \( J_n(\alpha \rho') \to 0 \) for \( n \neq 0 \). Equation (6.66) is then seen to converge to the free-space scalar Green's function, since

\[
G(r|r') = \frac{e^{-ik|r-r'|}}{|r-r'|} = \int_{-\infty}^{\infty} \alpha H_0^{(1)}(\alpha \rho) \frac{e^{-\alpha_1|r-r'|}}{\alpha_1} \, d\alpha. \tag{6.68}
\]
Chapter 7

The 3D Forward Problem

Knowledge may give weight, but accomplishments give lustre, and many more people see than weigh.

Philip Dolmer Stanhope
Earl of Chesterfield 1694 - 1773

A fully three-dimensional flaw model was developed using the theory from Chapters 5 and 6. This flaw model approximates solutions to the 3D forward problem: predicting signals from a known flaw, probe and inspection parameters. The implementation of this model is discussed in this chapter, along with various validation exercises that have been carried out in an attempt to help validate the theory and code. There is a short applications section at the end of the chapter discussing further applications of the code.

Since the volume integral method is inherently inflexible when it comes to variations in the workpiece geometry, these variations must be accounted for in the Green's functions used. Therefore, a specific collection of the more common NDE geometries, referred to as media types are specifically supported. They include a collection of media types with planar boundaries and another collection with cylindrical boundaries. The specific parameters and assumptions for these media types were discussed in Chapters 2 and 3.

7.1 Physical model

The general 3D forward problem must take into account variations in the $x$ or $y$ directions as well as the $z$ direction, which has already been examined. The new model must take into account this kind of variability. An approach to solving the 3D forward problem is to divide up the region containing the flaw into smaller sub-regions and then describe the eddy current interaction between each of these parts. There are three basic approximation techniques: finite elements, boundary elements and volume elements.
The finite element and the volume element approaches both divide the volume of the problem into elements. The finite element technique has the disadvantage of having to take into account volumes out to infinity, although variable element sizes are implemented easily with this method. Another disadvantage of finite elements is that the meshes are complicated and the number of unknowns is quite large. The volume element and the boundary element schemes both have the advantage of using a Green's function approach, that is the Green's function contains the boundary conditions that describe the basic geometry of the conducting volume.

The 3D forward model that has been developed is based on the volume integral approach. Only the flaw region is subdivided. For planar media physical models the defect volume is built up from a stack of cells of dimension $n_x \times n_y \times n_z$, each with a conductivity of $\sigma_{klm}$. This flaw physical model is shown in Figure 5.1. For cylindrical media types, the flaw volume is built up from a stack of cells of dimensions $n_x \times n_\theta \times n_r$.

### 7.2 Mathematical Model

It is assumed that the scattered field due to the defect resembles the field caused by a distribution of electric dipole densities $P_{klm}$, which are constant in each cell volume $V_{klm}$. The coil generates an incident field $P^{(i)}_{klm}$ in the conductor at each defect volume $P^{(i)}_{klm}$, again assumed constant on each cell volume $V_{klm}$. The incident field term for each of the probe models was discussed in Chapter 5. Cell interactions are determined by a matrix of elements $G$, discussed for each of the media physical models in Chapter 6, derived using the moment method with dyadic Green's functions. The resulting equation can be written as

$$GP = P^{(i)}; \quad \Delta Z = cP \cdot P^{(i)} \quad (7.1)$$

where $c$ is a constant. The second relationship uses the reciprocity theorem, which relates the field and dipoles in the conductor to the field and currents in the coil, to describe the change in impedance in the air-cored probe model, due to the presence of the defect.

Equation (7.1) can be rewritten in what may be a more familiar notation as

$$Ax = y; \quad \Delta Z = cxy \quad (7.2)$$

This linear system of equations represents the mathematical model of the 3D physical model for all of the media physical models and the air-cored probe.

### 7.3 Numerical Model

Figure 7.1 gives a conceptual representation of the numerical model and also acts as a flowchart for the computer program developed. Figure 7.1 shows the four major components necessary to solve the 3D forward model. The interaction between the individual cells
or elements is described by the free-space matrix elements and the reflection matrix elements. For the free-space elements the interaction is strictly between the elements themselves, for the reflection terms the elements interact by a reflection off the surface of boundaries of the host region. Combined, these matrices describe how the incident field is scattered by the defect. For a solution with small discretization errors, the smallest cell dimension should be no greater than a third of a skin depth.

### 7.3.1 Special structure of the matrix elements

Once the linear system has been determined it is solved using a conjugate gradient algorithm[57,7]. To solve equation (7.2), the algorithm calls for repeated evaluation of matrix-vector products of the form $Ax$ and $A^*s$, where $A^*$ is the conjugate transpose of $A$. These products can be carried out taking advantage of the Toeplitz structure of $A$. A Toeplitz matrix is a matrix where the coefficients are constant down each of the diagonals of the matrix. A $4 \times 4$ Toeplitz system can be written as

\[
\begin{bmatrix}
    y_1 \\
    y_2 \\
    y_3 \\
    y_4
\end{bmatrix} =
\begin{bmatrix}
    a_0 & a_1 & a_2 & a_3 \\
    a_{-1} & a_0 & a_1 & a_2 \\
    a_{-2} & a_{-1} & a_0 & a_1 \\
    a_{-3} & a_{-2} & a_{-1} & a_0
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix}.
\] (7.3)

There are only 7 distinct values in the matrix $a_{-3}, \ldots, a_3$. More generally, for an $n \times n$ Toeplitz matrix there are only $2n - 1$ distinct values in the matrix. By taking advantage of this structure a dramatic reduction in storage requirements can be achieved.

A Hankel matrix is similar to a Toeplitz matrix except coefficients are constant on diagonals from the top, right corner to the bottom, left corner of the matrix. So a $4 \times 4$
Hankel matrix could appear as

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3 \\
  y_4
\end{bmatrix} = \begin{bmatrix}
  a_{-3} & a_{-2} & a_{-1} & a_0 \\
  a_{-2} & a_{-1} & a_0 & a_1 \\
  a_{-1} & a_0 & a_1 & a_2 \\
  a_0 & a_1 & a_2 & a_3
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix}.
\]

Carrying out the matrix-vector operation in equation (7.3) is a convolution, while the matrix-vector operation in equation (7.4) is a correlation. The convolution theorem states that for functions \( f(t) \) and \( g(t) \)

\[
(f \ast g) \Leftrightarrow \tilde{f} \tilde{g},
\]

where \( \Leftrightarrow \) match Fourier transform pairs. Forming sequences \( f = \{0, a_{-3}, a_{-2}, a_{-1}, \ldots, a_3\} \), \( f^\dagger = \{0, a_3^*, a_2^*, a_1^*, \ldots, a_{-3}^*\} \) and \( x = \{x_1, x_2, x_3, x_4, 0, 0, 0, 0\} \), the convolution and correlation can be carried out in Fourier space by \( \tilde{f} \tilde{x} \) and \( \tilde{f}^\dagger \tilde{x} \) respectively, where \( \tilde{\cdot} \) represents the discrete Fourier transform. These operations can be done very efficiently with fast Fourier transforms (FFT). The example shown here is one dimensional, but extends quite easily to the three spatial dimensions in the forward problem at hand requiring three-dimensional FFT’s.

There are several implications of this approach. The matrix \( A \) is never assembled, the solution vector \( x \) internally is 8 times larger due to the padding in each variable and the number of unknowns in each spatial dimension must be powers of two so that FFT’s can be used, however; the number of cells in the \( x \) and \( y \) direction in this formulation is odd, so \( n_x \) and \( n_y \) must be of form \( 2^i - 1 \) for some \( i \).

### 7.3.2 Conjugate gradients

The conjugate gradient algorithm has been implemented using the operators as described above. Due to the iterative nature of this algorithm, at each pass a solution and a rough estimate of the error is available. This allows for quick and dirty solutions, as well as allowing more iterations for very precise answers. The number of iterations and the maximum allowable error can be individually set.

### 7.3.3 Cubic spline interpolation

For planar media types and for fixed depth \( z_m \), the incident field \( P^{(i)}(\rho, z_m) \) is axially symmetric. The incident field must be computed on a raster grid, resolved into its \( x \) and \( y \) components. To avoid evaluating a Fourier integral for each pair \( (x_k, y_l) \), we sample the field at several points and then fit these points with a cubic spline. Then the field is simply interpolated to find \( P^{(i)}(\rho_{kl}, z_m) \). The Surrey functions, discussed in Chapter 6 are handled the same way.
7.4 Validation Exercises

To test whether the physical model and/or the computer code can reliably be used to predict eddy current probe signals extensive validation exercises have been carried out. The predictions of the computer model have been compared to analytic results and with international benchmark results and the comparisons appear in the following sections.

7.4.1 Disc in a constant field

In a constant unidirectional field, a crack of negligible opening acts as a surface barrier to the flow of eddy-current producing a discontinuity in the field that can be represented in terms of a surface distribution of current dipoles[12], $P(r) = e_n P(r)$ where $e_n \equiv e_x$ is the unit vector normal to the crack. The dipole distribution acts as the effective source of the scattered field which we express in integral form writing the total field as

$$E(r) = E^{(i)}(r) + i\omega\mu_0 \int_{\text{crack}} \tilde{G}^0(r|r') \cdot P(r') \, dS'$$  (7.6)
Where \( \mathcal{G}^0(\mathbf{r}|\mathbf{r'}) \) is the free-space dyadic Green's function for a source in the conductor and \( \mathbf{E}^{(i)}(\mathbf{r}) \) is the electric field in the absence of the flaw.

If an infinitely thin disc of radius \( a \) defines the crack and the incident field is a constant value \( \mathbf{E}^i = ik\mathbf{H}_0 \), normal to the disc, then a series solution for \( P(\mathbf{r}) \) to 7.6 can be developed[9]. The solution is radially symmetric and the first term of which is

\[
p(\rho) = \frac{4ia|k\mathbf{H}_0|}{c\pi} \sqrt{a^2 - \rho^2} \cdot \mathbf{e}_z,
\]

where \( c \) is the thickness of the disc. For cracks where \( a \ll \delta \), \( \delta \) being the skin depth, this term dominates the series and is a good approximation to the field response.

Although we cannot model 2-dimensional objects with the 3-D model, we can look at narrow cracks of openings \( c \), that approach the 2-dimensional idealization, if \( c \ll a \). This problem is attractive because the solution does not depend on the reflection matrix elements or the coil incident field. Therefore, it isolates 2 of the major computations in the code and provides us with an approximation to the dipole distribution \( P(\mathbf{r}) \) instead of simply the change in impedance.

The model was run with the following parameters: \( \delta = 1 \text{ m}, a = 10 \text{ mm}, c = 1 \text{ mm}, \sigma = 10^4 \text{ S/m}, f = \frac{250}{\pi} \text{ and } \mathbf{E}^i = \frac{25}{4\pi} \mathbf{e}_z \). According to the theory, the dipole distribution resulting from these values is a hemispheroid of height unity and a base the size of the disc. Figure 7.2(a) shows the calculated \( x \)-component of the dipole distribution. Figures 7.2(b) and (c) show the \( y \) and \( z \) components of the dipole distribution. In the limit of zero crack opening \( P_y \) and \( P_z \) are zero. The fact that the computed distributions are non-zero is due to the finite crack opening, but they are only 7% of the \( x \)-component values. The complimentary nature of these components would also be expected. Since the analytic result is a truncated series expansion, the exact error is unknown; however, the model’s prediction is within 4% at the centre of the disc. Figure 7.2(d) shows the model prediction compared to the analytical approximation at the centre of the disc. It clearly indicates the piece-wise nature of the solution.

### 7.4.2 Half disc embedded in half-space in constant field

If we now consider the case where there is an infinitely thin half-disc of radius \( a \) embedded at the surface of half-space conductor with the same incident field as before. The dipole density at the flaw is again given by equation (7.6). The calculation must contain the effect of the reflection at the surface. This can be done by adding the reflection half-space dyadic Green's function \( \mathcal{G}^R(\mathbf{r}, \mathbf{r'}) \) as

\[
\mathbf{E}(\mathbf{r}) = \mathbf{E}^{(0)}(\mathbf{r}) + i\omega\mu_0 \int_{\text{crack}} \left[ \mathcal{G}^0(\mathbf{r}, \mathbf{r'}) + \mathcal{G}^R(\mathbf{r}, \mathbf{r'}) \right] \cdot \mathbf{P}(\mathbf{r'}) \, dS'
\] (7.8)

At low frequencies the reflection term should be simply the term that is added if the boundary was removed and the half-disc reflected into a whole disc. Therefore, the
solution to equation (7.6), in the conductor, should be the same as the entire disc for $z \leq 0$ and zero for $z > 0$.

Using the same parameters as before, the solution should be half a hemispheroid of unit height with the half-disc base. The $x$-component of the computed dipole distribution is shown in Figure 7.3(a), which in the low frequency limit should be the same as the previous solution for $z \leq 0$. The $y$ and $z$ components are shown in (b) and (c) of the same Figure. Again, these components are zero in the analytic expression, but due to the finite opening they should be the same, but defined over half the range. Figure 7.3(b) shows the input incident field. Notice how the constant field has been modified at the boundary by the flaw volume fraction of the boundary elements. It is also important to notice that the free-space matrix elements are exactly the same as before. Since the conjugant gradient solution and the free-space matrix element computation have been confirmed for the whole disc problem, the half-disc problem isolates the reflection matrix element terms. Therefore, providing strong evidence that at low frequencies the reflection terms have been computed correctly and have been included in the 3D-model correctly.
Figure 7.3(d) shows the comparison of the $x$-component at the top of the half-disc, again indicating the piece-wise constant nature of the approximation.

### 7.4.3 Layered half-space

![Graph showing impedance response comparison](image)

Figure 7.4: Impedance response of the 1D model compared to 3D model for a layered half-space

![Graph showing phase response comparison](image)

Figure 7.5: Phase response of the 1D model compared to 3D model for a layered half-space
Another analytic result is available, the layered media forward model. In order to use the results of the layer model to compare with the 3D model the defect volume must be extended at least one skin depth beyond the edge of the coil to avoid edge effects. A layer of IGA was simulated by two $15 \times 15$ layers of cells each of dimension $0.7 \times 0.7 \times 0.35$ mm with a constant conductivity $0.65 \times 10^6$ S/m. A coil of outside radius 1.8 mm and inside radius 0.8 mm was placed in the centre of this flaw and the response computed at three frequencies. Figures 7.4 and 7.5 show the absolute impedance and phase response of the coil to this layer of 0.7 mm deep IGA. The layer model was run for the same coil parameters and with a layer of 0.7 mm depth and the same conductivity as above. As can be seen in the figures, there is very good agreement between predictions for the 3D model and the layer model.

### 7.4.4 The infinite strip

Suppose we have an infinite strip in the complex $z$-plane ($z = x + iy$) with a width $2a$, centred at the origin. If we determine the electric field $E(r)$, the current distribution follows from $J(r) = \sigma E(r)$ when $\sigma$ is constant, i.e. the material is isentropic. All that it is necessary to know about $E$ can be derived from a complex potential $\Psi(z)$ whose real and imaginary parts are $u(z)$ and $v(z)$, so

$$
\Psi(z) = u(z) + iv(z). \quad (7.9)
$$

The function $u(z)$ is the ordinary electric potential which satisfies

$$
E(r) = -\nabla u = \left( \frac{\partial u}{\partial x} \hat{x} + \frac{\partial u}{\partial y} \hat{y} \right) \quad (7.10)
$$

and a line defined by $u(z) = c$, for a constant $c$, is an equipotential line. The lines defined by $v(z) = c$ are the field lines, they cut the equipotentials at right angles[48].

We are interested only in the case of $E$ being uniform as $|z| \to \infty$. Then the complex potential is

$$
\Psi(z) = E_0 \left[ z \cos \theta - i\sqrt{z^2 - a^2} \sin \theta \right] \quad (7.11)
$$

and

$$
u(z) = E_0 \left[ x \cos \theta + \text{Im}(\sqrt{z^2 - a^2} \sin \theta) \right]. \quad (7.12)
$$

Consider points where $z = x + i\epsilon$, where $\epsilon$ is a positive and real such that $\epsilon \ll a$ and $|x| < a$. Then

$$
\sqrt{z^2 - a^2} = (x^2 - \epsilon^2 - a^2 + 2i\epsilon x)^{\frac{1}{2}} \\
\approx (x^2 - a^2)^{\frac{1}{2}} \left[ 1 - \frac{i\epsilon x}{a^2 - x^2} \right] \\
= (a^2 - x^2)^{\frac{1}{2}} \left[ \frac{\epsilon x}{a^2 - x^2} + i \right]. \quad (7.13)
$$
The change in the tangential field across the strip is
\[
\Delta \left( \frac{\partial u}{\partial x} \right) = \left( \frac{\partial u}{\partial x} \right)_+ - \left( \frac{\partial u}{\partial x} \right)_-
\]
\[
\frac{2\pi E_0}{\sqrt{a^2 - x^2}} = -\Delta E_t, \tag{7.14}
\]
since \( \theta = \pi/2 \). The surface dipole density distribution across the infinite strip is then
\[
\begin{align*}
p &= \sigma_0 E_0 \int \Delta E_t \, dx \\
&= \sigma_0 E_0 \int \frac{2\rho}{\sqrt{a^2 - x^2}} \, dx \\
&= -2\sigma_0 E_0 \sqrt{a^2 - x^2}.
\end{align*}
\tag{7.15}
\]
The volumetric dipole density distribution is
\[
P(r) = -\frac{2\sigma_0 E_0}{\Delta c} \sqrt{a^2 - \rho^2} \cdot \hat{z}, \tag{7.16}
\]
where \( \Delta c \) is the crack opening.

This result is similar to the disc and half-disc problem used for validating the free-space and half-space matrix element computation. The major difference is that here, the barrier is infinite in one of its dimensions. This result can still be used to validate the matrix element computations for the slab.

We will study a non-conducting strip, infinite in length, with a half-width of 10 mm and a thickness of 1 mm embedded in a conductor with conductivity \( \sigma = 10000 \) S/m. Equation (7.16) describes the dipole density distribution that will be set up across this non-conducting region when a constant field of strength \( E_0 \) and frequency of 25.3333 Hz is injected normal to the strip. The skin depth in this case is 1 m and the strip is 10 times wider than thick, so the assumptions that \( a \ll \delta \) and \( \Delta c \ll a \) are satisfied. The shape of the dipole density distribution will be that of half of an ellipsoid with a base of radius \( a \) and height of \( 20\sigma_0 E_0 \). If the injected field strength is chosen to be \( \sigma_0/20 \), the height of the semi-ellipse should be unity.

The 3D model cannot be used to study infinite objects; however, the infinite strip can be approximated by studying a long strip. If the strip is long enough (long enough for most of the eddy currents to flow around the strip and not go over and under the ends) the results in the centre will be valid. Figure 7.6 (a) show the free-space dipole density distribution for a strip that is half a metre long. The strip was divided into 1 element of 1 mm across the opening, 15 elements of 1.3333 mm each along the width of the strip and 16 elements of length 31.25 mm each along its length. Only the last three rows of elements at each end of the strip show any deviation from the semi-ellipse shape, which peaks as expected at unity. The dipole distribution from an infinite non-conducting strip embedded in a half-space conductor is shown in Figure 7.6 (b). Due
to the low frequency limiting behaviour, the reflection matrix elements act as perfect reflectors and the dipoles at the surface of the strip should show the same behaviour as the centre line of the infinite strip.

Figure 7.6 (c) shows predicted dipole distribution for a square of half-length 10 mm normal to a constant field, while Figure 7.6 (d) shows the dipole distribution for half of the square embedded in a half-space conductor. Again the distribution at the surface is the same as along the centre line of the distribution from the non-conducting square in free-space. Notice how the dipole distribution has dropped below unity due to the fact that eddy currents now flow around sides of the square in free-space and underneath the half square in half-space.

One way of preventing the currents from flowing underneath the half square is by adding another surface at its bottom, This can be done by embedding half the square into a slab that is 10 mm thick. Just like the top surface, the bottom surface will act as a perfect reflector, because of the low frequency. Unlike the half-space case the reflections will interact, providing multiple reflections. The net result should be that...
the dipole density distribution of half of a non-conducting square embedded through a slab should have the same cross-section as the infinite strip in free-space. Figure 7.7 shows the predicted dipole distribution for this case and we see that the profile is again a semi-ellipse of height unity. By comparing Figures 7.7 and 7.6 (d) it is clear that the additional surface of the slab has increased the dipole density to unity at the lower surface, but the multiple reflections have contributed in just the right way to increase the dipole distribution at the top surface to unity.

This result provides a strong validation test for the matrix elements for slabs and hence all media types involving multiple reflections in the source region. Although the correction terms involving the Surrey functions have not been validated by this exercise, due to the low frequency, they are only second order terms. Their validation will have to be made in conjunction with comparisons to experimental results.

### 7.4.5 Air-cored coil over a machined slot

The Applied Computational Electromagnetic Society (ACES) and a series of Testing Electrical Analysis Methods (TEAM) Workshops have tried to address the problem of lack of benchmark tests for electromagnetic codes. In a joint effort, both have adopted an experiment conducted by S. K. Burke of an air-cored rectangular cross-sectioned coil above a block of aluminium with a rectangular machined slot[15,11,12,38,31]. The size of the slot and coil used is larger than those typically found in NDE, but this enabled the dimensions to be determined accurately. Measurements of impedance were made at regular intervals with the axis of the probe in the plane of the crack and the lift-off
constant. At the frequency used in making the measurements, 900 Hz, the skin depth was 3.03 mm and therefore of the same order of magnitude as the slot depth. The parameters of the experiment were as follow:

![Comparison of experimental results and predicted responses for various resolution of defect volumes](image)

Figure 7.8: Comparison of experimental results and predicted responses for various resolution of defect volumes

**Coil parameters**

- Inner radius = 6.15 ± 0.05 mm.
- Outer radius = 12.4 ± 0.05 mm.
- Height = 6.15 ± 0.1 mm.
- Number of turns = 3790
- Lift-off 0.88 mm.
- Frequency = 900 Hz.

**Specimen parameters**

- Conductivity = 3.06 ± 0.02 × 10⁷ S/m
- Thickness = 12.22 ± 0.02 mm.

**Flaw parameters**

- Length = 12.60 ± 0.02 mm.
• Depth = 5.00 ± 0.05 \text{ mm}.

• Width 0.28 ± 0.01 mm.

Burke’s data is presented in Figure 7.8 as o symbols. The predictions from the 3D model for various meshes of the flaw region are presented. The coarsest grid is 1 × 7 × 4; each finer grid gives a better response. The 1 × 15 × 8 and 3 × 15 × 8 grids which have similar traces. The best signal, which duplicates most of Burke’s result is the finest grid 3 × 15 × 16. This is a very encouraging result.

![Graph showing experimental results and predicted responses for various resolutions of defect volumes.](image)

Figure 7.9: Comparison of experimental results and predicted responses for various resolutions of defect volumes

The phase response is presented in Figure 7.9. The phase predictions for each of the grid resolutions are also presented, again showing very good agreement, especially with the finer grids. It is important to note that the coarse grid which has only 84 unknowns is very quickly solved, while the finest grid has 2160 unknowns and therefore requires much more effort to solve.

### 7.4.6 Differential coil over a machined slot

Another benchmark test from TEAM Workshops involves a differential probe[63,61, 27,29,30,4]. This probe has a driving coil surrounding two identical pick-up coils. The coil is shown in Figure 7.10. The physical model does not need to be changed for this problem since the pick-up coils are not driven and the dipole distribution is determined as before. The potential is determined in each of the pick-up coils by taking the inner product of the dipole distribution with the incident field created by each pick-up coil.
7.4. VALIDATION EXERCISES

Figure 7.10: Differential probe

Figure 7.11: Comparison of measured and predicted magnitude of differential voltage responses

from a virtual current in the coils. This field is generating using reciprocity theorem and is explained in more detail in Chapter 6.

A 0.5mm x 40mm x 10 mm slot was machined into a 285mm x 330mm x 30 mm slab of austenitic steel with a conductivity of 1.4 \times 10^6 S/m. Figure 7.11 shows the normalized absolute voltage measured by Vérité[63] for the scan parallel to the slot and for the scan perpendicular to the slot. The line joining the centres of the pick-up coils is always in the direction of the scan. The data is determined to within a scalar multiple and a fixed rotation, so the model predictions for two different mesh sizes and the data shown in Figure 7.11 have been normalized to unity. The model shows good agreement.

Figure 7.12 shows the phase response for parallel and perpendicular scans. Again there is general agreement, both experimental traces show large edge effects as the probe nears the edges of the slab and these probe positions were not modelled. The thickness
of the slab is 30 mm and with the skin depth at 19 mm for these scans the reflection of the bottom surface does not contribute much to the signal. The effect of this surface can be evaluated by comparing the half-space model to the slab model.

Figure 7.13 compares the predicted results using the two physical models for the media. One pair of curves in the Figure represent the predicted differential voltages when a half-space model is used for the normal and parallel scans. A second pair of curves in the Figure represent the predicted differential voltages when a slab model was used. Both pairs are plotted against the measured results. Since the plate is approximately 1.5 skin depths deep, the presence of the bottom surface is not an important effect, which is shown by the Figure. The parallel scan shows that there is very little difference between the models, while the normal scan does give a better data match than the half-space model.
The validation exercises have shown that the 3D model does indeed provide high quality quantitative responses to NDE problems. The model can now be used for studies that would be expensive in time and money to run in the laboratory. Three examples of such studies follow.

### 7.5.1 Probe response as a function of crack opening

The Burke slot and coil were used to study the effect of slot opening on the probe response. A 7 x 7 flaw grid was used and after each run another layer in the yz plane of the flaw grid was reset to the base conductivity $\sigma_2$, thereby narrowing the slot. The impedance was measured with the coil centred on the slot and with the coil 9 mm along the slot where the peak impedance response is measured. The probe response is plotted against slot opening in Figure 7.14 at the centre of the slot and at scan point with maximum signal.

![Graphs showing probe responses as a function of crack opening](image)

**Figure 7.14: Probe responses as a function of crack opening**

The results show a linear sort of behaviour, excluding the case where there is only
one element across the slot (the dipole distribution is harder to approximate with only one cell across the slot). The curves predict a non-zero limit with respect to slot opening.

### 7.5.2 Probe response as a function of cross crack bridging

The effect of cross crack bridging was studied, again using the Burke slot and coil. For this example a $1 \times 7 \times 4$ grid was used with four of the top seven elements set to the host material conductivity $\sigma_2$. This simulates smearing or bridging that might occur during surface preparation. Half of the slot is sub-surface, although the bottom of the slot is of the same depth. The predicted response of the probe scanning along the slot is presented in Figure 7.15 plotted against the predicted response with no bridging.

![Figure 7.15: Probe responses in impedance plane with bridged and unbridged slot](image)

### 7.5.3 Probe response to IGA

Unlike cracks or slots which are modelled with cells with zero conductivity, IGA can be modelled with cells of lower conductivity than the host material. Complex shapes like a hemispheroid, with circular cross-sections in the $xy$ plane and semi-ellipse cross-sections in the $xz$, can be modelled by using volume fractions for weighting the conductivity in each cell for the cells that intersect the boundary. A hemispheroid of base radius 7 mm and height of 1.75 mm was modelled. The predicted raster scan response is shown in Figure 7.16.

Any shape of IGA make up of rectangular cells can be investigated. The IGA profile on the left of Figure 7.17 was raster scanned and the predicted response is shown on the right side of the figure.

### 7.6 Predicted ACPD signals
7.7 CYLINDRICAL MEDIA RESULTS

Figure 7.16: Probe response to hemispheroid of IGA

Figure 7.17: Complex IGA patch and predicted probe response

Figure 7.18 shows the ACPD predicted signal for a 0.28 mm × 12.6 mm × 5 mm flaw in aluminium[14]. The predicted shape is similar to those reported in the literature[54, 38,39]. Notice how the step change in the signal occurs only when the probe spans the flaw and goes to a constant value from the flaw. Also, when the legs of the probe pass over the flaw, there is no signal, which accounts for the null point in the signal.

Figure 7.19 shows the variation of the ACPD signal V(0) as the length of a slot with a width of 0.28 mm and depth of 5 mm is varied. Three different meshes were used to generate the data. Figure 7.20 shows the variation of the ACPD signal V(0) as the depth of a slot with a width of 0.28 mm and length of 30 mm is varied. In this case only two different meshes were employed to cover the range of values used.

7.7 Cylindrical Media Results
Suppose that there is a flaw of dimensions 0.28 mm × 12.6 mm × 5 mm slot protruding outwards from the centre of a 0.8 mm diameter bore-hole in austenitic steel with conductivity $1.3 \times 10^6$ S/m and relative permeability of unity. Suppose further that the coil #7, described in Table 4.1, is inserted into the bore-hole and centred on the slot and assumed to be sufficiently far away from the ends of the bore at all times. The solid curve in Figure 7.21 traces out the predicted magnitude of the impedance change $Z$ in the coil as the coil is scanned along the bore axis. In the Figure, the slot end would occur at 6.3 mm.

The other two curves in the Figure 7.21 correspond to the same slot and probe in two different tubes. The first tube has a 5 mm wall thickness, hence the flaw is through wall, while the second tube has a 6 mm wall thickness and the flaw has only an 83.33% through wall extent. The tube media type was used for both predicted curves.
7.7. CYLINDRICAL MEDIA RESULTS

Figure 7.19: ACPD signal variation vs. slot depth

Figure 7.20: ACPD signal variation vs. slot length
Figure 7.21: Cylindrical media model predictions
Chapter 8

The Inverse Problem: Applications to IGA

Not only is there one way of doing things rightly, but there is only one way of seeing them, and that is, seeing the whole of them.

John Ruskin 1819 - 1900

This chapter presents a novel approach at solving the inverse problem for inverting multi-frequency impedance measurements to determine layer thickness and material properties. The approach uses the layered half-space forward model to compute predicted probe responses for a set of layer parameters and then uses the Levenberg-Marquardt method to minimize an error function to invert this non-linear equation. Eaton[19] uses an alternative approach for inverting non-linear systems, but the Levenberg-Marquardt scheme was chosen because of its simplicity. This method requires gradient information from the forward model. Norton and Bowler[42] approximate the gradient by solving an associated adjoint problem, which is computationally equivalent to solving the forward problem itself. This is an attractive approach, but due to the simple algebraic expressions of the model, gradients are performed analytically and then approximated. The layers half-space conductor was the only media type used; however, there is no reason this model cannot be extended to other layered planar types or to cylindrical media types.

The application is driven by the desire to measure the depth of intergranular attack (IGA) in large diameter austenitic piping using eddy current inspection techniques. The corrosion is cause by environmental effects on the exterior of the pipe. The corrosion is detectible because the attack at the grain boundaries decreases the conductivity locally[41,45]. It is this variation, reported by Babcock Power Ltd. to be from 98 to 13
percent of the host conductivity[49], that is used for detection. Therefore, detection of IGA using eddy current techniques is well established.

In order to form an inverse model, there must be a physical model of IGA. The one adopted is just the layered half-space media type described in Section 2.7.1. This model assumes that at least locally, on the scale of the probe diameter, that the IGA appears to be a uniform layer. The inverse model then takes the impedance measurements, from possibly several frequencies and predicts the model parameters, \( l, \sigma \) and \( \mu \), where these parameters represent the layer depth, conductivity and permeability, respectively.

This physical model of IGA was adopted and the following section describes inverse model and its implementation. Later in this chapter several validation exercises are carried out with experimental and synthetic data to validate the layered half-space inverse model, the IGA inverse model. It is shown that model is accurate if assumptions are made on the size and boundaries of the IGA relative to the probe size. Finally, it is shown how the layered half-space forward model can be used to optimize coil size and scanning parameters for IGA inspection.

8.1 Layered half-space media inverse problem

8.1.1 Mathematical model

Suppose that \( m \) measurements of the absolute impedances, \( |\Delta Z|_\alpha \), at frequencies \( \omega_\alpha \), \( \alpha = 1, \ldots, m \) are taken. Suppose further that these measurements can have random noise from various sources. The inverse problem, in this situation, is the problem of finding the parameters \( \sigma, \mu \) and \( c \) that will satisfy equation (4.1) at each frequency or in some sense give the best approximation at each frequency.

Referring again to equation (4.5), the response \( \Delta Z \) depends on the desired parameters in a non-linear way. Non-linear inversion is not a well understood problem. It is certainly no longer clear if the problem is one-to-one and therefore whether it is well-posed. In order to avoid these issues another approach was implemented. This approach is described in the next section.

8.1.2 Levenberg-Marquardt least square algorithm

Due to inadequacies in the physical model and errors in the data it would be unreasonable to expect equation (4.1) to be satisfied exactly at any or all frequencies measured. A measure of the quality of the fit is needed. One method is to define a merit function \( \chi^2 \) which measures the degree of fit to the data, being small if the fit is good, large otherwise. Then the inverse problem can be recast as a minimization of the merit function over the parameter space to find the best fit.

A good example which illustrates this approach is the task of fitting a straight line
to some data. Suppose that the two sets of data \( \{x_i, f_i\} \) shown in Figure 8.1 have been collected and that the errors in the data are normally distributed. The 1, 2 and 3 standard deviation bands are shown around each point in Figure 8.1. Fitting a straight line implies a mathematical model of the form

\[
\begin{align*}
    f(x) &= ax + b, \\
    \chi^2 &= \sum_{i=1}^{3} (f_i - f(x_i))^2,
\end{align*}
\]

minimizing \( \chi^2 \) for both data sets will find \( a = 0 \) and \( b = 1 \). These fits, the lines, are shown in Figure 8.1. The question of how well the model fits the data can be answered quantitatively as discussed in Section 8.1.3

Because the function \( f(\omega; \sigma_1, \mu_1, c) \) is non-linear in its parameters, the Levenberg-Marquardt method was used. This is the standard method used for non-linear least square problems[44]. The Levenberg-Marquardt method is a weighted combination of two algorithms, the inverse Hessian and the steepest descent method. The relative weights given to each method is controlled by the use of the parameter \( \lambda \). It assumes that the topology of the parameter space is locally linear and uses gradient information to find the appropriate direction of search. The parameter \( \lambda \) allows the algorithm to search beyond local minima.

Figure 8.1: Linear model \( f(x; a, b) = a + bx \), where \( a = 0 \) and \( b = 1 \). Each circle is a multiple of one standard deviation from the data point.
### 8.1.3 Statistical measures

The quantitative measure of the *goodness-of-fit*, is the probability function \( Q(\chi^2|\nu) \), where \( \chi^2 \) is the achieved value of the merit function and \( \nu \) is the number of *degrees of freedom* (the number of data points minus the number of parameters). This function measures the probability that the data set could have occurred for the given model and error distributions. For the first data set in Figure 8.1, the goodness-of-fit would be quite large, approaching unity, as can be seen by the fact that the line falls within one standard deviation (the inner circles in the figure) of all the data points; however, for the second set of data \( Q(\chi^2|\nu) < 1 \), because the line does not pass even within 3 standard deviations (the outer circles) of all the data points. The small value of \( Q(\chi^2|\nu) \) implies that the model is inappropriate to the data, or that the data is bad. Goodness-of-fit measures below 0.1 are questionable, while goodness-of-fit measures consistently around 1 are suspiciously high[44].

Once the global minimum has been found the Levenberg-Marquardt algorithm allows the correlation matrix to be computed. This square matrix has the dimension of the number of search parameters. The correlation matrix provides information about the independence of each parameter and also gives the variance of each of the search parameters. This information can be used to put *confidence bands* on the search parameter for a given confidence level. For example, the depth of IGA could be found to be \( 8 \pm 0.1 \text{mm} \) at a confidence level of 66.6%. It is important to note that the confidence bands are only meaningful if the goodness-of-fit measure says the model is appropriate.

### 8.1.4 Numerical model

Figure 8.2 gives a conceptual representation of the numerical model and also acts as a flow chart for the computer programme that has been written. The output of this algorithm is the physical and material properties of the layer along with the statistical
8.1. LAYERED HALF-SPACE MEDIA INVERSE PROBLEM

Information discussed in 8.1.3.

The Levenberg-Marquardt $\chi^2$ method cannot be implemented without having information about the first derivatives of the response function with respect to search parameters. The response function in equation (4.5) can be differentiated with respect to $\sigma_1$, $\mu_1$ and $c$. These derivatives are

\[
\frac{\partial f}{\partial \sigma_1} = \frac{\omega \pi \mu_0 n^2}{A^2} \left[ \int_0^\infty \frac{\Psi^2}{\alpha^2} \left[ e^{-\alpha l_2} - e^{-\alpha l_1} \right]^2 \left( \frac{\partial \Gamma}{\partial \sigma_1} - \Gamma_0 \right) \, d\alpha \right] \tag{8.3}
\]

\[
\frac{\partial f}{\partial \mu_1} = \frac{\omega \pi \mu_0 n^2}{A^2} \left[ \int_0^\infty \frac{\Psi^2}{\alpha^2} \left[ e^{-\alpha l_2} - e^{-\alpha l_1} \right]^2 \left( \frac{\partial \Gamma}{\partial \mu_1} - \Gamma_0 \right) \, d\alpha \right] \tag{8.4}
\]

\[
\frac{\partial f}{\partial c} = \frac{\omega \pi \mu_0 n^2}{A^2} \left[ \int_0^\infty \frac{\Psi^2}{\alpha^2} \left[ e^{-\alpha l_2} - e^{-\alpha l_1} \right]^2 \left( \frac{\partial \Gamma}{\partial c} - \Gamma_0 \right) \, d\alpha \right] \tag{8.5}
\]

Each of these derivatives is an improper integral and can be approximated in the same way as equation (4.5); however, since the derivative information is used only to determine the direction of search, these derivatives need not be computed with high precision.

8.1.5 Merit function in a complex world

The Levenberg-Marquardt algorithm is a real algorithm and the impedance responses are complex, therefore the algorithm must be modified or adapted.

When collecting data from an eddy current probe there is a considerable amount of noise due to probe positioning and surface irregularities. This noise, however, tends to be scattered in a co-linear way in the direction of lift-off. Lift-off is the direction the signal takes in the impedance plane when the probe is lifted off the surface. The cleaner signal is the signal normal or in-quadrature to this direction. Each measurement can be divided into the two signals, the in-phase and in-quadrature signals, each with its own standard deviation. If the data was presented as in Figure 8.1 the standard deviation bands would be ellipses not circles, with the minor axis the in-quadrature deviation and the major axis the in-phase deviation.

In light of the above, the merit function is defined as

\[
\chi^2 = \sum_{j=1}^m \left[ \left( \frac{\text{quad}(\Delta Z_i - f(\omega_i; \sigma_1, \mu_1, c))}{q_j} \right)^2 + \left( \frac{\text{phase}(\Delta Z_i - f(\omega_i; \sigma_1, \mu_1, c))}{p_j} \right)^2 \right]. \tag{8.6}
\]

where quad() and phase() give the in-quadrature and in-phase components of their arguments with respect to lift-off direction and $m$ is the number of frequencies. $q_j$ and $p_j$ are the standard deviations of the data for the quadrature and in-phase components respectively.

One immediate advantage of this approach is that one frequency inversions can be made for the two unknowns, depth $c$ and conductivity of the layer $\sigma_1$, since we have two relationships per frequency. Another advantage is the in-phase component, having
a much higher standard deviation, will be given less weight in the merit function. Using this $x^2$ merit function provides a search algorithm that has been found to be very robust and to converge quickly to the global minimum of the merit function.

8.2 Validation of the numerical model

8.2.1 Uniform layers in laboratory specimens

Three hexagonal blocks of austenitic steel were uniformly attacked with a corrosive agent to form IGA on all their surfaces. The depth and uniformity of the attack was then measured destructively on some of the edges. Coils #6 and #7 were used repeatedly to measure $\Delta Z$ due to the presence of the IGA at three different frequencies. The data was then split into its in-phase and in-quadrature signals with respect to the theoretical lift-off direction. The theoretical value is used because it is in good agreement with experimental results and it simplifies the use of the model. It eliminates a separate experimental measurement at each frequency. The results are presented in Table 8.1[33]. Due to the sample sizes, and the shallow depths of penetration of attack, the larger coils were unsuitable for this experiment.

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<th>In-phase Avg.</th>
<th>Quad SDev.</th>
<th>In-phase SDev.</th>
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Table 8.1: Laboratory data, $|\Delta Z|$(Ohms) for Probe #6

The data in Table 8.1 was then used with the IGA inverse model and the depth and conductivity of the corrosion were predicted. These inverted parameters are presented in Table 8.2 along with the goodness-of-fit measure. The confidence level used was 99%. However, only if the goodness-of-fit measure is large can the parameters be used with any degree of confidence. The discussion in Section 8.1.3 points out that numbers greater
8.3. VALIDATION OF THE PHYSICAL MODEL FOR GENERAL IGA

than or equal to 0.1 are reasonable. The depth predictions are in good agreement with the depths that were measured destructively.

| Coil # 6 | Sample | Inv depth (mm) | Inv σ₁ (S/m) | \(Q(\chi^2|ν)\) |
|----------|--------|----------------|--------------|------------------|
| 0.3 mm   | 0.305 ± 0.066 | 0.86 \times 10^6 | 0.11         |
| 0.4 mm   | 0.296 ± 0.078 | 0.88 \times 10^6 | 0.13         |
| 0.6 mm   | 0.812 ± 0.076 | 0.79 \times 10^6 | 0             |

| Coil # 7 | Sample | Inv depth (mm) | Inv σ₁ (S/m) | \(Q(\chi^2|ν)\) |
|----------|--------|----------------|--------------|------------------|
| 0.3 mm   | 0.216 ± 0.040 | 0.69 \times 10^6 | 0             |
| 0.4 mm   | 0.354 ± 0.053 | 0.94 \times 10^6 | 0.08         |
| 0.6 mm   | 1.04 ± 0.085 | 0.82 \times 10^6 | 0.05         |

Table 8.2: Inverted results for Probes #6 and #7

Figure 8.3 presents the data from Table 8.1 in the impedance plane for the 0.3 mm sample using coil #6. Figure 8.3(a)-(c) compare the experimental results with the inverted results at each of the test frequencies. The major and minor axes of the 1 standard deviation ellipses are also presented in the Figures, these ellipses correspond to the circles in Figure 8.1. The major axes at all frequencies are in the direction of lift-off. Notice how the scatter is almost entirely in-phase with lift-off at the two lower frequencies, but the highest frequency has much more scatter. At 400 kHz, the ellipse has degenerated into a line and the model is forced to select a point along the line. If a point had been selected that did not fall on the line, the goodness-of-fit measure would be very small.

These experimental results show that the numerical model does characterize the depth of the IGA of these samples. If the noise level in the data could be reduced, the model should provide even better predictions. This model does implement the mathematical model, minimizing the merit function, and gives good agreement with real world measurements with coils and with layered media problems.

8.3 Validation of the physical model for general IGA

In austenitic stainless steels, IGA is a form of environmentally assisted cracking that occurs in the grain structure; therefore, it is a reasonable to assume that there will be no subsurface patches of IGA. A feature of this type of cracking is that it tends to form as layers achieving a mature depth. From this description and the fact that "thick" objects are to be inspected, the layered half-space media would be a reasonable physical model for IGA. However, in practice multiple initiation sites lead to isolated patches which can then interlink. How can a physical model that assumes the IGA is a finite depth hope to
cope with situations like these? To understand how the model is effective, even in this more general setting, it is important to understand the electric field produced by the coil in the workpiece. Of course, the electric field is proportional to the eddy currents produced in the workpiece.

On surfaces of the conductor the electric field strength is zero at the origin, peaks under the coil and dies away in one to two skin depths in the radial direction or with depth. Figure 8.4 shows the average field strength over the first 0.35 mm of steel for coil #6 at 135 kHz. The skin depth at this frequency is 1.2 mm and the outer radius of the coil is 1.8 mm. As can be seen in the figure, the field has substantially died away after 5 to 6 mm. The field dies exponentially with depth, penetrating only around 2 skin depths. The important thing to notice is that the fields are highly localized. Material over 2 skin depths away has very little interactions with the induced eddy currents. Therefore the physical layer model really models annular piece of IGA on a half-space conductor. The impedance change is caused by the average over this volume; however, it is the material
8.3. VALIDATION OF THE PHYSICAL MODEL FOR GENERAL IGA

directly under the windings themselves that has the greatest influence. In section 8.1.1 it was stated that a necessary condition for the inverse problem to be well-posed was that it be one-to-one. One way of meeting this requirement is to use multi-frequency data, since at each frequency the volume of sampling is different.

![Incident field strength at surface from coil #6](image)

**Figure 8.4: Incident field strength at surface from coil #6**

8.3.1 Synthetic data from patches of IGA

![Synthetic data for a stepped IGA profile](image)

**Figure 8.5: Synthetic data for a stepped IGA profile**

One way of testing how well the physical model models different types of IGA is to use the 3D forward model to produce responses to known flaws. The first example is a 10.5 mm × 10.5 mm × 0.35 mm block of IGA next to a block of dimensions 10.5 mm × 10.5 mm × 0.7 mm. A scan was taken down the center of the blocks (y direction) starting
4.9 mm to the left of the blocks and calculating impedance at 0.7 mm increments until the probe is 4.9 mm to the right of the second block. The extent of the IGA in the $x$ direction is sufficient to prevent edge effects, hence this situation models IGA which is constant in the $x$ direction and has 3 step changes in the $y$ direction. The responses were computed at 2 frequencies and the results are plotted on the impedance plane in figure 8.5.

![Image](image)

**Figure 8.6:** 1 parameter inversion for a stepped IGA profile

This data was then inverted for depth only, keeping the conductivity of the layer fixed at 50% of the host material (the value used for the 3D forward model), first using only one frequency, then using both frequencies. The inverted results are shown in Figure 8.6 along with the depth of the IGA used in the 3D model. The inverted depths show very good agreement with the actual depth. There is very little change by the use of the additional frequency.

Although the data used for this inversion is not real world data, it was calculated using a volume integral technique which is different to the mathematical model at the heart of the inversion algorithm. There is, however, no noise in the signals. Since there is no data scatter, the standard deviation parameters $q_j$ and $p_j$ were arbitrarily chosen to be one tenth and one third of the $|\Delta Z|$, respectively, to simulate the fact that the component in phase with lift-off variations shows more noise.

The depth profile in Figure 8.6 appears to be a convolution of the actual depth with some sort of response function. This effect has been explored, although there is no theoretical basis for this convolution. To find this response function the impedance signal was computed for a pulse profile of IGA, an array with only one column of IGA. This data was then inverted and the results are shown in figure 8.7(a). This profile is the exact opposite of a layer, instead it resembles an infinite slot full of some conducting material. The layer model, not surprisingly has trouble inverting this profile.

The inverted depth profile is, in a sense, the point spread function for the coil at the
8.3. VALIDATION OF THE PHYSICAL MODEL FOR GENERAL IGA

Figure 8.7: Deconvolution with the inverted response from a pulse of IGA

specified frequency. Deconvolving the predicted depth signal with the response function implies an assumption that a super-position principle holds, which assumes that the incident field in the conductor is not affected by the presence of IGA, the Born approximation. Figure 8.7(b) shows the result of deconvolving the predicted depth profile by the response function in Figure 8.7. Although the signal is noisier, it does give an improved prediction.

Figure 8.8: 2 parameter inversion of stepped IGA profile

The IGA inversion model was run again using first one and then two frequency data from the 3D model, but this time both depth and conductivity of the layer were predicted. The algorithm first searches for the depth keeping conductivity fixed and, if the depth is greater than 0.01 mm, it then does a 2 parameter search. This avoids the degenerate case of inverting for the conductivity of a layer with zero depth. The results of this inversion are shown in Figure 8.8. Again there is little difference whether one or two frequencies are used. The conductivity curves in the figure are constant while the probe is over the IGA, but change abruptly near its boundaries.

Several other one-dimensional profiles were studied[31] by using the 3D model to predict $|\Delta Z|$ and then inverting for the depth. All profiles studied were at least as good
as the stepped profile and the smoother and more gradual the transitions the better the prediction. For example, the raster scan response to the hemispheroid of IGA shown in Figure 7.16 was inverted and the results are shown in Figure 8.9. The depth is in good agreement with the actual depth of 1.75 mm. The conductivity signal again indicates the boundary of the IGA with its abrupt change there and steadily falls near to the actual value of half the host conductivity.

In an attempt to understand the influence of surface roughness and other random noise sources, experimental data was taken both in the laboratory and in the field to find the background noise level in the absence of IGA. This is a statistical measure and it can be used in the IGA inverse model. It was found that by varying the lift-off randomly between -0.05 mm and 0.05 mm and varying the conductivity of the IGA randomly from 40% to 50% reduction of the host material, the amount of scatter could be
duplicated numerically. The resulting surface is shown in Figure 8.10. This patch of IGA was scanned, staying near the centre to avoid edge effects and the response predicted. The data and its standard deviations are shown in Table 8.3. The IGA inverse model predicted an IGA depth of $0.8 \pm 0.17$ mm with a conductivity of $0.726 \times 10^6$ S/m with a goodness-of-fit of a respectable 0.39, which compares very well to the actual depth of 0.7 mm and conductivity range of $0.62 \times 10^6 - 0.78 \times 10^6$ S/m.

<table>
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</tbody>
</table>

Table 8.3: Simulated noisy data, $|\Delta Z| (\text{Ohms})$ for Probe #6

8.4 Applications

8.4.1 Error distribution for IGA inverse model

![Figure 8.11: Relative error for inversion of depth and conductivity vs depth of IGA](image)

The 3D model was used to predict the probe response for different depths of IGA layers at 135 kHz using probe #6. These signals were input into the IGA inverse model to predict the depth and conductivity of the layer. The relative error in the inversion is plotted in Figure 8.11 against the thickness of the IGA. Although this study was conducted at one frequency the results should hold more generally.

The predicted depths are most accurate when the depth is just larger than one skin depth, with the predictions becoming more unreliable as the depth of the IGA increases or decreases from this neighbourhood. For depths very much smaller than a skin depth, the algorithm has trouble finding the global minimum in the parameter space due to the large derivative values there. This leads to the erratic behaviour of the error term.
in Figure 8.11(a). For depths larger than two skin depths, the field from the coil barely penetrates to the host material, so the inversion routine begins to think that the sample is just a half-space of IGA, i.e. the depth goes to infinity. This accounts for the dramatic increase in the error in this vicinity.

The error in inverting the conductivity of the layer is shown in Figure 8.11(b). The error behaviour for this term is quite different. For deep layers of IGA, as indicated before, only the IGA is sampled; therefore, the conductivity of the IGA is inverted quite accurately. However, as the depth of the layer decreases, there is more error in the inverted conductivity parameter; the model must now distinguish between the host material and the IGA. As indicated in Section 8.3.1 detecting a change in conductivity appears to be a good indicator of the presence of IGA. Therefore, the probability of detection (POD) curve for this model should be the inverse of the curve in Figure 8.11, i.e. will be small for shallow layers and approach unity for layer depths considerably smaller than a skin depth.

8.4.2 Optimizing operating frequency for a fixed coil

![Figure 8.12: Phase separation between lift-off and depth angle](image)

There is a large amount of lift-off noise in probe signal when using eddy current probes to scan for IGA. This noise is caused from positioning errors, side loading of the probe, surface irregularities, etc. To maximize the signal to noise ratio for the signal normal to this lift-off direction, the lift-off angle, the probe should be operated at a frequency so that the signal caused by a change in depth of the IGA, the depth angle, is normal to the lift-off angle. However, the depth angle is a function of the depth of the IGA.

The previous study indicates that the smallest inversion errors occur when the depth of the IGA is around one skin depth. Figure 8.12 plots the phase separation for coil #6 at 3 frequencies against the depth of the IGA layers in this neighbourhood. Assuming
that a scan is to be done to show no IGA was deeper than some fixed amount \( c \), then the frequency should be selected so the inversion is most sensitive at this depth. Figure 8.12 indicates that the frequency should be approximately \( 1.2c - 1.4c \), because at this frequency the inversion errors are minimal and the lift-off signal is in quadrature to the depth signal.

### 8.4.3 Probe design

The last section was concerned with optimizing the operating frequency for a fixed coil. This section describes how the IGA model can be used to aide in the design of the probe itself. Again assume that the objective of the eddy current scan is to show there is no IGA deeper than some predetermined depth \( c \). The coil then should be designed so that at its operating frequency the skin depth is \( c \). Therefore the operating frequency \( f \) is

\[
f = \frac{1}{c^2 \pi \mu_0 \sigma_2} \quad (8.7)
\]

For best coupling, the coil should be as flat as possible. To keep the fields focused the coil should have small outer radius.

The idea is to design the probe so that it has an impedance of 50 \( \Omega \) at the operating frequency. Input the coil dimensions and \( f \) into the IGA forward model with the number of turns \( n \) set to one. The model will predict the free-space impedance \( Z_0 \). Set the number of windings \( \sqrt{\frac{50}{Z_0}} \). If the number of windings is large then the design may have to be changed. The wire diameter \( r = \sqrt{\frac{A}{n \pi}} \), where \( A \) is the cross-sectional area of the coil. For large \( n \) the wire diameter could be too small to be manufactured or cause inter-winding capacitance which could cause the coil to resonate at the operating frequency (this is not predicted by the model).
Chapter 9

Conclusions and future work

Enough, if something from our hands have power
To live, and act, and serve the future hour;

William Wordsworth 1770 - 1850

An important goal in achieving quantitative understanding of a complex physical phenomenon like the probe/workpiece interaction in eddy current nondestructive evaluation is the availability of a good model. Such a model can be used to give predictions on probe response quickly and inexpensively to expected NDE situations in the field. Two types of forward models have been developed to meet this need: the layered media forward model predicts the probe/workpiece interactions for unblemished stratified isotropic materials, the 3D forward model predicts probe responses from volumetric flaws inside a particular region of a stratified conductor.

An inverse model provides NDE inspectors with important information when making an inspection, it allows them to “see” inside the material and hence improve decision making. A layered half-space inverse model has been developed to predict the depth of IGA on the outside surface of half-space conductors.

9.1 Achievements

A consistent theory has been developed for analyzing electromagnetic properties of isotropic linear homogeneous stratified conductors with planar or cylindrical boundaries in the presence of an air-cored eddy current probe. This theory can be used to analyze the impedance changes in a coil due to the structure and material properties of a nearby conductor. This theoretical work is the basis on which a layered media forward model has been built.
The layered media forward model was implemented in FORTRAN and can accurately predict the signals from layers of IGA in austenitic steel and from other planar or cylindrically stratified media. The code has been validated using experimental measurements of free-space inductance of the eddy current probes, phase changes with changes in lift-off and laboratory measurements of uniform layers of IGA.

The theory has been extended to allow for one of the regions in the conductor to contain a volumetric flaw. The flaw is represented in discrete form by subdividing it into elements on a regular grid with constant conductivity over each element. This model allows for a wide range of defects to be simulated. This theory developed by Bowler, Sabbagh and Sabbagh[13] is extended to allow for layered planar or cylindrical geometries. This new theory then could be applied to model the probe/flaw interactions.

In order to gain precision, increase computational speed and reduce memory requirements, matrix elements were evaluated using a real space integration instead of a Fourier space integration as was done earlier[13]. A point matching scheme was used to evaluate the incident field in the flaw volume, also achieving a significant savings in memory and computation time. The numerical model, the implementation of the entire flaw model on a computer, is significantly faster and smaller than its Fourier equivalent.

The 3D forward model was coded in FORTRAN and gives good predictions of eddy current probe responses to a wide range of defects. The code has been extensively tested. Validation has been carried out using analytic results, the layered media forward model predictions and with international benchmark tests established for this purpose. Indeed, for the differential probe results shown in Chapter 7, this implementation was able to compute the differential voltage in under 2 minutes/point on a 4 MByte 386 PC. In comparison Vérité[64] used 20 minutes on a Cray and Takagi[5] used 78 Mbytes of pre-computed results, and gave inferior results.

The layered half-space inverse model and the numerical code provides reasonable inverted depth indications on the limited amount of experimental measurements available. However, when applied to predicted signals from the 3D model, highly accurate predictions are obtained. Using the 3D model predictions is not as "satisfying" as using actual measurements; however, some degree of confidence is obtained because the theoretical basis of the 3D model is different from that of the 1D model and the 3D model has been successfully validated.

All of the numerical models developed are now available and running on computers at Nuclear Electric PLC. They are powerful tools and should be very useful: in designing eddy current probes, optimizing eddy current scanning parameters and for the analysis of eddy current scan data.
9.2 Potential future work

There are several areas for further development, with a wide range in effort required. The easiest would be the extension of the inverse layered half-space model to other media types. This would allow for prediction of layer coatings thicknesses or depth of corrosion on thin plates or tubing.

The layered half-space inversion model should be validated against *real world data*. The inversion model must be shown to work in situations of small isolated patches and/or interlinked patches of IGA. The model can be extended very simply by the addition of "shape" parameters or by some sort of deconvolution or other filtering.

The 3D flaw forward model could be extended to allow for the use of ferrite cored probes. The presence of the magnetic core couples the fields, therefore adding another layer of complexity to the incident field computation. However, assuming the influence of the flaw on the core is negligible, only the incident field theory need be changed, the matrix elements need not be changed.

The last two areas of future work are the most complex. Modelling ferrous materials and the full 3D inverse problem. Can the volume integral method be extended to model this large class of materials or is a "thin skin" theory more appropriate, is an open question. Another open question is the feasibility of solving the 3D inverse problem using a volume integral approach. Since this work has presented an efficient, accurate solution to the 3D forward problem, can the inverse be too far behind?
Bibliography


