THE DEVELOPMENT AND APPLICATION OF A FINITE ELEMENT PROGRAM FOR THE SOLUTION OF GEOTECHNICAL PROBLEMS.

A Thesis Submitted to the University of Surrey for the Degree of Doctor of Philosophy in the Department of Civil Engineering

By

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ABSTRACT

THE DEVELOPMENT AND APPLICATION OF A FINITE ELEMENT PROGRAM FOR THE SOLUTION OF GEOTECHNICAL PROBLEMS.

(a Ph.D. thesis, Department of Civil Engineering, University of Surrey)

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A program based on the finite element displacement method is developed, tested and applied to the solution of various geotechnical problems. It is written in ALGOL and is suitable for plane stress, plane strain and axi-symmetric analysis, in general, of isotropic, cross-anisotropic and orthotropic media.

The program is particularly suitable to study soil deformations due to excavation around diaphragm walls. In addition to displacements and stresses, it evaluates the consistent nodal forces relevant for the following stages of excavation, the variations of loads in anchors or struts, bending moments, shear forces and axial forces along the neutral axis of the wall, horizontal and vertical distributed pressures over the surfaces of the wall, assuming either no friction or no sliding along each surface of contact between soil and wall.

The linear strain triangular element of six nodes is employed for plane stress and plane strain analysis, while the corresponding ring-type triangular element is employed for axi-symmetric analysis. The element matrices are evaluated by exact integration, assuming linear elastic material with constant Poisson's ratio and Young's moduli varying linearly in any direction, over the element.
The accuracy and convergence of results are tested on numerous small or medium size problems, by comparing the computed solutions with the theoretical solutions when available or with the solutions obtained by other finite element programs and by field measurements.

The program is also applied to the solution of four major geotechnical problems:

1. Settlement of a circular loaded area on the surface of an elastic, incompressible, homogeneous or heterogeneous medium resting on a smooth or rough rigid base;

2. The behaviour of the heterogeneous foundations of two 100000 ton oil tanks, 79.2m diameter and 19.8m high;

3. Comparison of field measurements with the predictions by the program for a diaphragm wall, 13m deep and 0.61m thick, with four rows of anchors;

4. Preliminary parametric study of a diaphragm wall to investigate the influence of some relevant factors.
TO MY PARENTS
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NOTATION

Principal symbols used in this thesis are defined in this list. Other symbols are defined in context.

\( A \) = Area of a triangular element or cross-section of a linear element.

\( \textbf{A} \) = Auxiliary matrix of constants.

\( \bar{A} \) = Coefficient matrix of a system of linear equations.

\( \textbf{A}_0 \) = Vector of constants defined by equations (3.76).

\( \textbf{a} \) = Vector of constants defined by equations (3.76).

\( a_1, a_2, a_3 \) = Components of vector \( \textbf{a} \).

\( \textbf{B} \) = Strain matrix relating \( \varepsilon \) to \( \textbf{U}_n \).

\( \textbf{B} \) = Load vector of the structure (FEM program).

\( \textbf{B}_{un} \) = Matrix of horizontal components of the element nodal body forces matrix \( \textbf{P}_b \).

\( \textbf{B}_{vn} \) = Matrix of vertical components of the element nodal body forces matrix \( \textbf{P}_b \).

\( \textbf{b} \) = Body forces vector.

\( \textbf{b} \) = Body forces vector (static loading only).

\( \textbf{b} \) = Vector of constants defined by equations (3.76).

\( b_1, b_2, b_3 \) = Components of vector \( \textbf{b} \).

\( b_x, b_y, b_z \) = Components of the body forces vector \( \textbf{b} \).

\( \textbf{C} \) = Elastic compliance matrix or auxiliary matrix of constants.

\( \textbf{C}_1, \textbf{C}_2, \textbf{C}_3 \) = Elastic compliance matrices of orthotropic, cross-anisotropic and isotropic bodies, respectively, in plane stress analysis.

\( \textbf{C}_1', \textbf{C}_2', \textbf{C}_3' \) = Elastic compliance matrices of orthotropic, cross-anisotropic and isotropic bodies, respectively, in plane strain analysis.
\( C_2, C_3 \) = Elastic compliance matrices of cross-anisotropic and isotropic bodies, respectively, in axi-symmetric analysis.

\( \mathbf{C} \) = Auxiliary matrix of constants.

\( \mathbf{D} \) = Elastic rigidity matrix.

\( \mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3 \) = Elastic rigidity matrices of orthotropic, cross-anisotropic and isotropic bodies, respectively, in plane stress analysis.

\( \mathbf{D}_1', \mathbf{D}_2', \mathbf{D}_3' \) = Elastic rigidity matrices of orthotropic, cross-anisotropic and isotropic bodies, respectively, in plane strain analysis.

\( \mathbf{D}_2'', \mathbf{D}_3'' \) = Elastic rigidity matrices of cross-anisotropic and isotropic bodies, respectively, in axi-symmetric analysis.

\( \mathbf{d} \) = Displacement vector of the structure.

\( d \_i \) = Displacement corresponding to generalized coordinate \( i \).

\( d_{px} \) = Displacement of node \( p \) in the direction of \( x \)-axis.

\( d_x \) = Horizontal component of the displacement of a point.

\( d_y \) = Vertical component of the displacement of a point.

\( E \) = Young's modulus of an isotropic body.

\( \mathbf{E} \) = Row matrix of components \( E \_i \).

\( E_H \) = Horizontal Young's modulus of a cross-anisotropic body with a horizontal plane of isotropy.

\( E_i \) = Value of vertical Young's modulus \( E \_y \) at corner node \( i \) of a triangular element, i.e. element of matrix \( E \).
$E_x', E_y', E_z$ = Young's moduli for tension (compression) along the principal directions of elasticity $x, y, z$, respectively.

$E_u$ = Undrained Young's modulus.

$E_v$ = Vertical Young's modulus of cross-anisotropic body with a horizontal plane of isotropy.

$f$ = Load vector of the structure.

$G$ = Shear modulus of an isotropic body or number of system equations (FEM program).

$G_{HH}$ = Non-independent shear modulus of a cross-anisotropic body characterizing the changes of angles between any two perpendicular directions of the horizontal plane of isotropy.

$G_{VH}$ = Shear modulus of a cross-anisotropic body characterizing the changes of angles between the vertical direction and any direction of the horizontal plane of isotropy.

$G_{yz}, G_{zx}, G_{xy}$ = Shear moduli characterizing the changes of angles between the principal directions of elasticity $y$ and $z$, $z$ and $x$, $x$ and $y$, respectively.

$H$ = Height of a rectangular cross-section.

$h$ = Width of a rectangular cross-section.

$I$ = Moment of inertia.

$I$ = Unit matrix.

$K$ = Element stiffness matrix.

$K$ = Stiffness matrix of the structure.

$\overline{K}$ = Primary stiffness matrix of the structure (before implementing the boundary conditions).

$K$ = Bulk modulus.

$KE$ = Element stiffness matrix (FEM program).
\( K_{ij}, k_{ij} \) = Elements of \( K \).
\( K_{ij} \) = Submatrices of \( K \).
\( k_{ij} \) = Submatrices of matrix \( K \).
\( K^t \) = Submatrices of \( K^t \).
\( L \) = Lower triangular matrix of constants.
\( \ell \) = Length of a linear element or of one side of a triangular element.
\( M \) = Consistent element mass matrix.
\( M \) = Bandwidth of the stiffness matrix of the structure (FEM program).
\( M \) = Consistent mass matrix of the structure.
\( M_{ij} \) = Submatrices of \( M \).
\( m \) = Ratio \( G_{xy}/E_y \) for an orthotropic material or \( G_{VH}/E_y \) for a cross-anisotropic material (equal to unity for an isotropic material).
\( m_{ij} \) = Submatrices of matrix \( M \).
\( N \) = Number of nodes of the structure.
\( N_A \) = Number of anchors or struts (FEM program).
\( N_B \) = Number of boundary conditions of the type \( d_i = 0 \) to be explicitly enforced (FEM program).
\( NDF \) = Number of degrees of freedom per node (FEM program).
\( N_E \) = Number of triangular elements (FEM program).
\( N_H \) = Number of elements under surface loading due to distributed horizontal pressure (FEM program).
\( N_L \) = Number of linear elements (FEM program).
\( N_LD \) = Number of load cases (FEM program).
\( N_P \) = Number of nodal points (FEM program).
NV = Number of elements under surface loading due to distributed vertical pressure (FEM program).

n = Ratio $E_x/E_y$ for an orthotropic material or $E_H/E_V$ for a cross-anisotropic material (equal to unity for an isotropic material).

$n_e$ = Total number of elements of a structure.

$n_1, n_2, ..., n_s$ = Actual node numbers of an element n of the structure.

$P$ = Consistent element force matrix.

$P$ = Consistent force matrix of the structure.

$\bar{P}$ = Consistent force matrix of the structure including the reactions (before implementing the boundary conditions).

$P_b$ = Element nodal body forces matrix.

$P_i$ = Submatrices of $P$.

$P_i$ = Element of matrix $P$ when it reduces to a vector.

$P_o$ = Consistent element force matrix due to initial strain or stress.

$P_{vun}$ = Submatrix of horizontal components of force matrix $P_o$.

$P_{vvn}$ = Submatrix of vertical components of force matrix $P_o$.

$P_P$ = Element nodal surface forces matrix.

$P_{un}$ = Submatrix of horizontal components of force matrix $P_P$.

$P_{vn}$ = Submatrix of vertical components of force matrix $P_P$.

$F$ = Surface forces vector.

$\bar{F}$ = Surface forces vector (forces independent of time).
\( P_x, P_y, P_z \) = Components of vector \( P \).

\( P_{xn} \) = Matrix of nodal values of the horizontal distributed surface forces over the sides of an element.

\( P_{yn} \) = Matrix of nodal values of the vertical distributed surface forces over the sides of an element.

\( q \) = Distributed surface pressure.

\( q_i \) = Undetermined parameters (called generalized coordinates in the Rayleigh-Ritz method).

\( q_x, q_y \) = Horizontal and vertical components, respectively, of surface pressure.

\( R \) = Transformation matrix.

\( S \) = Total boundary of a body.

\( S_u \) = Boundary of a body on which displacements are prescribed (known) or undrained shear strength.

\( S_o \) = Boundary of a body on which stresses are prescribed (known).

\( r, \theta, z \) = Cylindrical coordinates.

\( T \) = Temperature increment.

\( T_i \) = Matrix of temperature increments at all nodes of a triangular element.

\( T_{i} \) = Element of matrix \( T \) corresponding to node \( i \) of an element.

\( \Gamma_e \) = Strain transformation matrix to transform the strains from cartesian system \( X, Y, Z \) to cartesian system \( X^*, Y^*, Z^* \).

\( \Gamma_o \) = Stress transformation matrix to transform the stresses from cartesian system \( X, Y, Z \) to cartesian system \( X^*, Y^*, Z^* \).

\( U \) = Upper triangular matrix of constants.
\( \mathbf{U} = \) Nodal displacement matrix of the structure.

\( \mathbf{U}_i = \) Element of matrix \( \mathbf{U} \) when it reduces to a vector.

\( \mathbf{U}_i = \) Submatrix of matrix \( \mathbf{U} \) containing the components of displacement of node \( i \).

\( \mathbf{U}_n = \) Element nodal displacement vector.

\( \ddot{\mathbf{U}}_n = \) Second derivative of \( \mathbf{U}_n \) with respect to time.

\( u = \) Pore water pressure.

\( \mathbf{U} = \) Displacement vector.

\( \ddot{\mathbf{U}} = \) Second derivative of \( \mathbf{U} \) with respect to time.

\( u, v, w = \) Displacements in the direction of the cartesian axes, components of vector \( \mathbf{U}_i \).

\( \mathbf{U}_i = \) Nodal displacement matrix for node \( i \).

\( \mathbf{U}_n = \) Vector containing the horizontal components of \( \mathbf{U}_n \).

\( u_r, u_\theta, u_z = \) Components of displacement \( u \) of a point in cylindrical coordinates.

\( u_1, u_2, \ldots, u_6 = \) Horizontal nodal displacements (components of vector \( \mathbf{U}_n \)).

\( \mathbf{V} = \) Total strain energy.

\( v = \) Strain energy density.

\( \mathbf{V}_n = \) Vector containing the vertical components of \( \mathbf{U}_n \).

\( v_1, v_2, \ldots, v_6 = \) Vertical nodal displacements (components of vector \( \mathbf{V}_n \)).

\( X, Y, Z = \) Cartesian global reference frame.

\( X^*, Y^*, Z^* = \) Cartesian system of axes different from the global cartesian system \( X, Y, Z \).

\( x = \) Displacement vector of the structure.
x', y', z = Variables in the directions of cartesian axes X, Y, Z, respectively.

y = Auxiliary displacement vector.
z = Depth of a geotechnical medium.

WD = Work done by the internal forces.
WE = Work done by the external forces.

α = Thermal strain coefficient of an isotropic body.

Κ = Matrix of displacement parameters for an element.

αH, αV = Thermal strain coefficients of a cross-anisotropic body in any direction of the horizontal plane of isotropy and in the vertical direction, respectively.

αX, αY, αZ = Thermal strain coefficients of an orthotropic body in the principal directions of elasticity X, Y, Z, respectively.

Γ = Auxiliary row matrix defined by equation (3.161).
γ = Unit weight of a body.
Δl = Variation in length l.
ΔU = Vector of virtual displacements.

Δu, Δv, Δw = Components of vector ΔU.

δE = Vector of strain increments.

δεx, δεy, δγxy = Components of the vector δE of strain increments.

δX = Incremental total potential energy.

Ε = Strain vector.

Ε0 = Initial strain vector.

Ε01, Ε02, Ε03 = Initial strain vectors for orthotropic, cross-anisotropic and isotropic bodies, respectively, in plane stress analysis.
\[ \varepsilon_{01}', \varepsilon_{02}', \varepsilon_{03}' \] = Initial strain vectors for orthotropic, cross-anisotropic and isotropic bodies, respectively, in plane strain analysis.

\[ \varepsilon_x', \varepsilon_y', \varepsilon_\theta', \gamma_{xy} \] = Components of strain vector \( \varepsilon \), in axi-symmetric analysis.

\[ \varepsilon_x', \varepsilon_y', \ldots, \gamma_{xy} \] = Components of strain vector \( \varepsilon \), in three-dimensional analysis.

\[ \theta \] = Rotation of the element frame with respect to the global frame or inclination of a linear element or cylindrical coordinate.

\[ \lambda \] = Lame's constant or increase of Young's modulus \( E_y \) per unit depth.

\[ \mu \] = Lame's constant

\[ \mu_1, \mu_2, \mu_3 \] = Components, in general (whatever the system of coordinates and type of analysis), of the initial strain vector.

\[ \nu \] = Poisson's ratio of an isotropic body.

\[ \nu_{HH} \] = Poisson's ratio of a cross-anisotropic body characterizing the decrease (increase) of strain in a direction of the horizontal plane of isotropy when tension (compression) is applied in a perpendicular direction of the same plane.

\[ \nu_{VH} \] = Poisson's ratio of a cross-anisotropic body characterizing the decrease (increase) of strain in any direction of the horizontal plane of isotropy when tension (compression) is applied in the vertical direction.

\[ \nu_{yx} \] = Poisson's ratio characterizing the decrease (increase) of strain in the principal direction \( x \) when tension (compression) is applied in the principal direction \( y \).

Similarly for Poisson's ratios \( \nu_{xy}', \nu_{zx}', \nu_{xz}', \nu_{yz}', \nu_{zy} \).
\[ \xi = \text{Vector of homogeneous (triangular, area, oblique) coordinates of a triangular element.} \]

\[ \xi_1, \xi_2, \xi_3 = \text{Components of vector } \xi. \]

\[ \rho = \text{Mass density.} \]

\[ \Upsilon = \text{Stress vector.} \]

\[ \sigma_H = \text{Horizontal total stress.} \]

\[ \sigma_H' = \text{Horizontal effective stress.} \]

\[ \sigma_{\text{MAX}} = \text{Major principal stress.} \]

\[ \sigma_{\text{MIN}} = \text{Minor principal stress.} \]

\[ \sigma_V = \text{Vertical total stress.} \]

\[ \sigma_V' = \text{Vertical effective stress.} \]

\[ \Sigma_0 = \text{Initial stress vector.} \]

\[ \sigma_{x'}, \sigma_{y'}, \sigma_{\theta'}, \tau_{xy} = \text{Components of stress vector } \Upsilon \text{ in axi-symmetric analysis.} \]

\[ \sigma_{x'}, \sigma_{y'}, \ldots, \tau_{xy} = \text{Components of stress vector } \Upsilon, \text{ in three-dimensional analysis.} \]

\[ \Phi = \text{Interpolation matrix containing } \Phi \text{ as submatrix.} \]

\[ \Phi = \text{Row matrix of interpolation functions.} \]

\[ \phi = \text{Angle of friction.} \]

\[ \phi_d = \text{Drained angle of friction.} \]

\[ \phi_i = \text{Prescribed functions of the coordinates, components of the row matrix } \Phi. \]

\[ \phi_u = \text{Undrained angle of friction.} \]

\[ \chi = \text{Total potential energy functional.} \]

\[ \psi = \text{Auxiliary matrix defined by equation (3.98).} \]

\[ \Omega = \text{Potential energy of the applied forces.} \]
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Chapter 1
GENERAL INTRODUCTION

1.1 The Finite Element Method in Geotechnical Engineering

1.1.1 Historical Background

Over the past few years, the finite element method (FEM) has been found to be an extremely powerful tool for solving a wide variety of complex engineering problems.

Applications of the FEM to problems in soil and rock mechanics followed almost immediately the method's development and applications to aerospace and structural engineering.

In structural and aerospace engineering, the first formulation which led to the FEM was given by Turner et al (1956) and Clough (1960) introduced the use of the term finite element method.

Argyris (1960, 1965) gave impetus to the method by relating it to energy principles and the matrix techniques have provided a compact basis for handling the algebra of the FEM in the computer. Oden (1972) provided a basis for the generalization of the method to various areas of mathematical physics and nonlinear problems.

The literature about the FEM is large and has been rapidly increasing. Back in 1972, Akin, Fenton and Stoddart listed no less than 1096 references.

Problems in geotechnical engineering pose complexities that may not exist in other areas of application. The main complexities arise from the multiphase nature of soils and
rocks, and from factors such as residual stresses and discontinuities (joints and fissures). Most of the applications, therefore, have been accomplished by adopting relatively simple and often most elementary forms of characterization to account for the factors involved.

Most of the applications have also been made by adopting or modifying formulations and programs developed for structural and continuum mechanics. Hence, the initial applications that can be linked to geotechnical engineering involve such problems as plane stress analysis of a concrete dam and foundation (Clough, 1962) and linear elastic stress analysis in a loaded half-space (Clough and Rashid, 1965). Subsequently the method has been extensively used and comprehensive reviews of its applications have been presented by Radhakrishna and Reese (1970) and Desai (1972).

1.1.2 Past Applications in Geotechnical Engineering

The use of variational principles permitted generalization of the FEM and provided physical insight leading to significant convenience in the formulation of the element and system equations for a number of different problems. In spite of certain limitations the method has proven to be of significant usefulness and has gained tremendous popularity.

Some of its merits reside in the fact that it can easily tackle problems with complex geometries and material properties, it is simple to introduce the boundary conditions into the system equations, it can handle complex nonlinear, hysteretic, time and temperature dependent material behaviour, and is specially suitable to benefit from the extensive facilities offered by the computer.
The generality of the basic principles on which the formulation of the method is based makes it possible to develop highly versatile computer programs. For instance, a program derived for elastic plane stress analysis can be conveniently modified for solution of fluid flow in porous media.

The finite element displacement method, based on the principle of minimum potential energy, is most commonly employed in geotechnical engineering. There are hardly any applications that employ other principles: e.g. the principle of minimum complementary energy and the Hellinger-Reissner mixed principle (Pian and Tong, 1969; Dunham and Pister, 1966).

Applications of the FEM to geotechnical engineering can be classified as structural and non structural (Desai, 1972). Fig. 1.1 shows such classification. Some of the applications will be mentioned, such as:

(a) Static and dynamic analyses of slopes, dams, embankments and their foundations, sometimes using the concept of sequential construction;

(b) Interaction between soil and structure in cases such as beams and plates on elastic foundations, footings, retaining walls and pavement systems;

(c) Analyses of tunnels, cavities and boreholes;

(d) Analyses of various laboratory test specimens of soils and rocks;
(e) One-dimensional, two-dimensional and three-dimensional consolidation;
(f) Dynamics and earthquake analyses of dams and wave propagation resulting from blast loads in geologic media;
(g) Seepage through porous media;
(h) Mining engineering, land subsidence and geological modelling.

1.1.3 Constitutive Models of Engineering Media

The development of the finite element technique may have reached its final plateau. Although there is scope for future research, a point of diminishing returns has been reached, at least in the matter of basic formulation. However, there is an area still not adequately investigated which is the characterization of material behaviour. Pages of output from a computer run can be of no significance if the material was not properly characterized in the first place.

A constitutive model (stress-strain law) for an engineering medium depends on a number of factors such as density, residual stresses, stress history, existence of discontinuities, existence of liquids and gases in the pores, time and temperature dependence, etc.

Ideally the constitutive model should be derived from a number of direct field measurements. For economic reasons or lack of adequate instrumentation this approach is often not feasible. Hence, one has to rely frequently upon a number of laboratory tests simulating the field conditions.
The laboratory measurements may not always be correct or representative of the field situation. Besides, the general constitutive model should be able to predict or define the behaviour of the medium under all possible states of stress and strain, not just those corresponding to the tests from which it was derived.

When using a model, although there are other factors to be considered, the most important question is: will it give reliable predictions of stresses, strains and displacements? Taking into account the degree of precision that can be attained in geotechnical engineering, one should strive for the simplest model that is accurate and realistic to a degree compatible with the practical use intended.

The simplest constitutive model is the linear elastic isotropic model in which the stress-strain relations are assumed to be linear. These relations are derived from the generalized Hooke's Law and can be written, for a finite element, as

$$\sigma = D \varepsilon$$

$$\varepsilon = C \sigma$$

(1.1)

This model has been widely used in geotechnical engineering with matrices C and D always containing a pair of parameters: e.g. Young's modulus E and Poisson's ratio ν, bulk modulus K and shear modulus G, or Lame's constants λ and μ = G.
Some of the constitutive models used in geotechnical engineering are listed below:

(a) Linear elastic models;

(b) Nonlinear (piecewise linear) elastic isotropic models in which the (tangent) Young's modulus is considered as a function of the state of stress or state of strain, using various techniques: the incremental, the iterative and the step-iterative or mixed method;

(c) Elastic-plastic models with various yield criteria (Tresca, von Mises, Mohr-Coulomb);

(d) Viscoelastic models for media dependent on rate of deformation in addition to strains.

The models used so far are generally suitable for linear elastic, nonlinear elastic, perfectly plastic and work hardening behaviour. However, they are not so suitable to handle such behaviour as work softening (e.g. natural clay deposits) and dilatancy (e.g. sands).

The existence of a nonzero state of stress (in situ or initial stress) prior to loading can affect the deformation behaviour of an engineering medium. The vertical initial stress is usually computed as the overburden pressure due to gravity and the initial horizontal stress is evaluated through the use of the coefficient of earth pressure at rest $K_0$. The value of $K_0$ depends on many factors such as overconsolidation, differential settlements, faults, folds and other tectonic effects. The only reliable way of
estimating the value of $K_o$ is by means of field measurements but, because these have not been available in many cases, most of the finite element applications have adopted $K_o$ as determined by the elastic theory ($K_o = \nu/(1 - \nu)$).

Another difficulty results from the fact that the behaviour of engineering media is path dependent and, therefore, the stresses, strains and displacements should ideally be obtained by proper integrations along the path of loading. Loading under different initial conditions, dynamic loading, loading and unloading of media exhibiting hysteretic effects, sequential construction are examples of different stress paths. The incremental load procedure can handle stress path dependent behaviour and is becoming commonly used in the finite element applications when the material behaviour is represented by the elastic model or the piecewise elastic model.

Some applications of the FEM in geotechnical engineering have used a simple approach to account for low (or zero) tensile strength by assigning arbitrarily a small stiffness to those elements in which tensile stresses are induced. This seems to be a reasonable approximation for clays since they usually possess some tensile strength.

Another method for the same purpose is to redistribute the tensile stresses in the neighbouring zones until the physical requirement of no tension is restored. This is done by an iterative procedure: in the first iteration the material is assumed to be linear elastic and in the next stage the
elements having tensile stresses are marked, self-equilibrating nodal forces are computed and applied to these elements and the iterations continue until the zone is relieved of the tensile stresses.

1.1.4 Material and Geometry Nonlinearity

In addition to the material nonlinearity resulting from nonlinear stress-strain relations, geometric nonlinearity can also occur due to changes in the geometry which may make the deformed structure significantly different from the initial (undeformed) structure. This is possible in unconsolidated soft soil and at higher loadings.

If the deformations are small, only the first-order terms in the strain-displacement relations need to be considered. These relations for $\varepsilon_x$ and $\gamma_{xy}$, for instance, can be written as follows, including the second-order terms:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$$

If the second-order terms are retained, large strains will be permitted. This together with large displacements leads to the most general case of geometric nonlinearity which can occur in some special cases.

The iterative procedure can be used for large displacement analysis, revising the coordinates of nodes in the mesh after each cycle of application of the load but the accumulated round off errors may be too large to yield a reliable solution.
Combination of material nonlinearity with geometric nonlinearity is easier to handle if an incremental elasticity matrix can be established (Marčal, 1969).

1.1.5 Practical Applications

The FEM possesses certain disadvantages for its current use in geotechnical engineering. For example, it cannot as yet handle properly discontinuous media, the problem of separation of contacts between two media and propagation of fractures or fissures has to be further investigated, and the search for adequate constitutive models requires further work. In addition to these and other difficulties, a large amount of computer memory and time is required, making it available only to those fortunate enough to have access to relatively large and high-speed computers.

However, it is such a powerful tool that its application by the geotechnical engineer will no doubt increase in the future, contributing to its further development and to increase the confidence of the design engineer to use the method for practical problems.

The FEM can find useful applications for various aspects of practical applications, some of which are mentioned below:

(a) Design

In such cases as shallow and deep foundations for instance, the method can now be used as a sole basis of design. As adequate verifications against test data and other conventional solution procedures are obtained, the use of the FEM for this purpose will increase;
(b) Design Analysis
The method can be used to analyse a number of alternatives of loading and geometries and to complement designs produced by other methods. It can also be used to analyse existing structures whose behaviour is known through instrumentation, and failed structures or foundations from the knowledge of the loading that caused the failure. In both cases this could provide useful information for improvements in design methods;

(c) Evaluation of Material Parameters
By back analysis of practical cases in which the behaviour is known through instrumentation the FEM can find by trial and error the corresponding material parameters. This knowledge can be very useful in the future for the design of similar structures to be built in analogous soil or rock masses;

(d) Location of Instrumentation
Pre-analysis of a structure (e.g. earth dam or embankment) can show the critical areas where the installation of instruments is of utmost interest;

(e) Monitoring and Control during Construction
The method can be used to predict the behaviour of a structure during construction, according to a planned construction sequence. These predictions can be compared with the observed behaviour as the construction proceeds, resulting in increased confidence in the stability of the structure;
(f) Analysis of Laboratory Test Specimens
This permits the evaluation of the quality of simulation by a given test for a field situation and also the study of the influence of some factors related to the testing technique (e.g. influence of friction due to end platens in a compression test);

(g) Design Charts and Tables
For structures of common occurrence, charts and tables can be prepared considering a number of different combinations of factors affecting the solution;

(h) Relative Importance of Factors Affecting the Solution of a Typical Problem
By considering different material parameters and geometries, it is possible to study the influence of each factor in the solution of a particular type of problem. This comparative study may show that the influence of the variation of some factors on the solution is negligible while small variations in the values of some other factors affect considerably the final results. Greater care must be taken to ensure that reliable values for the latter are available.
1.1.6 Availability and Efficiency of Programs

There are a number of general purpose programs based on the finite element method in current use in structural and continuum mechanics such as ASKA, DAISY, ELAS, ICES-STRUDL II, NASTRAN, SAMIS, SAP (Desai and Abel, 1972). These programs or parts thereof have been adopted for geotechnical applications.

Many authors have developed their own programs often specific for a class or subclass of geotechnical problems. However, even assuming that their validity has been established and they are readily available, the efficiency of their computational procedures sometimes leaves much to be desired. This is illustrated by a performance experiment run by Systems Development Corporation (David, 1971) in which twelve experienced programmers were asked to develop the solution to a given logic problem. The running times varied from 1 to 15, assuming unity for the best one.

A specific program which operates on a given computer at the least cost is the most efficient for that computer but it usually loses efficiency when adapted for a different computer. On the other hand not only the storage needs and running times but also the data preparation time must be considered in any evaluation of cost and efficiency.

For one or another reason, the geotechnical engineer sometimes has to develop his own program and usually uses a formulation based on the finite element method. But he may be bewildered, often rightly, by the great amount of information and sometimes conflicting deductions and conclusions,
because the choice of the most suitable scheme is governed by many different factors. However, what finally counts is the accuracy of results and the cost per solution which is not expressed only in terms of money.

This bewilderment was also experienced by the author when he accepted the challenge of developing his own program, in spite of having no previous experience of programming nor of using the finite element method.

The FEM program whose development and application are presented in this thesis was developed by the author and his supervisor. However, for simplicity of language, the author's supervisor will not be mentioned again explicitly, as a general rule.

1.2 Scope of this Thesis

1.2.1 Development of the Program

The purpose of the author was to develop and test a program which would satisfy the following main requirements:

1. To be capable of solving a variety of geotechnical problems, without considering dynamic loading, time dependency, material and geometric nonlinearity;
2. To be suitable for the analysis of deformations around diaphragm walls due to excavation as it involves some specific difficulties arising from the existence of anchors, interaction between soil and structure and also sequential construction which are common to other types of problems;
3. To use a simple constitutive model for the material which, nevertheless, would lead to a degree of accuracy compatible with the current knowledge of the behaviour of engineering media, in general;

4. To be easily modified and improved by future consideration of more sophisticated constitutive models, time dependency, dynamic loading, material and geometric nonlinearity, etc.;

5. To be most suitable for the computer ICL 1905F of the University of Surrey but easily adaptable to use in a different machine;

6. To reduce the human intervention in the preparation of input data and interpretation of output;

7. To be efficient, reducing both the running times and the storage needs to the minimum.

It was an easy decision to choose a formulation based on the finite element displacement method which has proven to be most suitable for engineering problems in general.

The basic formulation of the finite element displacement method will be presented in chapter 2 and its application in the program is explained in some detail in chapter 3.

The need for input data is reduced to a convenient level by leaving to the computer the task of calculating some of the necessary data, using the values given as input. The finite element mesh is not generated automatically but an algorithm was also developed to find the best node numbering system.
It will be presented in chapter 4 which is almost a copy of a paper approved for publication in the Journal of the Structural Division, ASCE, in February 1975, under the title "Node Numbering Optimization in Structural Analysis".

The author chose the linear elastic model as representative of the behaviour of isotropic, cross-anisotropic and orthotropic material. Taking into account the current knowledge of the behaviour of engineering media, in general, this seems to be a good enough model, especially for the evaluation of immediate settlements in saturated soils.

The consideration of a piecewise linear elastic model was disregarded because the relatively low capacity and low speed of the computer ICL 1905F made unpractical the solution of large problems using iterative procedures. For the same reason, the possibility of solving three-dimensional problems was also disregarded.

The author chose the linear strain triangular element of six nodes because it gives a much better accuracy, for the same solution time, than the widely used simplest triangular element. More sophisticated triangular elements are not justified by the possible degree of accuracy in the knowledge of engineering media and other elements are not so suitable for irregular geometries where, in addition, there are often stress concentration zones.

The triangular element is used for plane stress and plane strain analysis. The corresponding ring-type triangular element is used for axi-symmetric analysis.
Three-dimensional analysis has not been considered but, fortunately, most of the practical problems in geotechnical engineering fall in one of the categories considered in the program.

In the important case of immediate settlements of saturated clays, computational difficulties are expected from the use of soil parameters close to the values corresponding to the condition of no volume change or incompressibility. These difficulties have been noticed in certain cases forcing the use of "reduced" integration in order to be able to obtain a solution (Zienkiewicz, Taylor and Too, 1971; Naylor, 1974). When simulating the behaviour of soils exhibiting dilatancy by a piecewise linear elastic model (which will be later introduced in the program) it may be necessary to consider values of Poisson's ratio greater than 1/2 leading to the same computational difficulties. Keeping this in mind and knowing that exact integration is more accurate than numerical integration (in computer calculations this may be untrue in special circumstances), the author decided to evaluate all element matrices by exact integration. This is an unusual feature in finite element programs and was achieved considering isotropic, cross-anisotropic and orthotropic material in which the Young's moduli and shear moduli vary linearly in any direction over the element. Poisson's ratios, however, are assumed to be constant within each element.

The program deals with anchors or struts by considering them as linear elements connected to the diaphragm wall through any number of nodes. The variations of loads in
anchors, bending moments, shear forces, axial forces, horizontal and vertical pressures on the surfaces of the wall are automatically computed after each stage of excavation.

The interaction between soil and structure can be studied considering two extreme situations: no friction along the surface of contact between soil and structure or no sliding at all.

The system of equations is solved by a band solution technique based on the Cholesky decomposition of a matrix into a product of a lower triangular matrix by its transpose.

The use of disc backing store and magnetic tape makes it possible to solve large problems in a rather small computer, although the running times are obviously increased by the use of these peripherals. A second version of the program was prepared for small and medium size problems that can be solved in core.

The program is set up as a relatively short list of statements making use of a large number of routines. This facilitates the introduction of future improvements. For instance, the solution of three-dimensional problems can be obtained by the addition of a new routine corresponding to the evaluation of the element stiffness matrix for three-dimensional analysis.

When programming the routines, the author performed the mathematical operations whenever possible, leaving to the computer the easy task of evaluating the numerical values of expressions which are simple functions of a few parameters. This leads to efficiency in the computations, reducing the running time.
1.2.2 Tests of the Program

The program was entirely developed by the author and his supervisor. It was, therefore, vital to test it on a variety of problems and situations before one could gain enough confidence for its application in large practical problems of geotechnical engineering.

Chapter 5 presents the solutions of the problems listed below and their comparison with other solutions obtained either by a theoretical method or through the use of a different finite element program:

1. Four plane pin-jointed structures
2. Cantilever beam under end shear load
3. Plane strain triangle with two fixed nodes
4. Cantilever beam with linear elements
5. Elastic half-space under concentrated or strip load
6. Scammonden dam
7. Diaphragm wall of Britannic House
8. Solid and hollow cylinders under vertical or horizontal distributed pressure.

The results obtained demonstrate the convergence and accuracy of the solution and also the versatility of the program for the solution of a number of different types of problems.

Since only the last problem corresponds to a case of axi-symmetric analysis, a rather more complex axi-symmetric problem was also solved, but due to its importance the results are presented in a different chapter as an application of the program mentioned in the next section.
1.2.3 Applications of the Program

Chapter 6 investigates the influence of anisotropy, heterogeneity and position of the boundaries, on the computed deformation for a half-space under uniform circular load on the surface. The results are compared with the theoretical solution obtained by Gibson (1974) to show that the program gives satisfactory accuracy. It is also confirmed that the settlement under uniform surface load of a layer of Gibson soil resting on a smooth rigid base is independent of the depth, being constant under the loaded area and zero outside the loaded area.

Chapter 7 presents the results of the analysis of a heterogeneous soil medium loaded by two 100000 ton oil tanks of 79.2m diameter (Leggatt and Bratchell, 1973). The immediate settlements are compared with field measurements and the influence of variations in the values of the soil parameters is investigated.

Chapter 8 compares the prediction of displacements by the program with field measurements obtained by the Building Research Station during the construction of Neasden diaphragm wall. Simultaneously, the suitability of the program to analyse this rather complex type of structure is illustrated.

Chapter 9 presents a preliminary parametric study of a diaphragm wall. Factors considered include the influence of the position of the assumed boundaries, the number of elements of the triangular mesh, the conditions of no sliding or no
friction along the surface of contact between soil and wall and variations in the value of Poisson's ratio. The author could not carry this study further but is convinced it would be interesting to investigate the influence of other factors such as anisotropy and heterogeneity of the soil, rigidity of the wall, depth of the wall below the base of excavation, sequence of excavation and stressing of the anchors and also cross-section, length, position and inclination of the anchors.
APPLICATIONS OF THE FINITE ELEMENT METHOD IN GEOTECHNICAL ENGINEERING

STRUCTURAL

Static stress-strain analysis in continuous and discontinuous masses
Dynamic and earthquake analysis
Viscoelastic and others

NON STRUCTURAL

Fluid flow in porous rigid media
Temperature distributions
Gravitational and magnetic potential distributions
Dispersion in porous media

Fluid flow in deformable media
Geological modelling

Fig. 1.1 Classification based on Structural and Non Structural Applications of the FEM
Chapter 2

THE FINITE ELEMENT DISPLACEMENT METHOD
Basic Principles and General Procedure

2.1 Introduction

Finite element methods are techniques for approximating the governing differential equations for a continuous system by a set of algebraic equations relating a finite number of variables. The techniques were initially developed for structural problems but have since been extended to numerous other problems, in particular soil problems.

Conventional engineering structures can be visualized as an assemblage of structural elements interconnected at a finite number of nodal points. In an elastic continuum the true number of interconnection points is infinite but, using finite element techniques, one divides the domain into subdomains, called finite elements, and selects points, called nodes, on the inter-element boundaries and, sometimes, in the interior of the elements. Displacement and force values at the nodes are taken as discrete variables. Displacement and force values at points in the interior of an element are expressed in terms of the nodal variables using interpolation functions. Finally, the governing equations are generated by applying a variational principle.

There are two basic solution procedures of structural analysis, namely, the displacement (stiffness) method and the force (flexibility) method which can be deduced by applying the Principle of Minimum Potential Energy and the Principle of Minimum Complementary Energy, respectively.
The finite element displacement method (Zienkiewicz, 1971) is the most widely used finite element formulation and the large majority of computer programs available are based on it. The author has also chosen this type of formulation because it was the most suitable for his objective, viz to analyse soil deformations under static loading in general, and soil deformations due to excavation around diaphragm walls, in particular.

In this chapter the general procedure of the finite element displacement method will be outlined employing the Principle of Minimum Potential Energy to generate the governing equations, with special regard to two-dimensional continua. In chapter 3 it will be shown how the computer program based on this procedure can solve plane stress, plane strain and axi-symmetric problems.

2.2 Basic Principles

2.2.1 Principle of Virtual Displacement

Consider a body in equilibrium under a load system constituted by body forces $b_x$, $b_y$, $b_z$, surface forces $P_x$, $P_y$, $P_z$ and internal forces (stresses) $\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$. Imagine the body displaced from the equilibrium position and let $\Delta u$, $\Delta v$, $\Delta w$ define the virtual displacements.

If the initial position is an equilibrium position, the first-order work $\delta W_E$ done by the external forces is equal to the first order work $\delta W_D$ done by the internal forces during the virtual displacement, i.e.
\[ \delta W_E = \delta W_D \quad \text{for arbitrary } \Delta u, \Delta v, \Delta w \quad (2.1) \]

Equation (2.1) represents the Principle of Virtual Displacement. It is an alternative statement of the equilibrium conditions and is independent of material behaviour and magnitude of displacement. The three-dimensional form of (2.1) is

\[
\iiint_{\text{vol}} (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \sigma_z \delta \varepsilon_z + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{zx} \delta \gamma_{zx}) \, dx \, dy \, dz = \\
= \iiint_{\text{vol}} (b_x \Delta u + b_y \Delta v + b_z \Delta w) \, dx \, dy \, dz \\
+ \iint_{S_0} (p_x \Delta u + p_y \Delta v + p_z \Delta w) \, dS \quad (2.2)
\]

where \( \delta \varepsilon \) and \( \delta \gamma \) are the first-order strain increments due to the virtual displacements. The total boundary \( S \) consists of two zones: \( S_0 \) on which stresses are prescribed and \( S_u \) on which displacements are prescribed. \( S_u \) need not be considered since \( \Delta u, \Delta v \) and \( \Delta w \) vanish on it (they are prescribed as zero).

The first order strain increments, for the geometrically linear case, reduce to

\[
\begin{align*}
\delta \varepsilon_x &= \frac{3}{\partial x} \Delta u \\
\delta \gamma_{xy} &= \frac{3}{\partial y} \Delta u + \frac{3}{\partial x} \Delta v \\
\delta \varepsilon_y &= \frac{3}{\partial y} \Delta v \\
\delta \gamma_{yz} &= \frac{3}{\partial z} \Delta v + \frac{3}{\partial y} \Delta w \\
\delta \varepsilon_z &= \frac{3}{\partial z} \Delta w \\
\delta \gamma_{zx} &= \frac{3}{\partial z} \Delta u + \frac{3}{\partial x} \Delta w
\end{align*}
\quad (2.3)
\]
The three-dimensional form (2.2) of the Principle takes the compact form

\[ \int \int \int \sigma^T \delta \varepsilon \text{d}(\text{vol}) = \int \int \int \mathbf{D}^T \Delta \mathbf{u} \text{d}(\text{vol}) + \int_{S} \mathbf{P}^T \Delta \mathbf{u} \text{dS} \] (2.4)

using the following matrix notation

\[ \sigma = \{ \sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx} \} \]
\[ \delta \varepsilon = \{ \delta \varepsilon_x, \delta \varepsilon_y, \delta \varepsilon_z, \delta \gamma_{xy}, \delta \gamma_{yz}, \delta \gamma_{zx} \} \]
\[ \Delta \mathbf{u} = \{ \Delta u, \Delta v, \Delta w \} \] (2.5)

\[ \mathbf{P} = \{ P_x, P_y, P_z \} \]
\[ \mathbf{b} = \{ b_x, b_y, b_z \} \]

When the strain-displacement relations and boundary conditions are specified, (2.4) will give the equations of equilibrium.

2.2.2 Principle of Minimum Potential Energy

When the material is elastic and the deformation process is either isothermal or adiabatic, the work done by the external static forces is equal to the change in internal energy which is called strain energy.

Let \( V \) be the strain energy per unit volume, i.e. the strain energy density. \( \delta V \) will be equal to the first-order work per unit volume done by the stresses during the incremental displacement \( \Delta \mathbf{u} \) from an equilibrium position. Calling \( V \) the strain energy for the complete body one can write
\[ V = \iiint V \, d\text{(vol)} \]  
\[ \delta W_D = \delta V = \iiint \nabla^T \delta \varepsilon \, dx \, dy \, dz \]  

The external work can be expressed in terms of a force potential \( \Omega \) defined by

\[ \delta \Omega = -\iiint B^T \Delta y \, dx \, dy \, dz - \int_{S_0} P^T \Delta y \, dS = -\delta W_E \]  

\( \Omega \) could be obtained by integrating (2.7) with respect to the displacement components.

Let \( \chi \) be called the total potential energy defined by

\[ \chi(y) = V + \Omega \]  

Now the equilibrium requirement can be stated as

\[ \delta \chi = \delta V + \delta \Omega = 0 \]  

for any arbitrary \( \Delta y \) that satisfies the displacement boundary conditions. This means that an equilibrium position corresponds to a stationary value of the total potential energy.

It can also be shown (Washizu, 1968) that:

(a) In elastic situations the total potential energy is not only stationary but is a minimum;

(b) The total potential energy for an approximate displacement field which satisfies the boundary conditions is greater than the true value;

(c) If there is only a single force acting on the body, an approximate compatible displacement field corresponds to a structure which is
stiffer than the actual structure and therefore will give a lower bound on displacement (the calculated displacement is smaller than the real displacement).

2.2.3 Rayleigh-Ritz Method

This method is referred to because it is very closely related to the finite element method and the basic principles presented here will be found useful later.

Consider the problem of determining the unknown function \( y(x) \) which corresponds to a stationary value of the functional defined by

\[
I = \int_0^L f(y, \frac{dy}{dx}) \, dx \quad (2.10)
\]

\[y(0) = y(L) = 0\]

In the Rayleigh-Ritz method one approximates \( y \) with a finite expansion

\[
y = \sum_{i=1}^{n} q_i \phi_i(x) \quad (2.11)
\]

where \( q_i \) are undetermined constants called generalized coordinates and \( \phi_i(x) \) are prescribed functions of \( x \). Each of these functions must be continuous and satisfy the boundary conditions on \( y \) identically, i.e.

\[
\phi_i(0) = \phi_i(L) = 0 \quad i = 1, 2, \ldots, n \quad (2.12)
\]

Substituting (2.11) in (2.10) and requiring \( I \) to be stationary with respect to \( q_1, q_2, \ldots, q_n \) leads to \( n \) equations relating the generalized coordinates
\[ \frac{\partial I}{\partial q_j} = 0 \quad j = 1, 2, \ldots, n \quad (2.13) \]

These equations are linear in \( q_i \) when \( \phi_i \) are of second degree in \( y \) and \( dy/dx \), and can be solved to give \( q_1, q_2, \ldots, q_n \). The necessary conditions for convergence of the solution are (Brebbia, 1973):

(a) The approximating functions \( \phi_i \) must be continuous to one order less than the highest derivative in the integrand;

(b) The functions must individually satisfy the boundary conditions (displacement boundary conditions on \( Su \) when \( I = \chi \));

(c) The sequence of functions must be "complete", i.e. the mean square error vanishes in the limit

\[ \lim_{n \to \infty} \int_0^L \left( F - \sum_{i=1}^{n'} q_i \phi_i \right)^2 dx = 0 \]

Functions satisfying the conditions (a) and (b) are said to be admissible. Polynomials and trigonometric functions are suitable choices in the Rayleigh-Ritz method.

The Rayleigh-Ritz procedure of approximation frequently used in elastic analysis can be briefly summarized as follows:

1. The total potential energy expression given by (2.8) is formulated, assuming that the displacements vary with a finite set of undetermined parameters \( q_i \);
2. A set of simultaneous equations (2.13) minimizing the total potential energy with respect to the parameters \( q_i \) is set up;

3. The solution of the system of equations will determine the values of the parameters and hence the displacements and stresses.

2.2.4 Finite Element Methods

In the finite element displacement method the nodal displacements are taken as unknown and the displacement field is defined in terms of these variables, using the Principle of Minimum Potential Energy. Once the nodal displacements are known the strains are evaluated from the strain-displacement relations and, finally, the stresses are determined from the stress-strain relations. Thus the stresses are slightly less accurate than the displacements.

The finite element force method, based on the Principle of Minimum Complementary Energy, assumes that the stress field is continuous on the elements and that equilibrium is maintained inside the element and on the element boundaries. In this case the total complementary energy for the discretized (divided into elements) structure is larger than the true value. This implies that the results for displacement under a single concentrated load will converge towards the correct solution from above.

By applying both approaches it is possible to bracket the true solution for the displacement.
When comparing the Rayleigh-Ritz method with the finite element displacement method, its main differences are that:

(a) The generalized coordinates $q_i$ usually are not identified with the displacements at particular points (nodes);

(b) A single displacement expansion is employed for the entire domain.

Apart from these differences, the steps are the same but the finite element method has two important practical advantages:

1. It is easier (more economical) to solve the resultant system of equations than in the Rayleigh-Ritz method because, in this case, the coefficient matrix is full (no "band" occurs) due to the choice of a single expansion for the displacements which limits the method to relatively simple geometrical shapes of the total domain;

2. The use, in the finite element method, of associating each undetermined parameter with a particular displacement allows a simple physical interpretation invaluable to an engineer.

2.3. **General Procedure**

2.3.1 **Basic Assumptions**

This section will not go beyond the limits imposed by the use of the linear theory of elasticity which is based on the following assumptions:
(a) The change in orientation of a body due to displacement in negligible which leads to linear strain-displacement relations and also allows the equilibrium equations to be referred to the undeformed geometry;

(b) Linear material behaviour, i.e. linear stress-strain relations.

(c) Unless specifically stated otherwise, the discussion will also be restricted to the case of static loading and forces independent of time.

2.3.2 Finite Element Discretization

The subdivision of the continuum into elements is the most critical step of the method. The body is divided into volume elements with finite dimensions and certain points (nodal points or nodes) on the exterior boundary surfaces of the element are selected. Depending on the type of element chosen, sometimes points in the interior of the element are also selected. The elements and nodes are numbered and the element-node connectivity is specified by listing, for each element, the nodes associated with that element.

A variety of different elements is currently used. Quadrilateral and triangular elements are employed for plane strain, plane stress and plate bending. Shells are discretized with either flat or curved elements, solids of revolution with ring-type elements and three-dimensional solids with tetrahedral and hexahedral elements having straight or curved sides.
From past experience with the finite element method, some guidelines for the discretization of the structure have proved important (Brebbia, 1973):

(a) Irregularly shaped elements must be avoided. Equilateral triangles and "square" rectangles give the most accurate results;

(b) More nodes are required for stress concentration zones (high stress gradients) than for regions where the stresses vary smoothly;

(c) To evaluate the accuracy and convergence of results it is advisable to solve the same problem with a finer grid;

(d) The statics must always be checked.

The number and choice of displacement quantities is problem-dependent. For example, for plane strain and plane stress analysis the two in-plane displacements are an obvious choice. For three-dimensional analysis it is also convenient, in general, to choose the three displacement components $u$, $v$ and $w$.

2.3.3 Expansions for the Displacements Over the Element

If $\mathbf{U}_i$ is defined as the displacement matrix for node $i$, the elements of $\mathbf{U}_i$ are the nodal displacement quantities for node $i$ listed in some conventional order. For example, for the three-dimensional case,

$$
\mathbf{U}_i = \begin{bmatrix}
u \\ v \\ w
\end{bmatrix}
$$

(2.14)
The element chosen is assumed to have \( s \) nodal points to which one gives the numbers \( n_1, n_2, \ldots, n_s \). The node numbers can be specified in any order but, from a programming point of view, it is more convenient to list them according to a specified direction around the boundary. For instance, anti-clockwise direction for plane strain and plane stress elements.

The matrix containing as submatrices the nodal displacement matrices \( U_i \) for the element is called the element nodal displacement matrix and is denoted by \( U_n \):

\[
U_n = \begin{bmatrix}
U_{n1} \\
U_{n2} \\
\vdots \\
U_{ns}
\end{bmatrix}
\quad (2.15)
\]

Introducing expansions for the displacements over the element domain in terms of a set of parameters, one can write:

\[
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix} = A \alpha 
\quad (2.16)
\]

where \( A \) contains prescribed functions of \( x, y, z \)

\( \alpha \) contains the displacement parameters for the element.

Evaluating (2.16) for the nodes and substituting in (2.15) it is possible to relate \( U_n \) and \( \alpha \):

\[
U_n = C \alpha 
\quad (2.17)
\]
The order of $\mathbf{C}$ must be equal to or greater than the order of $\mathbf{U}_n$. The discussion will be restricted to the case where $\mathbf{C}$ is square and non-singular. Pre-multiplying (2.17) by $\mathbf{C}^{-1}$ and substituting in (2.16) one obtains

$$\mathbf{u} = \mathbf{A} \mathbf{C}^{-1} \mathbf{U}_n = \Phi \mathbf{U}_n \quad (2.18)$$

2.3.4 Interpolation Functions for the Displacements Over the Element

Instead of using (2.17) to derive (2.18) from (2.16), by inverting $\mathbf{C}$, one can obtain the same result directly by using interpolation functions. The procedure outlined in 2.3.3 is the "original" approach, the employment of interpolation functions is a subsequent innovation which is much more convenient. In this section, (2.18) will be obtained by this process limiting the discussion to the case of plane elements with two generalized displacements $u, v$ per node. It is assumed, in this case, that the in-plane displacements and external forces are constant over the thickness of the element.

The plane elements are suitable for plane strain and plane stress analysis and it will be shown later that only the strains $\varepsilon_x, \varepsilon_y$ and $\gamma_{xy}$ have to be considered. From (2.3) it can be obtained

$$\mathbf{C} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \quad (2.19)$$
Since the strain-displacement relations involve only first derivatives, inter-element displacement compatibility (see 2.2.3) requires only the displacement components \( u, v \) to be continuous across the element boundaries. Also, in order to be able to represent rigid body movement and constant strain states, the expansions for the displacements must contain a complete sequence of functions, say a complete polynomial. This means that the displacements must be expanded to

\[
\begin{align*}
  u &= a_1 + a_2 x + a_3 y + \text{additional terms} \\
  v &= a_4 + a_5 x + a_6 y + \text{additional terms}
\end{align*}
\]

Once the expansions are selected, the remaining operations are straightforward. In order to reduce the number of operations it is convenient to group \( u \) and \( v \) terms in \( \mathbf{U}_n \) separately as follows:

\[
\mathbf{U}_n = \begin{bmatrix} \text{\( u \) terms} \\ \text{\( v \) terms} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_n \\ \mathbf{v}_n \end{bmatrix}
\]

(2.20)

\[
\begin{align*}
  u &= \mathbf{\Phi} \mathbf{u}_n \\
  v &= \mathbf{\Phi} \mathbf{v}_n
\end{align*}
\]

where \( \mathbf{\Phi} \) is a row matrix containing the interpolation functions.

From (2.20) one can derive the matricial equation

\[
\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{O} \\ \mathbf{O} & \mathbf{\Phi} \end{bmatrix} \begin{bmatrix} \mathbf{u}_n \\ \mathbf{v}_n \end{bmatrix}
\]

(2.21)

or

\[
\mathbf{u} = \mathbf{\Phi} \mathbf{U}_n
\]

(2.22)

which is equivalent to (2.18) when
Using (2.20) it is also possible to write (2.19) as

$$\varepsilon = B U_n$$

or

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \phi'_{x} & \phi'_{y} \\ 0 & \phi'_{y} \\ \phi'_{x} & \phi'_{y} \end{bmatrix} \begin{bmatrix} U_n \\ V_n \end{bmatrix}$$

where

$$B = \begin{bmatrix} \phi'_{x} & \phi'_{y} \\ 0 & \phi'_{y} \\ \phi'_{x} & \phi'_{y} \end{bmatrix}$$

$$\phi'_{x} = \frac{3}{3x} \Phi = \left[ \frac{3+1}{3x}, \frac{3+2}{3x}, \ldots \right]$$

$$\phi'_{y} = \frac{3}{3y} \Phi = \left[ \frac{3+1}{3y}, \frac{3+2}{3y}, \ldots \right]$$

Once the element has been selected, the critical step in the finite element displacement method is the selection of the interpolation functions, i.e. $\Phi$ (see (2.22) and (2.23)) resulting from (2.20). If it includes all possible rigid body movements and all uniform strain states and if displacement compatibility along the boundaries between elements is satisfied, then the finite element solution represents an upper bound on the total potential energy and the solution for the displacements will converge, from below, to the correct value as the mesh size is decreased, i.e. the number of elements used for the same problem is increased.
Inter-element displacement compatibility requires that the assumed displacement field be continuous up to one order lower than the highest derivative appearing in the strain-displacement relations. For instance, the relations expressed by (2.19) for plane stress and plane strain analysis only involve first derivatives and therefore only the displacements must be continuous on the inter-element boundaries.

Complete polynomials satisfy all the requirements for convergence of the solution if they are, at least, of order equal to the highest derivative occurring in the strain-displacement relations. To obtain the necessary number of displacement parameters additional terms may be included.

Non-compatible elements do not provide a bound on the total potential energy but will give convergence to the true solution provided that all rigid body movements and uniform strain states are included.

2.3.5 Stationary Requirement for the Total Potential Energy Functional

From (2.8), using (2.6) and (2.7), one can write the expression of the incremental total potential energy for an element as

$$\delta X = \iiint \sigma^T \delta \varepsilon \, d\text{vol} - \iiint b^T \delta \mathbf{u} \, d\text{vol} - \int_{S} \mathbf{P}^T \delta \mathbf{u} \, ds$$

(2.27)

Restricting the discussion to linearly elastic behaviour one can express the stress-strain relations as
\[ \sigma = D (\varepsilon - \varepsilon_0) \quad (2.28) \]

where \( D \) is an elastic matrix (elastic rigidity matrix) containing the appropriate material properties;

\( \varepsilon \) are the strains produced by the stresses;

\( \varepsilon_0 \) are the initial strains not associated with stresses, e.g. strains due to variations in temperature.

In general \( D \) is symmetric and positive definite for a real material but it degenerates to a positive semi-definite matrix if the material is assumed to be incompressible. The form of \( D \) for orthotropic, cross-anisotropic and isotropic materials will be presented in chapter 3.

Using (2.28) and transposing the products of matrices in (2.27), one can write (2.27) as

\[ \delta \chi = \iiint (\delta \varepsilon^T D \varepsilon) \, d(\text{vol}) - \iiint (\delta \varepsilon^T D \varepsilon_0 + \delta \varepsilon^T b) \, d(\text{vol}) \]

\[ + \iint \delta \varepsilon^T P \, dS \quad (2.29) \]

The stationary requirement for the total potential energy functional, \( \delta \chi = 0 \) according to the Principle of Minimum Potential Energy, when applied to the whole structure leads to

\[ \sum_{n_e} \left[ \iiint (\delta \varepsilon^T D \varepsilon) \, d(\text{vol}) \right]_{\text{element } n} = \]

\[ = \sum_{n_e} \left[ \iiint (\delta \varepsilon^T D \varepsilon_0 + \delta \varepsilon^T b) \, d(\text{vol}) + \iint \delta \varepsilon^T P \, dS \right]_{\text{element } n} \quad (2.30) \]

where \( n_e \) denotes the total number of elements and \( P \) contains the prescribed external surface forces. Unless stated otherwise and according to conditions (c) of 2.3.1, it is
assumed \(b = \overline{b}\) (static loading) and \(p = \overline{p}\) (forces independent of time). For example, the dynamic case could be easily included in (2.30) by making

\[
b = \overline{b} - \rho \ddot{u}
\]  

(2.31)

where \(\rho\) is the mass density and \(\ddot{u}\) is the second derivative of \(u\) with respect to time.

2.3.6 Element Stiffness, Mass and Force Matrices

Since \(\overline{B}\) is independent of the displacements, from (2.24) one can obtain

\[
\delta \varepsilon = \overline{B} \delta \mathbf{u}_n
\]

(2.32)

Using (2.24) and transposing (2.32), one can arrive at

\[
\iiint \delta \varepsilon^T \overline{D} \varepsilon \ d(\text{vol}) = (\delta \mathbf{u}_n)^T \iint \overline{B}^T \overline{D} \overline{B} \ d(\text{vol}) \mathbf{u}_n
\]

\[
= (\delta \mathbf{u}_n)^T \mathbf{K} \mathbf{u}_n
\]

(2.33)

where

\[
\mathbf{K} = \iint \overline{B}^T \overline{D} \overline{B} \ d(\text{vol})
\]

(2.34)

is called the element stiffness matrix.

If (2.18) also applies to the dynamic case, then

\[
\ddot{u} = \frac{\partial^2}{\partial t^2} u = \Phi \dddot{u}_n
\]

and

\[
\iiint (-\rho) \delta \ddot{u}^T \ddot{u} \ d(\text{vol}) = -(\delta \mathbf{u}_n)^T (\iint \overline{\Phi}^T \overline{\Phi} \ d(\text{vol})) \dddot{u}_n
\]

\[
= -(\delta \mathbf{u}_n)^T \mathbf{M} \dddot{u}_n
\]

(2.36)
where
\[ M = \iiint \rho \hat{\Phi}^T \hat{\Phi} \, d(\text{vol}) \] (2.37)
is the consistent element mass matrix.

Similarly,
\[
\left[ \iiint (\delta \epsilon^T D \epsilon_o + \delta u^T b) \, d(\text{vol}) + \delta u^T P \, ds \right]_{\text{element } n} = \\
(\delta u_n)^T \left[ \iiint (B^T D \epsilon_o + \hat{\Phi}^T b) \, d(\text{vol}) + \iiint \hat{\Phi}^T P \, ds \right] \\
= (\delta u_n)^T P 
\] (2.38)

where
\[ P = \iiint (B^T D \epsilon_o + \hat{\Phi}^T b) \, d(\text{vol}) + \iiint \hat{\Phi}^T P \, ds \] (2.39)
is the consistent element force matrix.

\( P \) contains the prescribed external surface forces and the area integral involves only the exterior portion of the surface area for the element \( n \). Now, considering (2.31) and using the expressions (2.33), (2.36) and (2.38), the stationary requirement (2.30) can be expressed in a more compact way as
\[
\sum_{n_e} (\delta u_n)^T (K u_n + M \ddot{u}_n) = \sum_{n_e} (\delta u_n)^T P 
\] (2.40)

2.3.7 Transformation of Displacements from Local to Global Frame

In the derivation of the element matrices, the displacements are usually referred to a local element frame rather than the global frame common to all elements of the structure. However, (2.40) is valid only when the same frame is used for all elements. Therefore, it is necessary to transform the nodal displacements from the (local) element frame to the global frame.
Let the superscript \( 'o' \) indicate the global frame, i.e. \( U_i \) represents the displacement matrix (2.14) at node \( i \) referred to the local frame and \( U_i^o \) is the corresponding matrix referred to the global frame. These matrices are related by

\[
U_i = R U_i^o
\]

where \( R \) is the rotation matrix containing the direction cosines for the local frame with respect to the global frame. Fig. 2.1 illustrates the notation for the case of a plane triangular element. The form of \( R \) in this case is

\[
R = \begin{bmatrix}
\cos \theta & \sin \theta \\
-sin \theta & \cos \theta
\end{bmatrix}
\]

2.3.8 Assembly and Solution of System Equations

The system equations are assembled by superposing the contributions of the elements to each nodal equilibrium equation. The steps required for the \( n \)-th element will be described.

The first step is to expand the element nodal displacement matrix \( U_n \) in terms of the nodal displacement matrices in a suitable way:

\[
U_n = \{ u_{nj} \}_{j=1,2, \ldots, s}
\]

where \( s \) is the number of nodes of the element

\( nj \) is the actual node number, in the structure, of the \( j \)-th relative node number of the element.
Then $K, M$ and $P$ must be partitioned in a manner consistent with the partitioning of $U_n$:

$$K = [k_{ij}]$$

$$M = [m_{ij}]$$

$$P = [p_i] \quad i, j = 1, 2, \ldots, s \quad (2.44)$$

Using (2.41), the terms of (2.40) expand to

$$
(\delta U_n)^T K U_n = \sum_{i=1}^{s} (\delta u_{n1}^o)^T (\sum_{j=1}^{s} k_{ij} u_{nj}^o)
$$

$$
(\delta U_n)^T M \ddot{U}_n = \sum_{i=1}^{s} (\delta u_{n1}^o)^T (\sum_{j=1}^{s} m_{ij}^o u_{nj}^o) \quad (2.45)
$$

$$
(\delta U_n)^T P = \sum_{i=1}^{s} (\delta u_{n1}^o)^T p_i^o
$$

where

$$k_{ij}^o = R^T k_{ij} R$$

$$m_{ij}^o = R^T m_{ij} R$$

$$p_i^o = R^T p_i \quad (2.46)$$

If $R = I$ the operations defined by (2.41) are not necessary since $U_i = U_i^o$. Incidentally, it must be noted that, while $k_{ij}^o$ depends on the selection of the global frame, $k_{ij}$ is a natural property of the element. The same applies to $m_{ij}$.

When $R = I$, i.e. the local frame coincides with the global frame for all elements, expressions (2.45) will be simply
\[
\begin{align*}
(\delta \mathbf{U}_n)^T \mathbf{K} \mathbf{U}_n &= \sum_{i=1}^{S} (\delta \mathbf{u}_{ni})^T \sum_{j=1}^{S} k_{ij} \mathbf{u}_{nj} \\
(\delta \mathbf{U}_n)^T \mathbf{M} \ddot{\mathbf{U}}_n &= \sum_{i=1}^{S} (\delta \mathbf{u}_{ni})^T \sum_{j=1}^{S} m_{ij} \mathbf{u}_{nj} \\
(\delta \mathbf{U}_n)^T \mathbf{P} &= \sum_{i=1}^{S} (\delta \mathbf{u}_{ni})^T \mathbf{p}_i
\end{align*}
\]

(2.47)

The governing equations are obtained by substituting (2.45) in (2.40) and equating the coefficients of \( \delta \mathbf{u}_j^0 \) on the left and right hand sides. To simplify the notation, let the superscript 'o' be dropped on the various matrices, assuming all quantities are referred to the global frame. Let \( N \) be the total number of nodes of the structure and \( \mathbf{U} \) the system nodal displacement matrix defined by

\[
\mathbf{U} = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \} \quad \text{(2.48)}
\]

After expanding (2.40) by summing the contributions of the elements incident on each node, the resultant system equations could be represented in this compact manner

\[
(\delta \mathbf{U})^T (\mathbf{K} \mathbf{U} + \mathbf{M} \ddot{\mathbf{U}}) = (\delta \mathbf{U})^T \mathbf{P} \quad \text{(2.49)}
\]

Since (2.49) must be satisfied for arbitrary \( \delta \mathbf{U} \), one can write simply

\[
\mathbf{K} \mathbf{U} + \mathbf{M} \ddot{\mathbf{U}} = \mathbf{P} \quad \text{(2.50)}
\]

where \( \mathbf{K} \) and \( \mathbf{M} \) are the system stiffness and mass matrices and \( \mathbf{P} \) is the system nodal force matrix.
2.3.9 Direct Stiffness Method

The partitioned form of (2.50) is

\[
\begin{bmatrix}
M_{11} & M_{12} & \cdots & M_{1N} \\
M_{21} & M_{22} & \cdots & M_{2N} \\
\vdots & & & \\
M_{N1} & M_{N2} & \cdots & M_{NN}
\end{bmatrix}
\begin{bmatrix}
\vec{u}_1 \\
\vec{u}_2 \\
\vdots \\
\vec{u}_N
\end{bmatrix}
+
\begin{bmatrix}
K_{11} & K_{12} & \cdots & K_{1N} \\
K_{21} & K_{22} & \cdots & K_{2N} \\
\vdots & & & \\
K_{N1} & K_{N2} & \cdots & K_{NN}
\end{bmatrix}
\begin{bmatrix}
\vec{u}_1 \\
\vec{u}_2 \\
\vdots \\
\vec{u}_N
\end{bmatrix}
= \begin{bmatrix}
\vec{P}_1 \\
\vec{P}_2 \\
\vdots \\
\vec{P}_N
\end{bmatrix}
\tag{2.51}
\]

Since (2.50) is equivalent to (2.49) without involving \( \delta U^T \), this suggests a simple process to set up the system equations. It is called the "Direct Stiffness Method". It consists in forming \( K, M \) and \( P \), by superposing the contributions of each element. Considering these matrices partitioned as shown in (2.51), the process is straightforward and follows direct from (2.45). For instance, \( \vec{P}_1 \) must be added to row \( n_i \) of \( \vec{P} \) with \( i = 1, 2, \ldots, s \). The elements of \( \vec{P} \) are set to zero before starting the process.

The operations are carried out for all elements, taking advantage of the fact that \( K_{ij} \) is symmetric and \( K_{ij} = K_{ji}^T \), leading to symmetric \( K \). The same applies to \( M \) and this simplifies the operations, since only the upper triangles of the system stiffness and mass matrices have to be formed and stored.

It is easy to see now that it is possible and convenient, when using the direct stiffness method, to write for a single element,
\begin{align}
\mathbf{K} \mathbf{u}_n + \mathbf{M} \ddot{\mathbf{u}}_n &= \mathbf{P} = \mathbf{P}_o + \mathbf{P}_b + \mathbf{P}_d \\
\text{where } \mathbf{P}_o &= \int \mathbf{B}^T \mathbf{D} \mathbf{e}_o \, d\text{(vol)} \\
\mathbf{P} &= \int \mathbf{\Phi}^T \mathbf{b} \, d\text{(vol)} \\
\mathbf{P} &= \int \mathbf{\Phi}^T \mathbf{p} \, ds
\end{align}

(2.52)

are the initial, body and external surface force matrices. Of course, they reduce to vectors when a single load system is considered.

Disregarding the dynamic case which will not be considered again (i.e. (2.31) reduces to \( \mathbf{b} = \mathbf{b}_o \)), from (2.50) and (2.52) one has simply

\begin{align}
\mathbf{K} \mathbf{u}_n &= \mathbf{P} = \mathbf{P}_o + \mathbf{P}_b + \mathbf{P}_d \\
\mathbf{K} \mathbf{u} &= \mathbf{P}
\end{align}

(2.54)

(2.55)

2.3.10 Implementation of Boundary Conditions

It may appear that the displacements can be determined by solving the system of simultaneous linear equations represented by (2.55). However, this is not possible because \( \mathbf{P} \) includes the reactions (they are external surface forces) which are as yet unknown and (2.55) does not include any information regarding the manner in which the structure is supported. When (2.55) is modified to take into account the support conditions, the first difficulty is obviated so that prior knowledge of the reactive forces will not be necessary.
When the boundary conditions are imposed, some of the elements of \( \mathbf{U} \) are prescribed, i.e. known. The constraints, generally in the form of \( U_i = 0 \), must be introduced into (2.55). This could be done, for the particular but important case of \( U_i = 0 \) which will be considered in this section, by simply deleting the corresponding equations but that impliesrenumbering the rows and columns of the matrices concerned.

For practical programming reasons, it is usually preferred to preserve the size of the system of equations by replacing the equilibrium equations which should otherwise be deleted by modified equations enforcing the boundary conditions. The process consists in enforcing the condition \( U_i = 0 \) by replacing the off-diagonal elements of the \( i \)-th row and \( i \)-th column of \( \mathbf{K} \) and also the \( i \)-th row of \( \overline{\mathbf{P}} \) by zero. The diagonal element \( K_{ii} \) of \( \mathbf{K} \) could be given any non-zero value but, for computational reasons, it is more convenient to use a value roughly of the same order of the other diagonal elements of \( \mathbf{K} \). If there are other conditions of the type \( U_i = 0 \) to be implemented, this can be done by using repeatedly the process outlined above.

After introducing all boundary conditions (of any type), (2.55) will become

\[
\mathbf{K} \mathbf{d} = \mathbf{f} \tag{2.56}
\]

where \( \mathbf{K} \) is the stiffness matrix of the structure
\( \mathbf{d} \) is the displacement vector of the structure
\( \mathbf{f} \) is the load vector of the structure
Now it is easy to understand why some authors call $\mathbf{K}$ the "primary stiffness matrix of the structure" (prior to enforcing the boundary conditions) and $\mathbf{P}$ the "appended load vector of the structure" (containing all external loads and reactive forces).

The mass matrix of the structure is always positive definite. The stiffness matrix of the structure is positive definite when, under the load system, the structure is restrained with respect to rigid body movement.

Now (2.56) can be solved by any suitable method. There is a variety of solution techniques which will be referred in chapter 4.

### 2.3.11 Evaluation of Stresses

The strains and stress for each element can be obtained from the nodal displacements by using the following expressions (see (2.24) and (2.28)):

\[
\mathbf{\varepsilon} = \mathbf{B} \mathbf{U}_n
\]

\[
\mathbf{\sigma} = \mathbf{D} (\mathbf{\varepsilon} - \mathbf{\varepsilon}_o) = \mathbf{\sigma}_o + \mathbf{D} \mathbf{B} \mathbf{U}_n
\]

(2.57)

The stresses are usually evaluated at one or more points of the element, using various ways. One method is to evaluate (2.57) at the nodes of the structure and then to average, at each node, the values for the elements incident on that node. Another approach is to evaluate (2.57) at a particular point of the element, e.g. the centroid. A third approach consists in determining the stresses at a point, from the nodal displacements, by applying finite difference operators instead of using the strain expansions.
It is usual to arrange also for the computer to calculate the principal stresses and their directions for every element of the structure.

2.4 **Summary of the Method**

The finite element displacement method is one of various finite element techniques and very similar to the well known Rayleigh-Ritz method. It is the most widely used and can be summarized as follows:

1. Divide the structure into elements interconnected at certain nodal points;
2. Select the displacement expansions for the displacements over the element and generate the element stiffness, mass and nodal force matrices;
3. Superpose the contributions of each element to form the stiffness, mass and force matrices of the structure;
4. Introduce the displacement boundary conditions in the resultant system of simultaneous equations;
5. Solve the system of equations to determine the nodal displacements;
6. Determine the element strains from the nodal displacements and the stresses from the strains;
7. Determine the element principal stresses and their directions.

The procedure outlined in this chapter could be extended to include material and geometric non-linearity but these topics are beyond the scope of this thesis.
Fig. 2.1 Local and Global Frames for an Element
3.1 Elastic Rigidity Matrix

3.1.1 The Generalized Hooke's Law

For small deformations, a continuous medium which obeys the generalized Hooke's Law is widely used as the model of an elastic body. Thus the components of deformation are assumed to be linear functions of the components of stress.

An elastic body is called isotropic (Eekhnitskii, 1944) when its elastic properties are identical in all directions and anisotropic when its elastic properties are different for different directions. A body is called homogeneous when its elastic properties are identical in all parallel directions passing through any of its points, i.e. all identical elements in the shape of a rectangular parallelepiped with mutually parallel edges possess identical elastic properties.

Considering an arbitrary system of orthogonal coordinates \(x, y, z\), the generalized Hooke's Law can be expressed by the following system of linear equations:

\[
\begin{align*}
\varepsilon_x &= a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \sigma_z + a_{14} \tau_{yz} + a_{15} \tau_{zx} + a_{16} \tau_{xy} \\
\varepsilon_y &= a_{21} \sigma_x + a_{22} \sigma_y + a_{23} \sigma_z + a_{24} \tau_{yz} + a_{25} \tau_{zx} + a_{26} \tau_{xy} \\
\varepsilon_z &= a_{31} \sigma_x + a_{32} \sigma_y + a_{33} \sigma_z + a_{34} \tau_{yz} + a_{35} \tau_{zx} + a_{36} \tau_{xy} \\
\gamma_{xy} &= a_{61} \sigma_x + a_{62} \sigma_y + a_{63} \sigma_z + a_{64} \tau_{yz} + a_{65} \tau_{zx} + a_{66} \tau_{xy}
\end{align*}
\]

(3.1)

where \(a_{ij} = a_{ji}\) are the elastic constants or coefficients of deformation. The number of independent constants is 21 for the most general case of anisotropy.
An elastic body may have a plane of elastic symmetry, i.e. possess such properties that any two directions which are symmetrical with respect to the plane are equivalent in all respects. Bodies which have (or can be considered as having) three planes of elastic symmetry are encountered often in practice. Choosing a system of orthogonal coordinates $x, y, z$ with axes normal to the planes of symmetry, some elastic constants will become zero and (3.1) can be written simply as follows:

$$\varepsilon_x = a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z \quad \gamma_{yz} = a_{44}\tau_{yz}$$

$$\varepsilon_y = a_{21}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z \quad \gamma_{zx} = a_{55}\tau_{zx}$$

$$(3.2)$$

$$\varepsilon_z = a_{31}\sigma_x + a_{32}\sigma_y + a_{33}\sigma_z \quad \gamma_{xy} = a_{66}\tau_{xy}$$

The number of independent elastic constants is now nine. The directions normal to the planes of elastic symmetry are called principal directions of elasticity or, simply, principal directions. A homogeneous body with three mutually perpendicular planes of elastic symmetry passing through every point is called orthogonal-anisotropic, or simply, orthotropic.

An element of an orthotropic body in the shape of a rectangular parallelepiped with faces parallel to the planes of elastic symmetry remains a rectangular parallelepiped when subjected to tension or compression due to loading applied to two parallel faces, i.e. the length of the edges will change but the angles between the faces will remain unchanged.
Using the "engineering constants", equations (3.2) can be written in a more familiar way:

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E_x} \varepsilon_x x + \frac{v_{yx}}{E_y} \varepsilon_y y + \frac{v_{zx}}{E_z} \varepsilon_z z \\
\varepsilon_y &= \frac{v_{xy}}{E_x} x + \frac{1}{E_y} \varepsilon_y y + \frac{v_{zy}}{E_z} \varepsilon_z z \\
\varepsilon_z &= \frac{v_{xz}}{E_x} x + \frac{v_{yz}}{E_y} \varepsilon_y y + \frac{1}{E_z} \varepsilon_z z
\end{align*}
\]

where

\[
\gamma_{yz} = \frac{1}{G_{yz}}
\]

\(\gamma_{xz} = \frac{1}{G_{xz}}\)

\(\gamma_{xy} = \frac{1}{G_{xy}}\)

\(3.3\)

are the Young's moduli for tension (compression) along the principal directions of elasticity \(x, y, z\), respectively.

\(G_{yz}, G_{zx}, G_{xy}\) are the shear moduli characterizing the changes of angles between the principal directions \(y\) and \(z\), \(z\) and \(x\), \(x\) and \(y\), respectively.

\(\nu_{yz}\) is the Poisson's ratio characterizing the decrease (increase) of strain in \(x\)-direction when tension (compression) is applied in \(y\)-direction. Similarly, for other Poisson's ratios.

Due to the symmetry of coefficients \(a_{21}=a_{12}, a_{31}=a_{13}, a_{32}=a_{23}\) in the equivalent systems of equations (3.2) and (3.3), the Young's moduli and the Poisson's ratios must be related by

\[
\begin{align*}
E_x \nu_{yx} &= E_y \nu_{xy} \\
E_y \nu_{zy} &= E_z \nu_{yz} \\
E_z \nu_{xz} &= E_x \nu_{zx}
\end{align*}
\]

\(3.4\)
Because of these restrictions only nine of the engineering constants appearing in (3.3) are independent.

When a body has a plane of isotropy, i.e. through every point passes a plane in which all directions are equivalent with respect to the elastic properties, then the number of independent elastic constants is only five. A body possessing anisotropy of this type is called (Lyav, 1935) transversely-isotropic or cross-anisotropic. A direction normal to the plane of isotropy and all directions in this plane are principal directions of elasticity. Crystals of the hexagonal system exhibit this type of anisotropy and some natural soil deposits can be considered as cross-anisotropic bodies with a horizontal plane of isotropy (Barden, 1963).

Since there are only five independent elastic constants for the important case of cross-anisotropy, more suggestive notations (Gibson, 1974) will be introduced which one can define as follows, assuming that the plane of anisotropy is horizontal:

\[ \begin{align*} 
E_H & \quad \text{Horizontal Young's modulus} \\
E_V & \quad \text{Vertical Young's modulus} \\
\nu_{HH} & \quad \text{Poisson's ratio characterizing the decrease (increase) of strain in a direction of the horizontal plane when tension (compression) is applied in a perpendicular direction of the same plane} 
\end{align*} \]
$G_{VH}$ Shear modulus characterizing the changes of angles between the vertical direction and any horizontal direction.

The shear modulus in the horizontal plane (not independent) is given by $G_{HH} = E_H/[2(1+\nu_{HH})]$.

Taking the axis $y$ in the vertical direction and axes $x$ and $z$ in the horizontal plane of isotropy, one can relate the engineering constants for an orthotropic and a cross-anisotropic body as follows:

$$E_H = E_x = E_z \quad \nu_{HH} = \nu_{xz} = \nu_{zx} \quad G_{VH} = G_{yz} = G_{xy}$$

(3.5)

$$E_V = E_y \quad \nu_{VH} = \nu_{yx} = \nu_{yz} \quad G_{HH} = G_{xz} = G_{zx}$$

Using (3.4) and (3.5) it is possible to express (3.3) as

$$\varepsilon_x = \frac{1}{E_H \sigma_x} \nu_{VH} \sigma_y \nu_{HH} \sigma_z \quad \gamma_{xy} = \frac{1}{G_{VH}} \tau_{xy}$$

$$\varepsilon_y = \frac{1}{E_V \sigma_y} \nu_{VH} \sigma_x \nu_{HH} \sigma_z \quad \gamma_{yz} = \frac{1}{G_{VH}} \tau_{xy}$$

$$\varepsilon_z = \frac{1}{E_H \sigma_z} \nu_{VH} \sigma_x \nu_{HH} \sigma_y \quad \gamma_{zx} = \frac{2(1+\nu_{HH})}{E_H} \tau_{zx}$$

(3.6)

If a body is isotropic, every plane is a plane of elastic symmetry and every direction is a principal direction of elasticity. The generalized Hooke's Law will reduce to

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] \quad \gamma_{yz} = \frac{1}{G} \tau_{yz}$$

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_z + \sigma_x)] \quad \gamma_{zx} = \frac{1}{G} \tau_{zx}$$

(3.7)

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad \gamma_{xy} = \frac{1}{G} \tau_{xy}$$
where $E$ is the Young's modulus

$v$ is the Poisson's ratio

$$G = \frac{E}{2(1+v)}$$ is the shear modulus.

There are, in this case, only two independent constants $(E,v)$ which are related to the Lamé's constants $(\lambda, \mu)$ in the following way:

$$\lambda = \frac{E}{(1+v)(1-v)} \quad \mu = \frac{E}{2(1+v)} = G \quad (3.8)$$

For isotropic bodies the elastic constants do not depend on the system of coordinates. This is not valid for anisotropic bodies but the new elastic constants $a'_{ij}$ for the cartesian system of coordinates $x', y', z'$ can be expressed in terms of the constants $a_{ij}$ for another system of cartesian coordinates $x, y, z$.

It has been shown above that different types of bodies or materials have different numbers of independent elastic constants. However, in spite of being independent of each other they cannot take arbitrary values, as will be shown in 3.1.6.

3.1.2 Plane Stress Analysis

Consider the prismatic homogeneous solid shown in Fig. 3.1 in equilibrium under body forces and forces distributed on the lateral surface $S$. This solid is said to be in a state of plane stress if the following conditions are satisfied (Lekhnitskii, 1968):
1. The thickness \( h \) is small compared with the other dimensions

2. At each point of the solid there is a plane of elastic symmetry which is parallel to the middle plane \((x,y)\)

3. The forces applied to the lateral surface \( S \) and the body forces act within planes parallel to the middle plane, are distributed symmetrically with respect to this plane and can be considered constant over the thickness.

4. The deformations are small.

Since the middle plane does not bend while undergoing deformation it is reasonable to assume that \( \sigma_z, \tau_{zx} \) and \( \tau_{zy} \) are small in comparison with \( \sigma_x, \sigma_y, \tau_{xy} \) and the variation of all these quantities with respect to \( z \) is negligible.

The "plane stress" (or plate stretching) problem is based on these assumptions which can be represented by the following statements:

\[
\begin{align*}
u &= u(x,y) \\
v &= v(x,y) \\
\sigma_z &= \tau_{zx} = \tau_{zy} = 0
\end{align*}
\] (3.9)

The transverse displacement \( w \) is not independent and can be obtained from the stress-strain relation for \( \varepsilon_z \):

\[
\varepsilon_z = \frac{\partial w}{\partial z} = f(\sigma_x, \sigma_y, \tau_{xy})
\] (3.10)
Including the effect of variations in temperature, one can derive from (3.3) using (3.4), for an orthotropic material,

\[
\begin{align*}
\varepsilon_x &= \frac{1}{E_x} \sigma_x - \frac{\nu_{yx}}{E_y} \sigma_y - \frac{\nu_{zx}}{E_z} \sigma_z + \alpha_x T \\
\varepsilon_y &= -\frac{\nu_{yx}}{E_y} \sigma_x + \frac{1}{E_y} \sigma_y - \frac{\nu_{zy}}{E_z} \sigma_z + \alpha_y T \\
\varepsilon_z &= -\frac{\nu_{zx}}{E_z} \sigma_x - \frac{\nu_{zy}}{E_z} \sigma_y + \frac{1}{E_z} \sigma_z + \alpha_z T \\
\gamma_{xy} &= \frac{1}{G_{xy}} \tau_{xy} \\
\gamma_{yz} &= \frac{1}{G_{yz}} \tau_{yz} \\
\gamma_{zx} &= \frac{1}{G_{zx}} \tau_{zx}
\end{align*}
\]

(3.11)

where \( T \) is the temperature increment.

\( \alpha_x, \alpha_y, \alpha_z \) are the thermal strain coefficients.

Taking into account (3.9) one can express the relevant equations (3.11) as

\[
\begin{align*}
\begin{bmatrix}
\varepsilon_x - \alpha_x T \\
\varepsilon_y - \alpha_y T \\
\gamma_{xy}
\end{bmatrix} = &\begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\
-\frac{\nu_{yx}}{E_y} & \frac{1}{E_y} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\end{align*}
\]

(3.12)

or

\[
\varepsilon - \varepsilon_o = \mathcal{C}_1 \sigma
\]

(3.13)

for

\[
\mathcal{C}_1 = 
\begin{bmatrix}
\frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & 0 \\
-\frac{\nu_{yx}}{E_y} & \frac{1}{E_y} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{bmatrix}
\]

(3.14)
Inverting (3.12) leads to

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E_y}{(1-nv^2_y)} \begin{bmatrix}
n & nv_{yx} & 0 \\
nv_{yx} & 1 & 0 \\
0 & 0 & m(1-nv^2_{yx})
\end{bmatrix} \begin{bmatrix}
\varepsilon_x - a_x T \\
\varepsilon_y - a_y T \\
\gamma_{xy}
\end{bmatrix}
\]

(3.15)

where

\[ n = \frac{E_x}{E_y} \quad m = \frac{G_{xy}}{E_y} \]  

(3.16)

Matricial equation (3.15) for an orthotropic material can be written as

\[ \mathbf{C} = \mathbf{D}_1 (\mathbf{E} - \mathbf{E}_{01}) \]  

(3.17)

in which

\[ \mathbf{D}_1 = \frac{E_y}{(1-nv^2_y)} \begin{bmatrix}
n & nv_{yx} & 0 \\
nv_{yx} & 1 & 0 \\
0 & 0 & m(1-nv^2_{yx})
\end{bmatrix} \]

(3.18)

The corresponding matrices \( \mathbf{C}_2 \) and \( \mathbf{D}_2 \) for a cross-anisotropic material can be derived from \( \mathbf{C}_1 \) and \( \mathbf{D}_1 \) using relationships (3.5):

\[
\mathbf{C}_2 = \begin{bmatrix}
\frac{1}{E_H} & -\frac{\nu_{VH}}{E_V} & 0 \\
-\frac{\nu_{VH}}{E_V} & \frac{1}{E_V} & 0 \\
0 & 0 & \frac{1}{G_{VH}}
\end{bmatrix}
\]

(3.19)
\[
D_2 = \frac{E_V}{(1-nVH^2)} \begin{bmatrix}
n & nVH & 0 \\
VH & 1 & 0 \\
0 & 0 & m(1-nVH^2)
\end{bmatrix} \tag{3.20}
\]

for
\[
n = \frac{E_H}{E_V} \quad \text{and} \quad m = \frac{G_{VH}}{E_V} \tag{3.21}
\]

The corresponding matrices \(C_3\) and \(D_3\) for an isotropic material can be derived from \(C_2\) and \(D_2\) considering \(E_H = E_V = E\), \(V_{VH} = V_{HH} = V\) and \(G_{VH} = G_{HH} = G\):

\[
C_3 = \frac{1}{E} \begin{bmatrix}
1 & -V & 0 \\
-V & 1 & 0 \\
0 & 0 & 2(1+V)
\end{bmatrix} \tag{3.22}
\]

\[
D_3 = \frac{E}{(1-V^2)} \begin{bmatrix}
1 & V & 0 \\
V & 1 & 0 \\
0 & 0 & (1-V)/2
\end{bmatrix} \tag{3.23}
\]

As a verification it can be checked that \(C_3 D_3 = C_2 D_2 = C_1 D_1 = \mathbf{I}\).

For any type of elastic material one can write the general relationship

\[
\sigma = D (\epsilon - \epsilon_0) \tag{3.24}
\]

where \(D\) is called the elastic rigidity matrix and \(\epsilon_0\) is the initial strain vector. In this section \(D\) was derived for
the particular case of a plane stress problem; in the next section it will be derived for a plane strain problem. The same could be done for the general three-dimensional case in which \( D \) would appear as a 6 x 6 square matrix. In all cases it is called the elastic rigidity matrix and contains the relevant material properties. \( C = D^{-1} \) is called the elastic compliance matrix.

### 3.1.3 Plane Strain Analysis

Consider the cylindrical homogeneous solid of arbitrary cross-section of Fig. 3.1 in equilibrium under body forces and forces distributed on the lateral surfaces. This solid is said to undergo plane strain or plane deformation when the following conditions are satisfied (Lekhnitskii, 1968):

1. The solid is very long compared with the dimensions of the cross-section
2. At each point of the solid there is a plane of elastic symmetry (plane \( x,y \)) normal to the generator
3. The forces are acting in planes normal to the generator and can be considered constant along the generator
4. The deformations are small.

The cross-sections far away from the ends of the cylinder can be considered as plane after the deformation although, strictly speaking, this is only true for a cylinder of infinite length. The "plane strain" problem is based on these assumptions which can be represented by the following statements:
\[
\begin{align*}
\epsilon_x &= \frac{\partial u}{\partial x} \\
\epsilon_y &= \frac{\partial v}{\partial y} \\
\epsilon_z &= \gamma_{yz} = \gamma_{zx} = 0 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\

\text{In plane strain problems } \sigma_z \neq 0 \text{ but it can be determined by}
\end{align*}
\]

\[
\sigma_z = v_{zx} \sigma_x + v_{zy} \sigma_y - E_z a_z T
\]

obtained from (3.11) for \( \epsilon_z = 0 \).

Substituting (3.27) in (3.11) one obtains the relevant equations in matricial form:

\[
\begin{pmatrix}
\epsilon_x - (a_x + v_{zx} a_z) T \\
\epsilon_y - (a_y + v_{zy} a_z) T \\
\gamma_{xy}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{E_x} \frac{v_{zx}^2}{E_z} & -\frac{v_{yx} v_{zx} v_{zy}}{E_y E_z} & 0 \\
-v_{yx} \frac{v_{zx} v_{zy}}{E_y E_z} & \frac{1}{E_y} - \frac{v_{zy}^2}{E_z} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{pmatrix}
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix}
\]

(3.28)

This matricial equation can be written in a manner formally identical to (3.12) as
\[
\begin{pmatrix}
\varepsilon_x - a_x^T \\
\varepsilon_y - a_y^T \\
\gamma_{xy}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{E_x} & \frac{-\nu_{yx}}{E_y} & 0 \\
\frac{-\nu_{yx}}{E_y} & \frac{1}{E_y} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{pmatrix}
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix}
\]

(3.29)

In which

\[a_x' = a_x + \nu_{zx} a_z\]

\[a_y' = a_y + \nu_{zy} a_z\]

\[E_x' = \frac{E_x}{1-\nu_{zx}^2 E_x/E_z}\]

\[E_y' = \frac{E_y}{1-\nu_{zy}^2 E_y/E_z}\]

\[\nu_{yx}' = \nu_{yx} + \nu_{zx} \nu_{zy} E_y/E_z \]

Matricial equation (3.29) can be written in compact form as

\[\varepsilon - \varepsilon_o = \mathbf{C}' \mathbf{\sigma}\]

(3.31)

where

\[
\mathbf{C}' = \begin{pmatrix}
\frac{1}{E_x'} & \frac{-\nu_{yx}'}{E_y'} & 0 \\
\frac{-\nu_{yx}'}{E_y'} & \frac{1}{E_y'} & 0 \\
0 & 0 & \frac{1}{G_{xy}}
\end{pmatrix}
\]

(3.32)

Since this matrix \(\mathbf{C}'\) is formally identical to \(\mathbf{C}\) which, by inversion, led to \(\mathbf{D}_1\) one can obtain directly by comparison with (3.18):
The corresponding matrices $C'_2$ and $D'_1$ for a cross-anisotropic material can be derived from $C'_1$ and $D'_1$, using relationships (3.5), (3.30) and (3.34):

$$C'_2 = \begin{bmatrix} \frac{1-\nu^2_{HH}}{E_H} & -\frac{(1+\nu_{HH})\nu_{VH}}{E_v} & 0 \\ -\frac{(1+\nu_{HH})\nu_{VH}}{E_v} & \frac{1-\nu^2_{VV}}{E_v} & 0 \\ 0 & 0 & \frac{1}{G_{VH}} \end{bmatrix}$$

$$D'_2 = \frac{E_v}{(1+\nu_{HH})(1-\nu_{HH}-2\nu_{VH}^2)} \begin{bmatrix} n(1-\nu_{VH}^2) & \nu_{VH}(1+\nu_{HH}) & 0 \\ \nu_{VH}(1+\nu_{HH}) & 1-\nu^2_{HH} & 0 \\ 0 & 0 & m(1+\nu_{HH})(1-\nu_{HH}-2\nu_{VH}^2) \end{bmatrix}$$

The corresponding matrices $C'_3$ and $D'_3$ for an isotropic material can be derived from $C'_2$ and $D'_2$ considering $E_H=E_V=E$, $\nu_{VH}=\nu_{HH}=\nu$ and $G_{VH}=G_{HH}=G$. 

$D'_1 = \frac{E'_y}{1-n'(\nu_{yx}')^2} \begin{bmatrix} n' & n'\nu_{yx}' & 0 \\ n'\nu_{yx}' & 1 & 0 \\ 0 & 0 & m'[1-n'(\nu_{yx}')^2] \end{bmatrix}$  

where $n' = \frac{E'_x}{E'_y} = \frac{E_x}{E_y}$, $\frac{1-\nu^2_{yz}}{E_y/E_z}$, $\frac{1-\nu^2_{zx}}{E_x/E_z}$

$m' = \frac{G_{xy}}{E_y} = m(1-\nu^2_{yz} E_y/E_z)$  

$D'_2 = \frac{E_v}{(1+\nu_{HH})(1-\nu_{HH}-2\nu_{VH}^2)} \begin{bmatrix} n(1-\nu_{VH}^2) & \nu_{VH}(1+\nu_{HH}) & 0 \\ \nu_{VH}(1+\nu_{HH}) & 1-\nu^2_{HH} & 0 \\ 0 & 0 & m(1+\nu_{HH})(1-\nu_{HH}-2\nu_{VH}^2) \end{bmatrix}$ 

$C'_2 = \begin{bmatrix} 1-\nu^2_{HH} & \frac{(1+\nu_{HH})\nu_{VH}}{E_v} & 0 \\ -\frac{(1+\nu_{HH})\nu_{VH}}{E_v} & \frac{1-\nu^2_{VV}}{E_v} & 0 \\ 0 & 0 & \frac{1}{G_{VH}} \end{bmatrix}$
\[
\mathbf{C}_3' = \frac{1}{E} \begin{bmatrix}
1-v^2 & -v(1+v) & 0 \\
-v(1+v) & 1-v^2 & 0 \\
0 & 0 & 2(1+v)
\end{bmatrix}
\] (3.37)

\[
\mathbf{D}_3' = \frac{E}{(1+v)(1-2v)} \begin{bmatrix}
1-v & v & 0 \\
v & 1-v & 0 \\
0 & 0 & (1-2v)/2
\end{bmatrix}
\] (3.38)

As a verification it can be checked that

\[
\mathbf{C}_3' \mathbf{D}_3' = \mathbf{C}_2' \mathbf{D}_2' = \mathbf{C}_1' \mathbf{D}_1' = \mathbf{I}
\]

The matrices \(\mathbf{D}_3', \mathbf{D}_2', \mathbf{D}_1'\) are the elastic rigidity matrices, in plane strain problems, for orthotropic, cross-anisotropic and isotropic materials, respectively. Similarly, \(\mathbf{C}_3', \mathbf{C}_2', \mathbf{C}_1'\) are the elastic compliance matrices.

Since plane stress and plane strain problems differ only in the elements of the elastic rigidity matrix \(\mathbf{D}\) and the initial strain matrix \(\mathbf{E}_0\), only one formulation is needed and the same computer program can solve both types of problems. Therefore, unless explicitly stated otherwise, in what follows it will not be distinguished between the two problems and the stress-strain relations will be used as

\[
\bar{\mathbf{\sigma}} = \mathbf{D} (\mathbf{\varepsilon} - \mathbf{E}_0) = \bar{\mathbf{\sigma}}_0 + \mathbf{D} \mathbf{\varepsilon}
\]

\[
\mathbf{\varepsilon} = \mathbf{E}_0 + \mathbf{C} \bar{\mathbf{\sigma}}
\] (3.39)

in which \(\mathbf{C} = \mathbf{D}^{-1}\)
If the material is isotropic, a plane stress problem can be solved as a plane strain problem (and vice versa) provided Young’s moduli and Poisson’s ratios are related by the following expressions:

\[
E_2 = \frac{1}{1 - \nu_1} E_1
\]

\[
\nu_2 = \frac{1}{1 - \nu_1} \nu_1
\]

(3.40)

where \( E_1, \nu_1 \) refer to plane strain

\( E_2, \nu_2 \) refer to plane stress

Substituting \( E_1, \nu_1 \) in (3.38) and \( E_2, \nu_2 \) given by (3.40) in (3.23) it can be verified that \( D_3 \equiv D'_3 \). Both pairs of parameters \( E, \nu \) produce the same elastic rigidity matrix and therefore the solution of both problems will be numerically equal.

3.1.4 Transformation Matrices to Change from Local to Global Frame

Anisotropic materials present special difficulties because the matrices \( D \) and \( E_0 \) vary with the orientation of the reference frame. The element matrices are calculated using a local frame which may be different from the frame to which the material parameters are referred. An element of stratified (cross-anisotropic) material with inclined strata is an example in which the thermal strain coefficients, Young’s moduli and Poisson’s ratios are referred to the plane of the strata (plane of isotropy) and to the direction normal to this plane. The form of the elastic rigidity matrix \( D \) was derived for such a system of axes.
Let \( x^\ast, y^\ast \) be the frame to which the material properties and the previous expressions for \( D \) and \( E_0 \) refer (\( D^\ast \) and \( E_0^\ast \)) and \( x, y \) the global frame for which one wishes to find the corresponding matrices \( D \) and \( E_0 \).

The two-dimensional stress and strain transformations have the form

\[
\sigma^\ast = \mathbf{I}_\sigma \sigma
\]

\[
\varepsilon^\ast = \mathbf{I}_\varepsilon \varepsilon
\]

in which \( \mathbf{I}_\sigma \) and \( \mathbf{I}_\varepsilon \) are as yet unknown.

The first order work per unit volume is given by \( \sigma^T \delta \varepsilon \) and is invariant, i.e. does not depend on the system of coordinates. Thus,

\[
\sigma^T \delta \varepsilon = (\sigma^\ast)^T \delta \varepsilon^\ast = \sigma^T (\mathbf{I}_\sigma^T \mathbf{I}_\varepsilon) \delta \varepsilon
\]

Therefore

\[
\mathbf{I}_\sigma^T \mathbf{I}_\varepsilon = \mathbf{I} = \mathbf{I}_\varepsilon^T \mathbf{I}_\sigma
\]

(3.42)

From (3.24), using the notation of Fig. 3.21, one can write

\[
\sigma^\ast = D^\ast \varepsilon^\ast - D^\ast E_0^\ast
\]

(3.43)

Pre-multiplying (3.43) by \( \mathbf{I}_\varepsilon^T \), substituting \( \varepsilon^\ast \) from (3.41) and using (3.42) one arrives at

\[
\mathbf{I}_\varepsilon^T \sigma^\ast = \mathbf{I}_\varepsilon^T D^\ast \mathbf{I}_\varepsilon \varepsilon - \mathbf{I}_\varepsilon^T D^\ast \mathbf{I}_\varepsilon E_0
\]

or

\[
\sigma = D \varepsilon - D E_0
\]

(3.44)

where

\[
D = \mathbf{I}_\varepsilon^T D^\ast \mathbf{I}_\varepsilon
\]

(3.45)
Starting with (3.44) in a similar way one would obtain

\[ D^* = I_0 D I_0^T \]  

(3.46)

Using Fig. 3.2 one can easily verify that

\[
\begin{align*}
\sigma_x^* = & \begin{bmatrix} 
\cos^2 \theta & \sin^2 \theta & \sin 2\theta \\
\sin^2 \theta & \cos^2 \theta & -\sin 2\theta \\
-\sin \theta & \sin \theta & \cos 2\theta 
\end{bmatrix} \begin{bmatrix} 
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} \\
\sigma_y^* = & \begin{bmatrix} 
\sin^2 \theta & \cos^2 \theta & -\sin 2\theta \\
-\sin \theta & \sin \theta & \cos 2\theta 
\end{bmatrix} \begin{bmatrix} 
\sigma_y \\
\tau_{xy}
\end{bmatrix} \\
\tau_{xy}^* = & \begin{bmatrix} 
-\sin \theta & \sin \theta & \cos 2\theta 
\end{bmatrix} \begin{bmatrix} 
\tau_{xy}
\end{bmatrix}
\end{align*}
\]  

(3.47)

Comparing (3.47) with (3.41), one can write

\[
I_0 = \begin{bmatrix} 
\cos^2 \theta & \sin^2 \theta & \sin 2\theta \\
\sin^2 \theta & \cos^2 \theta & -\sin 2\theta \\
-\sin \theta & \sin \theta & \cos 2\theta 
\end{bmatrix}
\]  

(3.48)

and, using (3.42) and inverting \( I_0^T \),

\[
I_\epsilon = \begin{bmatrix} 
\cos^2 \theta & \sin^2 \theta & \frac{\sin 2\theta}{2} \\
\sin^2 \theta & \cos^2 \theta & -\frac{\sin 2\theta}{2} \\
-\sin 2\theta & \sin 2\theta & \cos 2\theta 
\end{bmatrix}
\]  

(3.49)

The form of \( \epsilon_i \) depends on the type of material and whether it is a plane strain or plane stress problem. The expressions for an orthotropic material can be obtained (using the notations of Fig. 3.2) from (3.15) and (3.28):
for plane stress 

\[
\varepsilon_{*01}^* = \begin{bmatrix}
\alpha_{x*} T \\
\alpha_{y*} T \\
0
\end{bmatrix}
\]

(3.50)

for plane strain 

\[
\varepsilon_{*01}^* = \begin{bmatrix}
(\alpha_{x*} + \nu_{z*x*} \alpha_{z*}) T \\
(\alpha_{y*} + \nu_{z*y*} \alpha_{z*}) T \\
0
\end{bmatrix}
\]

For a cross-anisotropic material these expressions become

for plane stress 

\[
\varepsilon_{*02}^* = \begin{bmatrix}
\alpha_{H*} T \\
\alpha_{V*} T \\
0
\end{bmatrix}
\]

(3.51)

for plane strain 

\[
\varepsilon_{*02}^* = \begin{bmatrix}
(1 + \nu_{H*H*}) \alpha_{H*} T \\
(\alpha_{V*} + n\nu_{H*H*} \alpha_{H*}) T \\
0
\end{bmatrix}
\]

For an isotropic material (3.51) reduce to:

for plane stress 

\[
\varepsilon_{*03}^* = \begin{bmatrix}
\alpha T \\
\alpha T \\
0
\end{bmatrix}
\]

(3.52)

for plane strain 

\[
\varepsilon_{*03}^* = (1 + \nu)\begin{bmatrix}
\alpha T \\
\alpha T \\
0
\end{bmatrix}
\]
The previous expressions for $\varepsilon_0$ show that no shear strains are caused by a thermal deformation. However, it must be emphasized that the shear component of initial strain will no longer be equal to zero if the transformation matrix $I_c$ is used. Therefore, in general,

$$\varepsilon_0 = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} * T \quad (3.53)$$

All elements of $D$ will also be different from zero if the transformation (3.45) is used. However, the elastic rigidity matrix will always be (in two-dimensional analysis) of the form:

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ (\text{sym}) & & D_{33} \end{bmatrix} \quad (3.54)$$

where $D_{ij}$ are constants.

3.1.5 Axi-symmetric Analysis

The problems of strain and stress distribution in bodies of revolution (axi-symmetric solids) under axi-symmetric loading are of considerable practical interest and very similar to those of plane stress and plane strain because they can be solved as two-dimensional (Clough, 1965).
Due to symmetry, the two components of displacements in any plane section containing the axis of symmetry define completely the state of strain and stress. However, any radial displacement automatically induces a strain in the circumferential direction and, as the stresses in this direction are certainly non-zero, this fourth component of strain (and of associated stresses) has to be considered. Here lies the essential difference in the treatment of the axi-symmetric situation.

Fig. 3.3 shows a portion of an axi-symmetric solid with the strains and stresses which have to be considered. The radial and axial coordinates are denoted by \( x \) and \( y \), respectively, and the corresponding displacements by \( u \) and \( v \).

The elastic rigidity matrix \( \mathbf{D} \) to be now derived is defined by

\[
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_\theta \\
\gamma_{xy}
\end{pmatrix} = \mathbf{D} \cdot (\varepsilon - \varepsilon_0) \quad (3.55)
\]

The most frequent case of initial strain is that due to a thermal expansion. The general case of anisotropy is incompatible with the assumption of axial symmetry. The only anisotropic case considered here is the case of cross-anisotropic material for which one can write
For an isotropic material $\varepsilon_o$ is obtained from (3.56) by making $\alpha_x = \alpha_y = \alpha$.

Considering the (horizontal) plane of isotropy perpendicular to the (vertical) axis $y$, one can write the relevant equations for axi-symmetric problems (see (3.6) and include the effect of temperature variations) when the material is cross-anisotropic:

$$\varepsilon_x = \frac{1}{E_H} \sigma_x - \frac{\nu_{VH}}{E_V} \sigma_y - \frac{\nu_{HH}}{E_H} \sigma_\theta + \alpha_x T$$

$$\varepsilon_y = -\frac{\nu_{VH}}{E_V} \sigma_x + \frac{1}{E_V} \sigma_y - \frac{\nu_{VH}}{E_V} \sigma_\theta + \alpha_y T$$

$$\varepsilon_\theta = -\frac{\nu_{HH}}{E_H} \sigma_x - \frac{\nu_{VH}}{E_V} \sigma_y + \frac{1}{E_H} \sigma_\theta + \alpha_\theta T$$

$$\gamma_{xy} = \frac{1}{G_{VH}} \tau_{xy}$$

or

$$\varepsilon - \varepsilon_o = C'' \gamma$$

(3.58)
By inverting $C_2''$ and using (3.21) one arrives at

$$D_2'' = \frac{E_v}{(1+\nu_{HH})(1-\nu_{HH}-2\nu_{VH}^2)} \begin{bmatrix} n(1-\nu_{VH}^2) & \nu_{VH}(1+\nu_{HH}) & n(\nu_{HH}+\nu_{VH}^2) & 0 \\ \nu_{VH}(1+\nu_{HH}) & 1-\nu_{HH}^2 & \nu_{VH}(1+\nu_{HH}) & 0 \\ n(\nu_{HH}+\nu_{VH}^2) & \nu_{VH}(1+\nu_{HH}) & n(1-\nu_{VH}^2) & 0 \\ 0 & 0 & 0 & m(1+\nu_{HH})(1-\nu_{HH}-2\nu_{VH}^2) \end{bmatrix}$$

(3.60)

The corresponding matrices $C_3''$ and $D_3''$ for an isotropic material are derived from $C_2''$ and $D_2''$:

$$C_3'' = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 \\ -\nu & 1 & -\nu & 0 \\ -\nu & -\nu & 1 & 0 \\ 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

(3.61)
\[ D' = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 \\ v & 1-v & v & 0 \\ v & v & 1-v & 0 \\ 0 & 0 & 0 & (1-2v)/2 \end{bmatrix} \] (3.62)

As a verification it can be checked that

\[ C''D'' = C''D'' = I \]

The strains are related to the displacements by

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_\theta \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} \\
\frac{u}{x} \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{bmatrix}
\] (3.63)

The strain \( \varepsilon_\theta \) can be derived by simple geometric considerations valid for small deformations. When, due to elastic deformation, the radius increases from \( x \) to \( x+u \) the perimeter increases from \( 2\pi x \) to \( 2\pi (x+u) \) and \( \varepsilon_\theta = \frac{2\pi u}{2\pi x} = u/x \).

3.1.6 Limiting Values of the Engineering Constants

Observing, for instance, (3.38) which is valid for an isotropic material in plane strain analysis, one can see that \( v = 1/2 \) makes the factor \( E/[(1+v)(1-2v)] \) common to all elements of \( D'_3 \) infinite (positive if \( E>0 \)). In this case the elastic rigidity matrix degenerates to a positive semi-definite
matrix. It will be seen that this situation arises when the material is incompressible.

From (3.3) one can obtain

\[ \varepsilon_x + \varepsilon_y + \varepsilon_z = (1 - \nu_{xy} - \nu_{xz}) \frac{1}{E_x} \sigma_x + (\nu_{yx} + 1 - \nu_{yz}) \frac{1}{E_y} \sigma_y + (\nu_{zx} - \nu_{zy} + 1) \frac{1}{E_z} \sigma_z \]

(3.64)

For small deformations the volumetric strain is given by

\[ \frac{\Delta V}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z \]

(3.65)

If the material is incompressible then there is no volume change, i.e. \( \Delta V/V_0 = 0 \), for all possible states of stress. This means that (3.64) must be identically equal to zero and, using (3.4), one can state

\[ 1 - \frac{E_x}{E_y} \nu_{yx} - \frac{E_x}{E_z} \nu_{zx} = 0 \]

\[ 1 - \nu_{yx} - \nu_{yz} = 0 \]

\[ 1 - \nu_{zx} - \frac{E_z}{E_y} \nu_{yz} = 0 \]

(3.66)

Solving (3.66) for \( \nu_{yx} \), \( \nu_{yz} \) and \( \nu_{zx} \), one arrives at

\[ \nu_{yx} = \frac{1}{2} \left( 1 + \frac{E_x}{E_y} \right) \]

\[ \nu_{yz} = \frac{1}{2} \left( 1 + \frac{E_y}{E_z} \right) \]

\[ \nu_{zx} = \frac{1}{2} \left( 1 + \frac{E_z}{E_x} \right) \]

(3.67)
Incompressibility therefore reduced the number of independent constants from nine to six, for an orthotropic material.

The equivalent relations for a cross-anisotropic material derived from (3.67) by using (3.5) are:

\[ v_{VH} = \frac{1}{3} \]  \hspace{1cm} (3.68)
\[ v_{HH} = 1 - \frac{1}{3} \frac{E_H}{E_V} \]

For a material in which \( E_H = E_V \), (3.68) reduce to the well known condition for isotropic material

\[ v = \frac{1}{2} \]  \hspace{1cm} (3.69)

The number of independent parameters has thus been reduced from five to three for cross-anisotropic material and from two to one for a material in which \( E_H = E_V \) by imposing the condition of no volume change or incompressibility. Since no restriction has been imposed on the independent parameter \( G_{VH} \), (3.69) not only applies to an isotropic material but also to a "pseudo-isotropic" material (Gibson, 1974) which possesses the same vertical and horizontal Young's moduli but is not isotropic (\( G_{VH} \) is independent).

Many natural soil deposits can be considered as cross-anisotropic bodies (with isotropy as a special case). Many authors in recent years have given attention to this matter, with special regard to establishing limits for the values of the independent parameters compatible with a real material (Barden, 1963; Pickering, 1970; Wardle and Gerrard, 1972; Gibson, 1974).
The theory of elasticity places certain restrictions on the elastic constants by the fact that the strain energy function must be positive for a real elastic material.

Expressing the strains as $\epsilon = C \sigma$, the strain energy density will be given by $\frac{1}{2} \sigma^T C \sigma$. If this quadratic form is positive definite, then the strain energy density will be positive. Following this line of thought, Pickering arrived at the following conditions.

$$E_H, E_V, G_{VH} > 0$$

$$-1 < v_{HH} < 1$$

$$\frac{E_H}{E_V} (1 - v_{HH}) - 2v_{HV}^2 > 0$$

For an incompressible cross-anisotropic material, from the expansion of the strain energy density which must be positive, Gibson arrived at the condition

$$0 < \frac{E_H}{E_V} < 4$$

Note that the conditions $v_{VH} = \frac{1}{2}, v_{HH} = -1$ and $n = 4$ make the factors $E_V/(1-nv_{VH}^2)$ and $E_V/[(1+v_{HH})(1-v_{HH}-2nv_{VH})]$ common to all elements of $D_2$ and $D_2', D_2''$, respectively, infinite. The elastic rigidity matrix $D$ is not positive definite and, therefore, the finite element displacement method as presented in chapter 2 cannot be used to study the deformation of incompressible cross-anisotropic or isotropic bodies. Although the analysis can be performed approximately, the solution will not converge if the values for the material parameters are chosen numerically too close to the conditions corresponding to incompressibility.
3.2 Element Stiffness Matrix for Two-Dimensional Analysis

3.2.1 Homogeneous Coordinates

The generation of interpolation functions for a triangular element and the calculation of the element stiffness matrices are considerably simplified if a local system of homogeneous coordinates (also called area coordinates, triangular coordinates and oblique coordinates) is used (Felippa, 1966). Such coordinates are indeed nearly always a natural choice for triangles (Zienkiewicz, 1971).

A system of homogeneous coordinates $\xi_1, \xi_2, \xi_3$ is defined by the following linear relations between the coordinates and a cartesian system $x, y, z$:

$$
\begin{align*}
x &= x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3 \\
y &= y_1 \xi_1 + y_2 \xi_2 + y_3 \xi_3 \\
1 &= \xi_1 + \xi_2 + \xi_3
\end{align*}
$$

(3.72)

They are not independent and can be interpreted as ratios of areas, according to Fig. 3.4

$$
\begin{align*}
\xi_1 &= \frac{A_1}{A} \\
\xi_2 &= \frac{A_2}{A} \\
\xi_3 &= \frac{A_3}{A}
\end{align*}
$$

(3.73)

where $A$ is the total area $A = A_1 + A_2 + A_3$ of the triangle.

The homogeneous coordinates do not depend on the position of the external reference system or the shape of the triangular element. The points 1 to 6 shown in Fig. 3.4 and the centroid of the triangle have the following coordinates:
<table>
<thead>
<tr>
<th>Point</th>
<th>( \xi_1 )</th>
<th>( \xi_2 )</th>
<th>( \xi_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1/2</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>6</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>Centroid</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Solving (3.72) for \( x \) and \( y \) gives

\[
\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} A_1^o & a_1 & b_1 \\ A_2^o & a_2 & b_2 \\ A_3^o & a_3 & b_3 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{3.75}
\]

where

\[
A_0 = \begin{bmatrix} A_1^o \\ A_2^o \\ A_3^o \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}
\]

\[
a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_2 - y_3 \\ y_3 - y_1 \\ y_1 - y_2 \end{bmatrix} \tag{3.76}
\]

\[
b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{bmatrix}
\]

are vectors of parameters depending on the global cartesian coordinates of the vertices of the triangular element. The area of the triangle is given by
\[ A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2} (a_2b_3 - a_3b_2) \]  

(3.77)

Having related the local triangular coordinates to the global cartesian coordinates, similar relations will be established for the derivatives and the integrals. These will appear in the expressions for the element matrices and it was shown in (2.47) that they must be in terms of the global frame before the assembly of the system of equations.

Writing (3.75) as

\[ \xi_i = \frac{1}{2A} \left( A_i^0 + a_i x + b_i y \right) \]  

(3.78)

one can evaluate

\[ \frac{3}{\partial x} \left[ f(\xi_1, \xi_2, \xi_3) \right] = \frac{3f}{\partial \xi_1} \frac{\partial \xi_1}{\partial x} + \frac{3f}{\partial \xi_2} \frac{\partial \xi_2}{\partial x} + \frac{3f}{\partial \xi_3} \frac{\partial \xi_3}{\partial x} \]

\[ = \frac{1}{2A} \sum_{i=1}^{3} a_i \frac{\partial f}{\partial \xi_i} \]  

(3.79)

and, similarly

\[ \frac{3}{\partial y} \left[ f(\xi_1, \xi_2, \xi_3) \right] = \frac{1}{2A} \sum_{i=1}^{3} b_i \frac{\partial f}{\partial \xi_i} \]  

(3.80)

By repeated application of the same process one can evaluate higher derivatives. For example:

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{(2A)^2} \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} \]  

(3.81)
The integral
\[ \iint f(\xi_1, \xi_2, \xi_3) \, dA \]
can be evaluated by using (3.72) and Fig. 3.5 to express \( \xi_3 \) and \( dA \) in terms of \( \xi_1 \) and \( \xi_2 \) as follows:
\[ \xi_3 = 1 - \xi_1 - \xi_2 \]
\[ dA = 2A \, d\xi_1 \, d\xi_2 \]  

(3.82)

Integrating with respect to \( \xi_1 \) and \( \xi_2 \) one can express the general formula as,
\[ \iint f(\xi_1, \xi_2) \, dA = 2A \int_0^1 \left[ \int_0^{1-\xi_2} f(\xi_1, \xi_2) \, d\xi_1 \right] \, d\xi_2 \]  

(3.83)

Using this expansion one can readily check the useful formula
\[ \iiint \xi_1^i \xi_2^j \xi_3^k \, dA = \frac{i! \cdot j! \cdot k!}{(i+j+k+2)!} \cdot 2A \]  

(3.84)

From Fig. 3.5 in which \( dL = l \, d\xi_1 \) over the side \( \overline{12} \) of length \( l \) one can derive the general expression for a linear integral
\[ \int_0^l f(\xi_1, \xi_2) \, dL = l \int_0^1 f(\xi_1, \xi_2) \, d\xi_1 \]  

(3.85)

and use it to check the general formula
\[ \int_0^l \xi_1^i \xi_2^j \, dL = \frac{i! \cdot j!}{(i+j+1)!} \cdot l \]  

(3.86)
3.2.2 **Interpolation Functions**

The formulation will be based on the expansion of the displacements in a complete second-degree polynomial. Thus there are twelve displacement parameters, six for the horizontal displacements and six for the vertical displacements.

Considering the six node triangular element of Fig. 3.4 with the node numbering system indicated and assuming the suitable interpolation function is not known in advance, one can start with a quadratic polynomial in \( \xi_1 \) and \( \xi_2 \)

\[
f = a_1 + a_2 \xi_1 + a_3 \xi_2 + a_4 \xi_1^2 + a_5 \xi_1 \xi_2 + a_6 \xi_2^2
\]

(3.87)

Taking the nodal values of \( f \) for the horizontal displacements, for instance, a system of equations can be formed as follows:

<table>
<thead>
<tr>
<th>node</th>
<th>( \xi_1 )</th>
<th>( \xi_2 )</th>
<th>( f = u_1 = a_1 + a_2 + 0 + a_4 + 0 + 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( f = u_2 = a_1 + 0 + a_3 + 0 + 0 + a_6 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>( f = u_3 = a_1 + 0 + 0 + 0 + 0 + 0 )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>( f = u_4 = a_1 + a_2 / 2 + a_3 / 2 + a_4 / 4 + a_5 / 4 + a_6 / 4 )</td>
</tr>
<tr>
<td>4</td>
<td>1/2</td>
<td>1/2</td>
<td>( f = u_5 = a_1 + a_2 / 2 + 0 + 0 + a_6 / 4 )</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1/2</td>
<td>( f = u_6 = a_1 + a_2 / 2 + 0 + a_4 / 4 + 0 + 0 )</td>
</tr>
</tbody>
</table>

where \( u_1, u_2, \ldots, u_6 \) are the horizontal nodal displacements.

This system of equations will become, in matricial form,
The coefficient matrix is dimensionless and can be inverted to give

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 & 1 & 0 & 0 & 0 \\
  -1 & 0 & -3 & 0 & 0 & 4 \\
  0 & -1 & -3 & 0 & 4 & 0 \\
  2 & 0 & 2 & 0 & 0 & -4 \\
  0 & 0 & 4 & -4 & -4 & -4 \\
  0 & 2 & 2 & 0 & -4 & 0 \\
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
  u_4 \\
  u_5 \\
  u_6 \\
\end{bmatrix}
\]

The coefficients \( a_i \) are now expressed in terms of \( u_1 \).

Substituting \( a_i \) in (3.87) and taking into account that

\[ \xi_3 = 1 - \xi_1 - \xi_2, \]

one arrives at

\[
u = \xi_1 (2\xi_1 - 1) u_1 + \xi_2 (2\xi_2 - 1) u_2 + \xi_3 (2\xi_3 - 1) u_3 + 4\xi_1 \xi_2 u_4 + 4\xi_2 \xi_3 u_5 + 4\xi_3 \xi_1 u_6
\]

or

\[
u = \phi_1 u_1 + \phi_2 u_2 + \phi_3 u_3 + \phi_4 u_4 + \phi_5 u_5 + \phi_6 u_6
\]

where

\[
\phi = \begin{bmatrix}
  \xi_1 (2\xi_1 - 1) & \xi_2 (2\xi_2 - 1) & \xi_3 (2\xi_3 - 1) & 4\xi_1 \xi_2 & 4\xi_2 \xi_3 & 4\xi_3 \xi_1 \\
\end{bmatrix}
\]
For all displacement components,

\[ u = \Phi \mathbf{u}_n \]

\[ v = \Phi \mathbf{v}_n \]

\[ \mathbf{u}_n = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} \quad \mathbf{v}_n = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} \]  

(3.93)

Finally,

\[ \mathbf{u} = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{pmatrix} \mathbf{u}_n \\ \mathbf{v}_n \end{pmatrix} = \Phi \begin{pmatrix} \mathbf{u}_n \\ \mathbf{v}_n \end{pmatrix} \]  

(3.94)

as shown in (2.22) and (2.32).

Noting that (2.25) is valid both for plane stress and plane strain, using (3.78) and (3.79), \( \epsilon_x \) can be expanded as follows:

\[ \epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( \sum_{j=1}^{6} \phi_j \mathbf{u}_j \right) = \frac{1}{2A} \sum_{j=1}^{6} a_i \sum_{i=1}^{3} \frac{\partial \phi_j}{\partial \xi_i} u_j \]  

(3.95)

After calculating the derivatives by using (3.92) and expanding, (3.95) will become

\[ \epsilon_x = \frac{1}{2A} \left[ a_1 \ a_2 \ a_3 \right] \begin{bmatrix} 4\xi_1 - 1 & 0 & 0 & 4\xi_2 & 0 & 4\xi_3 \\ 0 & 4\xi_2 - 1 & 0 & 4\xi_1 & 4\xi_3 & 0 \\ 0 & 0 & 4\xi_3 - 1 & 0 & 4\xi_2 & 4\xi_1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix} \]  

(3.96)
After using the same process for $\epsilon_y$ and $\gamma_{xy}$, it is possible to represent all three strains in a more compact manner, using the notations of (3.76):

$$
\epsilon_x = \frac{1}{2A} a^T \psi u_n, \quad \epsilon_y = \frac{1}{2A} b^T \psi v_n \quad (3.97)
$$

$$
\gamma_{xy} = \frac{1}{2A} (b^T \psi u_n + a^T \psi v_n)
$$

where

$$
\psi = [\psi_{ij}] = 
\begin{bmatrix}
\phi_{i1} \\
\phi_{i2}
\end{bmatrix} = 
\begin{bmatrix}
4\xi_1 - 1 & 0 & 0 & 4\xi_2 & 0 & 4\xi_3 \\
0 & 4\xi_2 - 1 & 0 & 4\xi_1 & 4\xi_3 & 0 \\
0 & 0 & 4\xi_3 - 1 & 0 & 4\xi_2 & 4\xi_1
\end{bmatrix} \quad (3.98)
$$

and $i = 1, 2, 3, j = 1, 2, \ldots, 6.$

Finally, (3.97) can be represented by a single matricial equation as in (2.24):

$$
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix} = \frac{1}{2A} 
\begin{bmatrix}
a^T & 0 \\
0 & b^T \\
b^T & a^T
\end{bmatrix} 
\begin{bmatrix}
\psi & 0 \\
0 & \psi \\
0 & \psi
\end{bmatrix} 
\begin{bmatrix}
u_n \\
v_n
\end{bmatrix} \quad (3.99)
$$

or

$$
\mathbf{E} = \mathbf{B} \mathbf{U}_n \quad (3.100)
$$

where

$$
\mathbf{B} = \frac{1}{2A} 
\begin{bmatrix}
a^T & 0 \\
0 & b^T \\
b^T & a^T
\end{bmatrix} 
\begin{bmatrix}
\psi & 0 \\
0 & \psi \\
0 & \psi
\end{bmatrix} \quad (3.101)
$$

The interpolation function $\phi$ has been determined by the general method outlined in 2.3.3. However, it is possible and quite easy to derive $\phi$ for a triangular element of any order.
For the constant strain triangular element (first order) it is obvious that

$$\Phi = \xi^T = [\xi_1 \xi_2 \xi_3]$$

(3.102)

is the appropriate interpolation function, since each individual function $\phi_i$ gives unity at one node, zero at the others and varies linearly everywhere. When the interpolation functions are available for a triangle of order $n$ ($n=1$ in this case), the interpolation functions for a triangle of order $n+1$ can be generated by recurrence (Irons and al., 1968).

Let $\phi_i$, $\xi_i$ refer to the $n$-th order triangle and $\phi_i^*$, $\xi_i^*$ to the $(n+1)$-th order triangle. From Fig. 3.6, by definition,

$$\xi_2 = \frac{\text{Area } P13}{\text{Area } 123} \quad \xi_2^* = \frac{\text{Area } P13^*}{\text{Area } 12*3^*}$$

Thus

$$\frac{\xi_2}{\xi_2^*} = \frac{\text{Area } P13}{\text{Area } P13^*} \cdot \frac{\text{Area } 12*3^*}{\text{Area } 123} = \frac{n}{n+1} \left(\frac{n+1}{n}\right)^2$$

or

$$\xi_2 = \frac{n+1}{n} \xi_2^*$$

(3.103)

Similarly,

$$\xi_3 = \frac{n+1}{n} \xi_3^*$$

(3.104)

Using successively the relation $\xi_1 = 1 - \xi_2 - \xi_3$ one arrives at

$$\xi_1 = \frac{1}{n} \left[ (n+1) \xi_1^* - 1 \right]$$

(3.105)
Each interpolation function $\phi_i^*$ for the (n+1)-th order triangle (corresponding to a node $i$ not lying on the base) must be of the form

$$\phi_i^* = c \xi_i^* \phi_1$$

(3.106)

provided the scaling factor $c$ is chosen as to make $\phi_i^* = 1$ at node $i$, because $\xi_i^*$ is zero along the base of the larger triangle. From the interpolation functions $\phi_i$ for the first order triangle presented in (3.102) one finds for n = 1:

for node 1* $\phi_1^* = c \xi_1^* \left[ \frac{1}{2} (2\xi_1^* - 1) \right] + \phi_1 = \xi_1^* (2\xi_1^* - 1)$

for node 4* $\phi_4^* = c \xi_4^* (\frac{2}{3} \xi_2^*) + \phi_4 = 4 \xi_4^* \xi_2^*$

for node 6* $\phi_6^* = c \xi_6^* (\frac{2}{3} \xi_3^*) + \phi_6 = 4 \xi_6^* \xi_3^*$

This process will not generate functions for nodes on the base but they can be obtained by a simple transposition of indices. By this method, the interpolation function $\phi$ in (3.92) could have been obtained more easily.

The search for the interpolation function $\phi$ could also be carried out by trial and error using the knowledge that each function $\phi_i$ must become zero for all nodes except node $i$ and must also have a number of variables equal to the required degree of $\phi$. For the triangular element of ten nodes, for instance, one would find:

for corner node 1

$$\phi_1 = \frac{1}{2} \xi_1 (3\xi_1 - 1) (3\xi_1 - 2)$$

for "midside" node 4

$$\phi_4 = \frac{9}{2} \xi_1 \xi_2 (3\xi_1 - 1)$$

for internal node 10

$$\phi_{10} = 27 \xi_1 \xi_2 \xi_3$$
3.2.3 **Exact Integration**

According to (2.34), the element stiffness matrix is given by

\[ K = \iiint \mathbf{B}^T \mathbf{D} \mathbf{B} \, d(\text{vol}) \]  \hspace{1cm} (3.107)

where \( \mathbf{D} \) is the elastic rigidity matrix and \( \mathbf{B} \) is defined by (3.101).

If the Young's modulus \( E_y \) varies linearly in any direction over the element, it will be given, at each point, by

\[ E_y = \mathbf{E} \xi = [E_1 \, E_2 \, E_3] \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = E_1 \xi_1 + E_2 \xi_2 + E_3 \xi_3 \]  \hspace{1cm} (3.108)

which is equivalent to

\[ E_y = \frac{1}{2A} \left[ (E_1 A_0) + (E_2 a)x + (E_3 b)y \right] \]  \hspace{1cm} (3.109)

where \( E_1, E_2, E_3 \) are the values of \( E_y \) at the corner nodes 1, 2 and 3, respectively.

Assuming that all Poisson's ratios are constant within the element, and all Young's moduli and \( G_{xy} \) vary linearly in such a way that \( E_x/E_y, E_z/E_y, G_{xy}/E_y \) are kept constant, if (3.54) and (3.108) are taken into account one can represent the elastic rigidity matrix by

\[ \mathbf{D} = (\mathbf{E} \xi) \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \]  \hspace{1cm} (3.110)

where \( D_{ij} \) are coefficients (constant within the element) depending on the type of material.
In two-dimensional analysis it is reasonable to evaluate the volume integral (3.107) assuming the thickness constant and the variables unchanged over it. Considering the thickness equal to unity for simplicity of notation, the volume integral (3.107) will be represented by an area integral

\[ K = \int B^T D B \, dA \]  \hspace{1cm} (3.111)

Using (3.101) and (3.110) the integrand can be expanded to

\[ B^T D B = \frac{1}{4A^2} \begin{bmatrix} \psi^T & 0 \\ 0 & \psi^T \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{bmatrix} a^T & 0 \\ 0 & b^T \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix} \]  \hspace{1cm} (E 5')

\[ = \frac{1}{4A^2} \begin{bmatrix} \psi^T & 0 \\ 0 & \psi^T \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \psi \\ 0 \end{bmatrix} \]  \hspace{1cm} (3.112)

where

\[ C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{bmatrix} a^T & 0 \\ 0 & b^T \end{bmatrix} \]  \hspace{1cm} (3.113)

is a matrix of constants depending on the global cartesian coordinates (see (3.76)) and on the type of material. Matrices \( \psi \) and (E 5') depend on the variables \( \xi_i \).
From (3.98) matrix \( \psi \) can be written simply as
\[
\psi = \begin{bmatrix}
\psi_{11} & 0 & 0 & \psi_{14} & 0 & \psi_{16} \\
0 & \psi_{22} & 0 & \psi_{24} & \psi_{16} & 0 \\
0 & 0 & \psi_{33} & 0 & \psi_{14} & \psi_{24}
\end{bmatrix}
\] (3.114)

and the submatrices of \( C \) in (3.113) will expand to
\[
C_{11} = D_{11} a a^T + D_{13} (a b^T + b a^T) + D_{33} b b^T
\]
\[
C_{12} = D_{12} a b^T + D_{23} b b^T + D_{13} a a^T + D_{33} b a^T
\]
\[
C_{21} = D_{12} b a^T + D_{13} a a^T + D_{23} b b^T + D_{33} a b^T
\]
\[
C_{22} = D_{22} b b^T + D_{23} (a b^T + b a^T) + D_{33} a a^T
\] (3.115)

Now the element stiffness matrix (3.111) can be represented in a more simple manner obtained from (3.112):
\[
K = \int \int B^T D B \ dA = \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix}
\] (3.116)

where
\[
K_{11} = \frac{1}{4A^2} \int \int (\psi^T C_{11} \psi) (E \xi) \ dA
\]
\[
K_{12} = \frac{1}{4A^2} \int \int (\psi^T C_{12} \psi) (E \xi) \ dA
\]
\[
K_{21} = \frac{1}{4A^2} \int \int (\psi^T C_{21} \psi) (E \xi) \ dA
\]
\[
K_{22} = \frac{1}{4A^2} \int \int (\psi^T C_{22} \psi) (E \xi) \ dA
\] (3.117)
Since $C21 = C12^T$ (see 3.115), $K21 = K12^T$ and only one of these two submatrices has to be evaluated. Because the element stiffness matrix $K$ is symmetric, only the elements on and above the principal diagonal have to be evaluated, i.e. 84 elements.

All submatrices $C_{ij}$ in (3.117) are matrices of constants and therefore all four integrals are of the form

$$RR = \frac{1}{4A^2} \iint (\hat{\psi}^T \Omega \hat{\psi}) (E \xi) \, dA$$

(3.118)

where $\Omega$ is a matrix of constants.

If one finds a general expansion for this integral in terms of the elements of $\Omega$, then it can be used to obtain the integrals (3.117). Making successively $\Omega = C11$, $\Omega = C12$ and $\Omega = C22$ one obtains $RR = K11$, $RR = K12$ and $RR = K22$, respectively.

As an example, three elements of $K$ will be evaluated using the relevant information contained in Appendix I.

Starting with element $K_{1,4}$,

from I.3: $I_7 = \iint \psi_{11} \psi_{24} (E \xi) \, dA = (14E_1 + 3E_2 + 3E_3) \, A/15$

$I_{10} = \iint \psi_{11} \psi_{14} (E \xi) \, dA = (3E_1 - 2E_2 - E_3) \, A/15$

from I.2: $R_{1,4} = Q_{1,1} \psi_{11} \psi_{14} + Q_{1,2} \psi_{11} \psi_{24}$

$$K_{1,4} = \frac{1}{4A^2} \iint (Q_{1,1} \psi_{11} \psi_{14} + Q_{1,2} \psi_{11} \psi_{24}) (E \xi) \, dA$$
\[
\frac{1}{4\pi^2} \left[ Q_{1,1} \int \psi_{11} \psi_{14} (E \xi) dA + Q_{1,2} \int \psi_{11} \psi_{24} (E \xi) dA \right]
\]

\[
= \frac{1}{4\pi^2} (Q_{1,1} I_{10} + Q_{1,2} I_{7})
\]

\[
= \left[ (3E_1 - 2E_2 - E_3) Q_{1,1} + (14E_1 + 3E_2 + 3E_3) Q_{1,2} \right] / (60\pi)
\]

(3.119)

\[K_{1,4}\] is an element of \(K_{11}\) and therefore \(\varrho = C_{11}\).

From I.1: \[Q_{1,1} = D_{11} a_1^2 + 2D_{13} a_1 b_1 + D_{33} b_1^2\]

(3.120)

\[Q_{1,2} = D_{12} a_1 a_2 + D_{23} b_1 b_2 + D_{13} (a_2 b_1 + a_1 b_2) + D_{33} b_1 b_2\]

Substituting these values in (3.119) the element \(K_{1,4}\) is determined.

For element \(K_{1,10}\), \(\varrho = C_{12}\) and therefore,

From I.1: \[Q_{1,1} = D_{12} a_1 b_1 + D_{23} b_1^2 + D_{13} a_1^2 + D_{33} a_1 b_1\]

(3.121)

\[Q_{1,2} = D_{12} a_1 a_2 + D_{23} b_1 b_2 + D_{13} a_1 a_2 + D_{33} a_2 b_1\]

The expression (3.119) used for \(K_{1,4}\) can now be used for \(K_{1,10}\) with these new values of \(Q_{1,1}\) and \(Q_{1,2}\).

It is known that \(K_{10,1} = K_{1,10}\) but, as a check, \(K_{10,1}\) will be evaluated using \(\varrho = C_{21}\).

From I.2: \[R_{4,1} = Q_{1,1} \psi_{11} \psi_{14} + Q_{2,1} \psi_{11} \psi_{24}\]

\[K_{10,1} = \frac{1}{4\pi^2} \int (Q_{1,1} \psi_{11} \psi_{14} + Q_{2,1} \psi_{11} \psi_{24}) (E \xi) dA\]
Since \( q = c_{21} \),

from I.1: \( Q_{1,1} = D_{12} a_1 b_1 + D_{13} a_1^2 + D_{23} b_1^2 + D_{33} a_1 b_1 \) \( \tag{3.122} \)

\( Q_{2,1} = D_{12} a_1 b_2 + D_{13} a_1 a_2 + D_{23} b_2 b_2 + D_{33} a_2 b_1 \)

The expansions of \( Q_{1,1} \) and \( Q_{2,1} \) to evaluate \( K_{10,1} \) are identical to the expansions of \( Q_{1,1} \) and \( Q_{1,2} \) in (3.121) to evaluate \( K_{1,10} \). Therefore \( K_{10,1} = K_{1,10} \) and the result above was correct.

3.3 Consistent System of Element Nodal Forces for Two-Dimensional Analysis

3.3.1 Surface Forces

According to (2.53), the consistent nodal forces due to surface loading are generated by

\[
\frac{P}{P} = \iiint \phi \ p \ ds = \iiint \begin{bmatrix} \phi^T & 0 \\ 0 & \phi^T \end{bmatrix} \begin{bmatrix} p_x \\ p_y \end{bmatrix} \ ds \tag{3.123}
\]

where \( \phi \) is a row matrix defined by (3.92).

Taking into account the assumptions made in 3.1.2 for plane stress and in 3.1.3 for plane strain analysis, and considering the constant thickness of the element equal to unity, the area integral (3.123) will become
\[
\begin{align*}
\mathbf{P} &= \begin{bmatrix} \mathbf{P}_{\text{un}} \\ \mathbf{P}_{\text{vn}} \end{bmatrix} = \int_0^s \begin{bmatrix} \Phi^T & 0 \\ 0 & \Phi^T \end{bmatrix} \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} \, ds
\end{align*}
\]

where \( s \) is the total length of the sides of the element, \( \mathbf{P}_{\text{un}} \) and \( \mathbf{P}_{\text{vn}} \) are, respectively, the matrices of horizontal and vertical components of the consistent system of nodal forces.

Since the displacement expansion is quadratic, i.e.

\[
\begin{align*}
\mathbf{u} &= \phi_1 u_1 + \phi_2 u_2 + \phi_3 u_3 + \phi_4 u_4 + \phi_5 u_5 + \phi_6 u_6 \\
\mathbf{v} &= \phi_1 v_1 + \phi_2 v_2 + \phi_3 v_3 + \phi_4 v_4 + \phi_5 v_5 + \phi_6 v_6
\end{align*}
\]

a quadratic expansion is also assumed for the distributed surface forces:

\[
\begin{align*}
\mathbf{P}_x &= \phi_1 \mathbf{P}_x^1 + \phi_2 \mathbf{P}_x^2 + \phi_3 \mathbf{P}_x^3 + \phi_4 \mathbf{P}_x^4 + \phi_5 \mathbf{P}_x^5 + \phi_6 \mathbf{P}_x^6 \\
\mathbf{P}_y &= \phi_1 \mathbf{P}_y^1 + \phi_2 \mathbf{P}_y^2 + \phi_3 \mathbf{P}_y^3 + \phi_4 \mathbf{P}_y^4 + \phi_5 \mathbf{P}_y^5 + \phi_6 \mathbf{P}_y^6
\end{align*}
\]

or

\[
\begin{align*}
\begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} &= \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \mathbf{P}_{xn} \\ \mathbf{P}_{yn} \end{bmatrix}
\end{align*}
\]

where \( \mathbf{P}_{xn} \) and \( \mathbf{P}_{yn} \) contain the nodal values of the distributed surface forces.

Now (3.124) can be written as

\[
\begin{align*}
\begin{bmatrix} \mathbf{P}_{\text{un}} \\ \mathbf{P}_{\text{vn}} \end{bmatrix} &= \int_0^s \begin{bmatrix} \Phi^T & 0 \\ 0 & \Phi^T \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \mathbf{P}_{xn} \\ \mathbf{P}_{yn} \end{bmatrix} \, ds
\end{align*}
\]
or after expansion,

\[ P_{un} = (\int_{s} \phi^T \phi \, ds) \, P_{xn} \]  

\[ P_{vn} = (\int_{s} \phi^T \phi \, ds) \, P_{yn} \]  

(3.128)

The same integral is used to find \( P_{un} \) and \( P_{vn} \). Therefore only one type of integral has to be calculated explicitly. In practice it is more convenient to consider separately the surface loads on each side of the element. This is the case of Fig. 3.7 in which the distributed surface loads are applied only on side \( \overline{l_2} \) of length \( s \).

Appendix I.4 shows the expansion of \( \Omega = \phi^T \phi \) and one of the individual integrals will be evaluated, as an example, using the formula (3.86) where \( \ell \) is the length of side \( \overline{l_2} \) in Fig. 3.7:

\[ \ell \Omega_{1,1} \, d\ell = \ell \xi_1^2 (2\xi_1 - 1)^2 \, d\ell = 4\ell \xi_1^4 \, d\ell - 4\ell \xi_1^3 \, d\ell + \ell \xi_1^2 \, d\ell = 4 \ell / 30 \]

After evaluating all integrals and taking into account that only side \( \overline{l_2} \) has surface loading, the horizontal nodal forces for side \( \overline{l_2} \) will be given simply by

\[ P_{un} = (\int_{\ell} \phi^T \phi \, d\ell) \, P_{xn} = \ell \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix} \begin{bmatrix} P_{x1} \\ P_{x2} \\ P_{x4} \end{bmatrix} \]  

(3.129)
Using the same process for the vertical nodal forces, one arrives at

\[
\begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix} \begin{bmatrix} p_{y1} \\ p_{y2} \\ p_{y4} \end{bmatrix} = \frac{1}{30} d_{\ell} \Phi^T \Phi \begin{bmatrix} y_l \end{bmatrix}
\]

The consistent nodal forces due to distributed loading on sides $\overline{23}$ and $\overline{13}$ can be evaluated by the same method.

It has been assumed that $\mathbf{p}_{xn}$ are the nodal values of the quadratically distributed horizontal loading over the length $\ell$ of the side $\overline{12}$. However, it is usually more convenient to use the horizontal distributed loading over the vertical projection $\ell_y$ of side $\overline{12}$ as in Fig. 3.7. In this case $\ell$ must be replaced by $\ell_y$ in (3.129). Similarly, for the vertical nodal forces $\mathbf{p}_{vn}$ one should replace $\ell$ in (3.130) by $\ell_x$.

If the distributed horizontal and vertical surface forces are constant over $\ell_y$ and $\ell_x$ of side $\overline{12}$ (Fig. 3.7), then $\mathbf{p}_{xi} = p_h$ and $\mathbf{p}_{yi} = p_v$, and (3.129) and (3.130) will reduce to

\[
\begin{bmatrix} p_{u1} \\ p_{u2} \\ p_{u3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \frac{\ell_y p_h}{6} \quad \mathbf{p}_{vn} = \begin{bmatrix} p_{v1} \\ p_{v2} \\ p_{v3} \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \frac{\ell_x p_v}{6}
\]
3.3.2 Body Forces

The consistent nodal forces due to distributed body forces are generated (see (2.53)) by

$$P_B = \iiint \Phi^T \mathbf{b} \ d(\text{vol}) = \iiint \begin{bmatrix} \Phi^T & \mathbf{0} \\ \mathbf{0} & \Phi^T \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \ d(\text{vol})$$  \hspace{1cm} (3.132)

where $\Phi$ is a row matrix defined by (3.92).

For the same reasons explained in 3.3.1 for the surface forces, this volume integral can be evaluated by the following area integral

$$P_B = \begin{bmatrix} B_{un} \\ B_{vn} \end{bmatrix} = \iint \begin{bmatrix} \Phi^T & \mathbf{0} \\ \mathbf{0} & \Phi^T \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} \ dA$$  \hspace{1cm} (3.133)

where $B_{un}$ and $B_{vn}$ contain the horizontal and vertical components, respectively.

Assuming a quadratic distribution for body forces and using the process outlined above for distributed surface forces, one will arrive at

$$\begin{bmatrix} B_{un} \\ B_{vn} \end{bmatrix} = \iint \begin{bmatrix} \Phi^T & \mathbf{0} \\ \mathbf{0} & \Phi^T \end{bmatrix} \begin{bmatrix} \phi & \mathbf{0} \\ \mathbf{0} & \phi \end{bmatrix} \begin{bmatrix} b_{xn} \\ b_{yn} \end{bmatrix} \ dA$$  \hspace{1cm} (3.134)

or, after expansion,

$$B_{un} = \iint \phi^T \phi \ dA \ b_{xn}$$  \hspace{1cm} (3.135)

$$B_{vn} = \iint \phi^T \phi \ dA \ b_{yn}$$
Since the same integral is used for both $B_{un}$ and $B_{vn}$ only one type of integral has to be calculated.

The expansion of $Q = \phi^T \phi$ is shown in Appendix 1.4 and, as an example, one of the individual integrals will be evaluated by using formula (3.84):

$$I_1 = \int Q_1 dA = \int \xi_1^2 (2\xi_1 - 1)^2 dA = 4\int \xi_1^4 dA - 4\int \xi_1^3 dA + \int \xi_1^2 dA = A/30.$$  

After evaluating all the integrals, the vertical components of the consistent system of nodal forces will be given by

$$B_{vn} = (\int \phi^T \phi \ dA) \ b_{vn} = \begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ -1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & 4 & 0 & 16 & 16 & 32 \end{bmatrix} \begin{bmatrix} b_{v1} \\ b_{v2} \\ b_{v3} \\ b_{v4} \\ b_{v5} \\ b_{v6} \end{bmatrix} = \begin{bmatrix} \frac{A}{180} \end{bmatrix} (3.136)$$

If the distributed body forces are constant within the element, then $b_{v1} = \gamma$ (say, the unit weight) and $B_{vn}$ will simply be given (deleting the zero forces) by

$$B_{vn} = \begin{bmatrix} b_{v4} \\ b_{v5} \\ b_{v6} \end{bmatrix} = \frac{\gamma A}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (3.137)$$

The same process could be used for $B_{un}$.

### 3.3.3 Forces Due to Initial Strain

The consistent nodal forces corresponding to initial strain are generated (see (2.53)) by

$$B_{vn} = \begin{bmatrix} B_{v4} \\ B_{v5} \\ B_{v6} \end{bmatrix} = \frac{\gamma A}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} (3.137)$$

The same process could be used for $B_{un}$. 
\[ P_o = \iiint B^T D \varepsilon_o \, d(\text{vol}) \quad (3.138) \]

where \( B \) is defined by (3.101), \( D \) by (3.110) and \( \varepsilon_o \) by (3.53).

For the same reasons presented in 3.3.1 for the surface forces, the volume integral (3.138) can be evaluated by an area integral

\[ P_o = \begin{pmatrix} P_{\text{oun}} \\ P_{\text{ovn}} \end{pmatrix} = \iiint B^T D \varepsilon_o \, dA \quad (3.139) \]

where \( P_{\text{oun}}, P_{\text{ovn}} \) contain the horizontal and vertical components of the consistent nodal forces.

Assuming a quadratic distribution of temperature variations \( T \) over the element, one can write

\[ T = \Phi \mathbf{T} = [\phi_1 \phi_2 \phi_3 \phi_4 \phi_5 \phi_6] \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} \quad (3.140) \]

where \( T_i \) is the temperature variation at node \( i \) and \( \Phi \) is defined by (3.92).

Using (3.53), (3.101) and (3.110) one can expand (3.139) to
\[
\left\{ \begin{array}{l}
P_{\text{oun}} \\
P_{\text{ovn}}
\end{array} \right\} = \int \int \frac{1}{2\pi} \left\{ \begin{array}{l}
\Psi^T \\
0
\end{array} \right\} \left[ \begin{array}{l}
a \ b \\
o \ b \ a
\end{array} \right] \left[ \begin{array}{ccc}
D_{11} & D_{12} & D_{13} \\
D_{12} & D_{22} & D_{23} \\
D_{13} & D_{23} & D_{33}
\end{array} \right] \left[ \begin{array}{l}
\mu_1 \\
\mu_2 \\
\mu_3
\end{array} \right] \ (\Phi^T) \ d\mathbf{A} \\
= \int \int \left\{ \begin{array}{l}
\Psi^T \\
0
\end{array} \right\} \left[ \begin{array}{l}
C_1 \\
C_2
\end{array} \right] \ (\Phi^T) \ d\mathbf{A} \\
\tag{3.141}
\]

where

\[
C = \left[ \begin{array}{c}
C_1 \\
C_2 \\
C_3 \\
C_4 \\
C_5 \\
C_6
\end{array} \right] = \left[ \begin{array}{c}
a \ b \\
o \ b \ a
\end{array} \right] \left[ \begin{array}{ccc}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{array} \right] \left[ \begin{array}{l}
\mu_1 \\
\mu_2 \\
\mu_3
\end{array} \right] \frac{1}{2\pi} \ d\mathbf{A} \\
\tag{3.142}
\]

Matrix \( C \) of constants will take the simplified form

\[
C = \left[ \begin{array}{c}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
c_6
\end{array} \right] = \left[ \begin{array}{c}
a_1 Q_1 + b_1 Q_3 \\
a_2 Q_1 + b_2 Q_3 \\
a_3 Q_1 + b_3 Q_3 \\
b_1 Q_2 + a_1 Q_3 \\
b_2 Q_2 + a_2 Q_3 \\
b_3 Q_2 + a_3 Q_3
\end{array} \right] \\
\tag{3.143}
\]

for \( Q = \left[ \begin{array}{c}
Q_1 \\
Q_2 \\
Q_3
\end{array} \right] = \frac{1}{2\pi} \left[ \begin{array}{c}
D_{11} \mu_1 + D_{12} \mu_2 + D_{13} \mu_3 \\
D_{12} \mu_1 + D_{22} \mu_2 + D_{23} \mu_3 \\
D_{13} \mu_1 + D_{23} \mu_2 + D_{33} \mu_3
\end{array} \right] \\
\tag{3.144}
\]
Using (3.114) and (3.142), one can now expand (3.141) to

\[
\begin{align*}
\{ \mathbf{P}_{\text{oun}} \} &= \left\{ \int \int \psi^T \mathbf{C}_1 \left( \mathbf{E} \frac{T}{2} \right) \left( \mathbf{\Phi} \mathbf{I} \right) dA \right\} \\
\{ \mathbf{P}_{\text{ovn}} \} &= \left\{ \int \int \psi^T \mathbf{C}_2 \left( \mathbf{E} \frac{T}{2} \right) \left( \mathbf{\Phi} \mathbf{I} \right) dA \right\}
\end{align*}
\] (3.145)

or

\[
\begin{align*}
\mathbf{P}_{\text{oun}} &= \int \int \begin{bmatrix}
\psi_{11} & 0 & 0 \\
0 & \psi_{22} & 0 \\
0 & 0 & \psi_{33}
\end{bmatrix}
\begin{bmatrix}
\mathbf{c}_1 \\
\mathbf{c}_2 \\
\mathbf{c}_3
\end{bmatrix}
\left( \mathbf{E} \frac{T}{2} \right) \left( \mathbf{\Phi} \mathbf{I} \right) dA \\
\mathbf{P}_{\text{ovn}} &= \int \int \begin{bmatrix}
\psi_{11} & 0 & 0 \\
0 & \psi_{22} & 0 \\
0 & 0 & \psi_{33}
\end{bmatrix}
\begin{bmatrix}
\mathbf{c}_4 \\
\mathbf{c}_5 \\
\mathbf{c}_6
\end{bmatrix}
\left( \mathbf{E} \frac{T}{2} \right) \left( \mathbf{\Phi} \mathbf{I} \right) dA
\end{align*}
\] (3.146)
(3.147)

Using the expansion of \( \psi^T \mathbf{C}_1 \) and the auxiliary integrals presented in Appendix I.6, it is now possible to find the horizontal components of the nodal forces

\[
\mathbf{P}_{\text{oun}} = \begin{bmatrix}
\mathbf{c}_1 I_1 \\
\mathbf{c}_2 I_2 \\
\mathbf{c}_3 I_3 \\
\mathbf{c}_1 I_5 + \mathbf{c}_2 I_4 \\
\mathbf{c}_2 I_6 + \mathbf{c}_3 I_5 \\
\mathbf{c}_1 I_6 + \mathbf{c}_3 I_4
\end{bmatrix}
\] (3.148)
Substituting in (3.148) $C_1$, $C_2$, $C_3$ by $C_4$, $C_5$, $C_6$, respectively, the vertical components $P_{ovn}$ of the nodal forces will be obtained.

3.4 Strains and Stresses in Two-Dimensional Analysis

3.4.1 Strains

After solving the system of equations (2.56) the nodal displacements are known and the strains can be evaluated using the expressions presented in (3.71) and (3.72):

$$\epsilon = Bu_n$$

$$B = \frac{1}{2A} \begin{bmatrix} a^T & 0 \\ 0 & b^T \\ b^T & a^T \end{bmatrix} \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix}$$

(3.149)

Matrix $\psi$ is a function of $\xi_1$, $\xi_2$, $\xi_3$ and therefore the numerical value of its non-zero elements depends on the coordinates of the particular point of the element at which the strains have to be calculated. For the centroid at which $(\xi_1, \xi_2, \xi_3) = (1/3, 1/3, 1/3)$ the expansion of $\psi$ given by (3.98) becomes

$$\psi = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 4 & 0 & 4 \\ 0 & 1 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 \end{bmatrix}$$

(3.150)

Now one can expand the first of equations (3.149) to
where $a_i, b_i$ are defined by (3.76) in terms of the global coordinates, $U_1 \ldots U_6$ are the horizontal nodal displacements and $U_7 \ldots U_{12}$ are the vertical nodal displacements corresponding to the chosen element node numbering system (e.g. as in Fig. 3.4). Expression (3.151) can be used for each triangular element of the structure.

### 3.4.2 Horizontal, Vertical and Shear Stresses

The stresses can be evaluated for every element by using repeatedly the first of equations (3.39) and (3.149):

$$
\sigma = \sigma_o + D B U_n
$$

(3.152)

The initial stresses are evaluated by means adequate to the particular type of problem. For instance, when they are the existing pressures in a natural soil medium, the following expression can be used:

$$
\sigma_o = \begin{bmatrix}
\sigma_{ox} \\
\sigma_{oy} \\
\tau_{oxy}
\end{bmatrix} = \begin{bmatrix}
K_o (-\gamma y - u) + u \\
-\gamma y \\
0
\end{bmatrix}
$$

(3.153)

where $K_o = \frac{\sigma_h}{\gamma}$ is the coefficient of earth pressure at rest

$\gamma$ is the unit weight

$u$ is the pore water pressure
\(-y\) is the depth.

Knowing that

\[
(E \xi) = E_1 \xi_1 + E_2 \xi_2 + E_3 \xi_3 = (E_1 + E_2 + E_3)/3
\]

at the centroid of the element, using the expansions of \(D\) in (3.110) and \(\varepsilon = B U_n\) in (3.151), the stresses will be given by

\[
\begin{align*}
\sigma_x &= \sigma_{ox} + C \\
\sigma_y &= \sigma_{oy} + C \\
\tau_{xy} &= \tau_{oxy} + C
\end{align*}
\]

with \(C = (E_1 + E_2 + E_3)/(18A)\)

3.4.3 Principal Stresses

It is usual to evaluate for every element not only the stresses but also the principal stresses and their directions, at the centroids. From the Mohr's circle one derives

\[
\begin{align*}
\sigma_{MAX} &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2} \\
\sigma_{MIN} &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_y - \sigma_x}{2}\right)^2 + \tau_{xy}^2} \\
\theta &= \arctan \left( \frac{\tau_{xy}}{\sigma_y - \sigma_{MIN}} \right)
\end{align*}
\]

where \(\sigma_{MAX}\) is the maximum stress for the element (algebraic value)

\(\sigma_{MIN}\) is the minimum stress for the element

\(\theta\) is the clockwise angle from the vertical of the line of action of the maximum stress.
It is convenient to multiply by 57.3 the value of $\theta$ given by (3.155) to express it in degrees.

### 3.5 Element Stiffness Matrix for Axi-symmetric Analysis

As shown by (2.34) the element stiffness matrix is given, in general, by

$$ K = \iiint B^T D B \, d(\text{vol}) \quad (3.156) $$

where $D$ is the elastic rigidity matrix defined in 3.1.5 for axi-symmetric analysis and $B$ is defined by

$$ \epsilon = B U_n = B \begin{bmatrix} u_n \\ v_n \end{bmatrix} \quad (3.157) $$

For plane stress and plane strain, matrix $B$ was shown to be

$$ B = \frac{1}{2A} \begin{bmatrix} a^T & 0 \\ 0 & b^T \\ b^T & a^T \end{bmatrix} \begin{bmatrix} \psi & O \\ O & \psi \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} a^T \psi & O \\ O & b^T \psi \\ b^T \psi & a^T \psi \end{bmatrix} \quad (3.158) $$

For axi-symmetric analysis, according to (3.63) the strains are related to the displacements by

$$ \epsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_\theta \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} \quad (3.159) $$

while, for plane stress and plane strain analysis, the strains are related to the displacements by
Introducing a new function $\Gamma$ defined by

$$\Gamma = \frac{2A}{x} \begin{bmatrix}
\xi_1 (2 \xi_1 - 1) & \xi_2 (2 \xi_2 - 1) & \xi_3 (2 \xi_3 - 1) & 4 \xi_1 \xi_2 & 4 \xi_2 \xi_3 & 4 \xi_3 \xi_1 \\
-1 & 1 & 1 & 2 & 2 & 2
\end{bmatrix}$$

$$= \begin{bmatrix}
\Gamma_1 & \Gamma_2 & \Gamma_3 & \Gamma_4 & \Gamma_5 & \Gamma_6
\end{bmatrix} = \frac{2A}{x} \Omega$$

(3.161)

and a new matrix $\mathcal{B}$ defined as

$$\mathcal{B} = \frac{1}{2A} \begin{bmatrix}
\mathcal{A}^T \psi & 0 \\
0 & \mathcal{B}^T \psi \\
\Gamma & 0 \\
0 & \Gamma
\end{bmatrix}$$

(3.162)

one can see, by comparing (3.159) with (3.160) and (3.162) with (3.158), that the expansions for strains $\varepsilon_x$, $\varepsilon_y$ and $\gamma_{xy}$ have not been affected. With regard to $\varepsilon_\theta$, according to (3.159),

$$\varepsilon_\theta = \frac{u}{x} = \frac{1}{2A} \Gamma u_n = \frac{1}{2A} \frac{2A}{x} \Omega u_n = \frac{1}{x} \Omega u_n$$

(3.163)

and since $u = \Omega u_n$ (see (3.93)) this proves that $\varepsilon_\theta$ is properly generated by (3.163) and thus $\mathcal{B}$ in (3.162) is suitable for axi-symmetric analysis.

The ring-type element used in this case is shown in Fig. 3.8 and the volume integral (3.156) will become

$$\mathbf{K} = 2\pi \int \int \mathcal{B}^T D \mathcal{B} x dA$$

(3.164)
and the integrand will expand to

\[ \mathbf{O} = \mathbf{B}^T \mathbf{D} \mathbf{B} = \frac{1}{4A} \left[ \begin{array}{ccc} \psi^T \mathbf{a} & 0 & \mathbf{I}^T \psi \mathbf{b} \\ 0 & \psi^T \mathbf{b} & 0 \end{array} \right] \]  

(3.165)

Expanding (3.165) and partitioning \( \mathbf{O} \) as

\[ \mathbf{O} = \begin{bmatrix} \mathbf{O}_{11} & \mathbf{O}_{12} \\ \mathbf{O}_{21} & \mathbf{O}_{22} \end{bmatrix} = \mathbf{B}^T \mathbf{D} \mathbf{B} \]  

(3.166)

one arrives at

\[ \mathbf{O}_{11} = \frac{1}{4A^2} (E_{k}) (D_{11} \psi \mathbf{a}^T \psi + D_{13} \mathbf{I}^T \psi \mathbf{a} + D_{13} \psi \mathbf{a} \mathbf{I} + D_{33} \mathbf{I}^T \psi + D_{44} \psi \mathbf{b} \mathbf{b}^T \psi) \]

\[ \mathbf{O}_{12} = \frac{1}{4A^2} (E_{k}) (D_{12} \psi \mathbf{a} \mathbf{b}^T \psi + D_{23} \psi \mathbf{b} \mathbf{b}^T \psi + D_{44} \psi \mathbf{a} \mathbf{b}^T \psi) \]

\[ \mathbf{O}_{21} = \frac{1}{4A^2} (E_{k}) (D_{12} \psi \mathbf{a} \mathbf{b}^T \psi + D_{23} \psi \mathbf{b} \mathbf{b}^T \psi + D_{44} \psi \mathbf{a} \mathbf{b}^T \psi) \]  

(3.167)

\[ \mathbf{O}_{22} = \frac{1}{4A^2} (E_{k}) (D_{22} \psi \mathbf{b} \mathbf{b}^T \psi + D_{44} \psi \mathbf{a} \mathbf{a}^T \psi) \]

As a verification it can be checked that \( \mathbf{O}_{21} = \mathbf{O}_{12}^T \). It must be noted, before going further, that the case of inclined strata of stratified (cross-anisotropic) material is not considered, since some of the elements of \( \mathbf{D} \) in (3.165) are assumed to be zero, i.e. no transformation matrices of the type presented in 3.1.4 will be used.
In order to facilitate the evaluation of (3.164), making use of previous results obtained while calculating the element stiffness matrix for plane analysis, the element stiffness matrix for axi-symmetric analysis represented by (3.164) will be partitioned and decomposed in a sum of matrices as follows

\[
\begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix} = \begin{bmatrix}
K_{A11} & K_{A12} \\
K_{A21} & K_{A22}
\end{bmatrix} + \begin{bmatrix}
K_{B11} & K_{B12} \\
K_{B21} & K_{B22}
\end{bmatrix} + \begin{bmatrix}
K_{C11} & K_{C12} \\
K_{C21} & K_{C22}
\end{bmatrix}
\]

or

\[
K = K_A + K_B + K_C
\]  

(3.168)

From (3.167), noting that

\[
K = \begin{bmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{bmatrix} = 2\pi \int \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix} x \, dA
\]  

(3.169)

one can write

\[
K_{A11} = \frac{2\pi}{4A^2} \int \psi^T (D_{11}a_1 + D_{44}b_1^T) \psi (E_t) \, x \, dA
\]

\[
K_{A12} = K_{A21}^T = \frac{2\pi}{4A^2} \int \psi^T (D_{12}a_2 + D_{44}b_2^T) \psi (E_t) \, x \, dA
\]  

(3.170)

\[
K_{A22} = \frac{2\pi}{4A^2} \int \psi^T (D_{22}b_2 + D_{44}a_2^T) \psi (E_t) \, x \, dA
\]

\[
K_{B11} = \frac{2\pi}{4A^2} D_{13} \int \rho \psi^T \rho + \psi^T a_1 \psi (E_t) \, x \, dA
\]

\[
K_{B12} = K_{B21} = \frac{2\pi}{4A^2} D_{23} \int \rho^T \rho \psi (E_t) \, x \, dA
\]  

(3.171)

\[
K_{B22} = 0
\]
\[ KC_{11} = \frac{2\pi}{4A^2} \int D_{33} / \Gamma T_x \ (E_x) \ xDA \]  

(3.172)

\[ KC_{12} = KC_{21} = KC_{22} = 0 \]

The function \( \Gamma \) (including the variable \( x = x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3 \) as denominator) does not appear in matrix \( KA \), appears once in matrix \( KB \) and twice in matrix \( KC \). Therefore \( KB \) is the easiest to evaluate by exact integration while \( KC \) is the most difficult, although it affects only one quarter of the elements of \( K \).

All individual integrals in \( KA \) have the form

\[ KA_{ij} = \frac{2\pi}{4A^2} \int \psi^T \Omega \psi \ (E_x) \ xDA \]  

(3.173)

where matrix \( \Omega \) is defined as follows, according to (3.170):

for \( KA_{11} \): \( \Omega = D_{11} \)

\[
\begin{bmatrix}
    a_1^2 & a_1 a_2 & a_1 a_3 \\
    a_1 a_2 & a_2^2 & a_2 a_3 \\
    a_1 a_3 & a_2 a_3 & a_3^2
\end{bmatrix} + D_{44} \begin{bmatrix}
    b_1^2 & b_1 b_2 & b_1 b_3 \\
    b_1 b_2 & b_2^2 & b_2 b_3 \\
    b_1 b_3 & b_2 b_3 & b_3^2
\end{bmatrix}
\]

(3.174)

for \( KA_{12} \): \( \Omega = D_{12} \)

\[
\begin{bmatrix}
    a_1 a_1^2 & a_1 b_1 & a_1 b_3 \\
    a_2 b_1 & a_2 a_2^2 & a_2 b_3 \\
    a_3 b_1 & a_3 b_2 & a_3 a_3^2
\end{bmatrix} + D_{44} \begin{bmatrix}
    a_1 b_1 & a_2 b_1 & a_3 b_1 \\
    a_1 a_2 & a_2 a_2 & a_3 a_2 \\
    a_1 a_3 & a_2 a_3 & a_3 a_3
\end{bmatrix}
\]

(3.175)

for \( KA_{22} \): \( \Omega = D_{22} \)

\[
\begin{bmatrix}
    b_1^2 & b_1 a_2 & b_1 a_3 \\
    b_1 a_2 & b_2^2 & b_2 a_3 \\
    b_1 a_3 & b_2 a_3 & b_3^2
\end{bmatrix} + D_{44} \begin{bmatrix}
    a_1^2 & a_1 a_2 & a_1 a_3 \\
    a_1 a_2 & a_2^2 & a_2 a_3 \\
    a_1 a_3 & a_2 a_3 & a_3^2
\end{bmatrix}
\]

(3.176)
Since the element stiffness matrix $K$ is symmetric it is not necessary to evaluate the elements below the principal diagonal. As an example, three elements of $KA$ will be determined using the auxiliary integrals presented in Appendix II.1 and the expansion of $\psi^TQ\psi$ presented in Appendix I.2.

$$k_{a_{1,3}} = \frac{2\pi}{4A^2} \int Q_{1,3} \psi_{11} \psi_{33} (E_\pi) \, xdA = \frac{2\pi}{4A^2} Q_{1,3} \, I_8$$

$$= \frac{2\pi}{4A^2} (D_{11}a_1a_3 + D_{44}b_3b_3) \, I_8$$

$$k_{a_{1,9}} = \frac{2\pi}{4A^2} (D_{12}a_1b_3 + D_{44}a_3b_1) \, I_8$$

$$k_{a_{7,9}} = \frac{2\pi}{4A^2} (D_{22}b_1b_3 + D_{44}a_1a_3) \, I_8$$

All individual integrals in $KB$ have either or both of the following forms:

$$J = \int \int \psi^TQ\psi (E_\pi) \, xdA$$

$$J = \int \int \psi^TQ\pi (E_\pi) \, xdA$$

where $Q$ is defined by (3.171).

From the expansions in Appendix II.2 and the auxiliary integrals in Appendix II.3 one can evaluate, for instance,

$$k_{b_{1,3}} = \frac{2\pi}{4A^2} D_{13} [\int Q_{13} \psi_{33} (E^T_\pi) xdA + \int \psi_{11} (E^T_\pi) xdA] =$$

$$= \frac{2\pi}{4A^2} D_{13} (a_3 \, I_{24} + a_1 \, I_{34})$$

(3.179)
\[ kb_{1,9} = \frac{2\pi}{4A^2} \frac{D_{23}}{b_3 I_{24}} \]

\[ kb_{7,9} = 0 \]

All individual integrals in KC have the form

\[ J = \frac{2\pi}{4A^2} D_{33} \int_{T}^{T} (E^T \xi) \, x \, dA \]

\[ J_1 = \frac{2\pi}{4A^2} D_{33} \int \frac{x}{1} \phi_j \phi_1 (E^T \xi) \, x \, dA \]

\[ = 2\pi D_{33} \int \frac{1}{x} \phi_j \phi_1 (E^T \xi) \, dA \]  

\[ = 2\pi D_{33} \int \frac{f(\xi_1, \xi_2)}{x_3 + (x_1 - x_3)\xi_1 + (x_2 - x_3)\xi_2} \]

\[ = 2\pi D_{33} \int F(\xi_1, \xi_2) \, dA \]  

All these integrals and thus all elements of KC were calculated by the author using formula (3.83). However, the calculations are not presented here because the exact integration does not lead to the best practical results in this particular case as the author has proved in a few test problems.

The final expressions of each element of KC are rather complex and involve differences between logarithms of numbers representing the cartesian coordinates (abscissae) of the corner nodes of the triangular element. When the corner node has abscissa equal to zero (i.e. lies on the axis y of symmetry) the logarithm becomes infinite. This
difficulty can be overcome by considering a hole of small
diameter along the axis y which is a reasonable approximation
avoiding the appearance of logarithm of zero. However,
another difficulty arises that cannot be obviated. When the
elements are far away from the axis y, the differences
between logarithms (i.e. logarithms of numbers very close
to unity) calculated by the computer are not sufficiently
accurate compared with other calculations and the round off
errors accumulate when evaluating the rather complex
expressions of the integrals leading to $KC$. The unavoidable
result is that the final values cannot be considered as
resulting from an exact integration.

The author is satisfied that better results can be
obtained in practice by using an approximate method of
integration to evaluate the elements of $KC$.

This consists in considering the function $1/x$ appearing
in the integrand of the individual integrals (3.180) as
constant and equal to $1/x_0$, $x_0$ being the abscissa of the
centroid of the element. The error in this approximation
increases when $x_0$ decreases, i.e. the error is maximum for
elements close to axis y. All elements of $KC$ have been
evaluated by this method in Appendix II.4.

Using (3.168), (3.177), (3.179) and Appendix II.4,
one can now write the final expressions for the three
elements of $K$ which have been considered as examples of
the process of calculation:
\[ k_{1,3} = \frac{2\pi}{4A^2} (D_{11}a_1a_3 + D_{44}b_1b_3) I_8 + \frac{2\pi}{4A^2} D_{13} (a_3i_{24} + a_1i_{34}) \]
\[ + (2\pi D_{33} A/x_0)(3E_1 - 2E_2 - E_3)/315 \]

\[ k_{1,9} = \frac{2\pi}{4A^2} (D_{12}a_1b_3 + D_{44}a_3b_1) I_8 + \frac{2\pi}{4A^2} D_{23} b_3 I_{24} \quad (3.181) \]

\[ k_{7,9} = \frac{2\pi}{4A^2} (D_{22}b_1b_3 + D_{44}a_1a_3) I_8 \]

The approximate integration used for \( KC \) only affects 6 x 6 of the 12 x 12 elements of \( K \). In practical problems, in most cases the numerical values of the elements of \( KC \) are of one order or even two orders less than the values of the corresponding elements of \( KA \) and \( KB \). This shows that, for all practical purposes, \( K \) can be considered as having been entirely obtained by exact integration.

Although the author has not considered it necessary, a better approximation could be readily used in the evaluation of \( KC \). Appendix II.5 shows suitable quadrature formulae suggested by Hammer et al.

### 3.6 Consistent System of Element Nodal Forces for Axi-symmetric Analysis

#### 3.6.1 Surface Forces

From (3.123), (3.124) and (3.127) one can derive directly the expression which generates the consistent system of nodal forces due to surface loading, in axi-symmetric analysis,
\[
\frac{P}{P} = \begin{pmatrix} \frac{P}{P_{un}} \\ \frac{P}{P_{vn}} \end{pmatrix} = 2\pi \int_{S} \begin{bmatrix} \phi \\ \phi^2 \end{bmatrix} \begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix} \begin{pmatrix} \frac{P}{P_{xn}} \\ \frac{P}{P_{yn}} \end{pmatrix} ds \tag{3.182}
\]

Since the element is not a plane triangular element but a ring-type triangular element, the only difference is the appearance of the constant factor \(2\pi = \int_0^{2\pi} d\theta\). Therefore it should not be necessary to add anything more to what was explained in 3.3.1.

However, because the case in which the vertical surface loading is constant over the (usually horizontal) surface of a ring-type triangular element is quite frequent, it is worthwhile to consider this particular case to obtain the simplified expression of \(\frac{P}{P} \).

From Fig. 3.9 it is easy to verify that a constant distributed pressure over a circular area is equivalent to a linearly increasing distributed loading \(P_x = 2\pi xp\) over the radius. In fact, the total load \(P\) can be evaluated by two different integrals leading to the same result:

\[
P = \int \int pdA = p \cdot \pi R^2
\]

\[
P = \int_0^R P_x dx = \int_0^R 2\pi xp dx = p \cdot \pi R^2 \tag{3.183}
\]

Using notations in Fig. 3.10, one can obtain directly from (3.130)

\[
\begin{pmatrix} P_{v1} \\ P_{v2} \\ P_{v4} \end{pmatrix} = \frac{x_1 - x_2}{30} \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix} \begin{pmatrix} 2\pi p \cdot x_1 \\ 2\pi p \cdot x_2 \\ \pi p \cdot (x_1 + x_2) \end{pmatrix} \tag{3.184}
\]
or

\[
\begin{pmatrix}
P_{v1} \\
P_{v2} \\
P_{v4}
\end{pmatrix} = \frac{p\pi(x_1 - x_2)}{3} \begin{pmatrix}
x_1 \\
x_2 \\
2(x_1 + x_2)
\end{pmatrix}
\]

(3.185)

3.6.2 Body Forces and Forces Due to Initial Strain

The consistent nodal forces due to distributed body forces are generated by the following expressions obtained directly from (3.132) and (3.135):

\[
B_{un} = 2\pi \int (f \phi^T \phi \times dA) b_{xn}
\]

\[
B_{vn} = 2\pi \int (f \phi^T \phi \times dA) b_{yn}
\]

(3.186)

Similarly, the consistent nodal forces due to initial strain can be obtained directly from (3.138) and (3.139):

\[
P_o = \begin{pmatrix}
P_{oun} \\
P_{oun}
\end{pmatrix} = 2\pi \int \int B \cdot D \cdot \varepsilon_o \cdot \times \ dA
\]

(3.187)

The consistent nodal forces are obtained by the process outlined in 3.3.2 and 3.3.3 for plane stress and plane strain analysis with the obvious differences resulting from the fact that:

(a) the constant factor $2\pi$ and variable $x$ appear in the expressions (3.186) and (3.187)

(b) matrices $B, D$ and $\varepsilon_o$ in (3.187) are now given by (3.162), (3.60) and 3.56).
3.7 Strains and Stresses in Axi-Symmetric Analysis

The strains can be evaluated by

\[ \varepsilon = B U_n \]  \hspace{1cm} (3.188)

where

\[ B = \frac{1}{2A} \begin{bmatrix} a^T \psi & 0 \\ 0 & b^T \psi \\ \frac{1}{r} & 0 \\ b^T \psi & a^T \psi \end{bmatrix} \]  \hspace{1cm} (3.189)

with \( r = \frac{2A}{x} \phi \)

To facilitate the calculations, matrix \( B \) will be expanded to

\[ B = \frac{1}{2A} \begin{bmatrix} a^T & 0 \\ 0 & b^T \\ 0 & 0 \\ b^T & a^T \end{bmatrix} \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix} + \frac{1}{x} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \phi \\ 0 & 0 \end{bmatrix} \]  \hspace{1cm} (3.190)

Considering the strains at the centroid of the element as in (3.151) for plane analysis, one can now expand (3.188) to

\[
\begin{align*}
\varepsilon_x &= \frac{1}{6A} \begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & b_1 & b_2 & b_3 \\
0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 + 4(U_4+U_6) \\
U_2 + 4(U_4+U_5) \\
U_3 + 4(U_5+U_6) \\
U_7 + 4(U_{10}+U_{12}) \\
U_8 + 4(U_{10}+U_{11}) \\
U_9 + 4(U_{11}+U_{12}) \end{bmatrix} \\
\varepsilon_y &= 0 \\
\varepsilon_\theta &= 0 \\
\gamma_{xy} &= \frac{1}{6A} \begin{bmatrix} b_1 & b_2 & b_3 & a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 + 4(U_4+U_6) \\
U_2 + 4(U_4+U_5) \\
U_3 + 4(U_5+U_6) \\
U_7 + 4(U_{10}+U_{12}) \\
U_8 + 4(U_{10}+U_{11}) \\
U_9 + 4(U_{11}+U_{12}) \end{bmatrix}
\end{align*}
\]
\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\sigma_\theta \\
\tau_{xy}
\end{pmatrix} = \frac{E_1+E_2+E_3}{3} \begin{pmatrix}
D_{11} & D_{12} & D_{13} & 0 \\
D_{12} & D_{22} & D_{23} & 0 \\
D_{13} & D_{23} & D_{33} & 0 \\
0 & 0 & 0 & D_{44}
\end{pmatrix} \begin{pmatrix}
\epsilon_x \\
\epsilon_y \\
\epsilon_\theta \\
\gamma_{xy}
\end{pmatrix}
\]  
(3.192)

3.8 Element Stiffness Matrix for Linear Elements in Two-Dimensional Analysis

Fig. 3.11 shows a linear element with the notations of nodal forces and displacements used for triangular elements. In this case one can write:

\[
\begin{align*}
\mathbf{u}_n &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} & \mathbf{v}_n &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & \mathbf{U}_n &= \begin{pmatrix} u_n \\ v_n \end{pmatrix} \\
\mathbf{P}_{un} &= \begin{pmatrix} P_{u1} \\ P_{u2} \end{pmatrix} & \mathbf{P}_{vn} &= \begin{pmatrix} P_{v1} \\ P_{v2} \end{pmatrix} & \mathbf{P}_D &= \begin{pmatrix} P_{un} \\ P_{vn} \end{pmatrix}
\end{align*}
\]  
(3.193)
It can be easily verified that the end forces and displacements of a linear element are related by

\[
\begin{bmatrix}
\cos^2 \theta & -\cos^2 \theta & \sin \theta \cos \theta & -\sin \theta \cos \theta \\
-\cos^2 \theta & \cos^2 \theta & -\sin \theta \cos \theta & \sin \theta \cos \theta \\
\sin \theta \cos \theta & -\sin \theta \cos \theta & \sin^2 \theta & -\sin^2 \theta \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & -\sin^2 \theta & \sin^2 \theta
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
P_{u1} \\
P_{u2} \\
P_{v1} \\
P_{v2}
\end{bmatrix}
\]  
(3.194)

or

\[
K U_n = P_P
\]
(3.195)

where

\[
K = \frac{EA}{L}
\]

is the element stiffness matrix for a linear element.

3.9 Implementation of the Condition \( d_i - d_j = 0 \)

3.9.1 Transformation of Coordinates

Let the displacement components \( \bar{d}^* \), the force components \( \bar{P}^* \) and the element stiffness matrix \( \bar{K}^* \) refer to a local coordinate system and \( d, P, K \) refer to the common coordinate system necessary for assembly. It was shown in (2.41) that the displacements are related by

\[
\bar{d}^* = R d
\]
(3.197)

and the element stiffness matrices, according to (2.46), are related by

\[
K = R^T K^* R
\]
(3.198)
As the force components must perform the same amount of work in either system of coordinates,

\[ P^T \mathbf{d} = P^* \mathbf{d}^* = P^T \mathbf{R} \mathbf{d} \]

Therefore

\[ P = R^T P^* \]  \hspace{1cm} (3.199)

The relation between force and displacement transformations typified by equations (3.197) and (3.199) is known in general as contragredience (Livesley, 1964).

In many complex problems, an external constraint of some kind may be imagined enforcing the requirement (3.197) with the number of degrees of freedom of \( \mathbf{d}^* \) and \( \mathbf{d} \) being quite different. Even in such instances the relations (3.197) and (3.199) continue to be valid.

3.9.2 Condition \( \mathbf{d}_i - \mathbf{d}_j = 0 \)

Imagine that it is required to enforce the condition \( \mathbf{d}_i^* - \mathbf{d}_j^* = 0 \), e.g. horizontal displacement of node \( p \) equal to horizontal displacement of node \( q \), in a complex structure whose stiffness was evaluated as \( K^* \) by assembly of the element stiffnesses.

Using the relationships

\[ \mathbf{d}_{px}^* = \frac{1}{2} (\mathbf{d}_{px} + \mathbf{d}_{qx}) \]

\[ \mathbf{d}_{qx}^* = \frac{1}{2} (-\mathbf{d}_{px} + \mathbf{d}_{qx}) \]  \hspace{1cm} (3.200)
it will be

\[ d_{px} = d^*_px - d^*_qx \]  \hspace{1cm} (3.201) \\
\[ d_{qx} = d^*_px + d^*_qx \]

Now it is easy to impose the condition \( d_{px} = 0 \) which is equivalent to \( d^*_px - d^*_qx = 0 \), i.e. \( d^*_i - d^*_j = 0 \).

Equations (3.200) can be written as

\[
\begin{pmatrix}
    d^*_i \\
    d^*_j
\end{pmatrix} =
\begin{pmatrix}
    d_{px} \\
    d_{qx}
\end{pmatrix} =
\begin{pmatrix}
    \frac{1}{2} & \frac{1}{2} \\
    -\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
    d_{px} \\
    d_{qx}
\end{pmatrix} =
\begin{pmatrix}
    \frac{1}{2} & \frac{1}{2} \\
    -\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
    d^*_i \\
    d^*_j
\end{pmatrix} \tag{3.202}
\]

or

\[ d^* = R \hat{d} \]

in which

\[
R =
\begin{pmatrix}
    \frac{1}{2} & \frac{1}{2} \\
    -\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \tag{3.203}
\]

Using (3.197) and (3.199) one can write

\[
K \hat{d} = P
\]

as

\[
R^T K^* R \hat{d} = R^T P^* \tag{3.204}
\]

Expanding (3.204) will lead to

\[
\begin{pmatrix}
    I & 0 & 0 \\
    0 & I & 0 \\
    0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
    K^* \\
    K^* \\
    K^*
\end{pmatrix}
\begin{pmatrix}
    I & 0 & 0 \\
    0 & I & 0 \\
    0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
    d^*_i \\
    d^*_j
\end{pmatrix} =
\begin{pmatrix}
    d^*_i \\
    d^*_j
\end{pmatrix}
\]
where

\[
\begin{pmatrix}
d_1 \\
\vdots \\
d_j \\
\vdots \\
d_N
\end{pmatrix} = \begin{pmatrix}
d_1 \\
\vdots \\
d_j \\
\vdots \\
d_N
\end{pmatrix} + \begin{pmatrix}
d_1^* \\
\vdots \\
d_j^* \\
\vdots \\
d_N^*
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
P_1 \\
\vdots \\
P_j \\
\vdots \\
P_N
\end{pmatrix} = \begin{pmatrix}
P_1 \\
\vdots \\
P_j \\
\vdots \\
P_N
\end{pmatrix} + \begin{pmatrix}
(P_1^* - P_j^*)/2 \\
\vdots \\
(P_j^* + P_j^*)/2 \\
\vdots \\
(P_N^* - P_N^*)/2
\end{pmatrix}
\]

Only two elements \(P_1^*\) and \(P_j^*\) of the force matrix \(P^*\) have to be modified as indicated in (3.207). The rows and columns \(i\) and \(j\) of the stiffness matrix \(K^*\) of the structure are modified as indicated in (3.206) prior to solving the system of equations. After solving this system of equations, only two of the final displacements have to be modified, according to (3.197), by using the expressions:
If there is more than one condition of the type \( d_i^* - d_j^* = 0 \) to be imposed, this can be done by modifying \( K^* \) and \( P^* \) successively for each condition as shown in (3.206) and (3.207) prior to implementing any other type of boundary conditions and solving the system of equations. Then equations (3.208) will be used successively for each pair of displacements \( d_i^* \) and \( d_j^* \).

### 3.9.3 Condition \( d_i^* - d_j^* = \text{constant} \)

After changing from variably \( d^* \) to \( d \) the condition \( d_i^* - d_j^* = C \) in which \( C \) is a constant different from zero can be easily imposed instead of \( d_i^* - d_j^* = 0 \). This is equivalent to imposing the condition \( d_{pi}^* = C \) and then, after solving the system of equations, \( d^* \) can be found by the equations

\[
\begin{align*}
d_i^* &= \frac{1}{2} (d_j + C) \\
d_j^* &= \frac{1}{2} (d_j - C)
\end{align*}
\]  

(3.209)

since it can be verified that \( d_i^* - d_j^* = C \).

Let \( K^* d^* = P^* \) be partitioned as follows:
If the condition $d_s^* = C$ is to be imposed one can expand (3.210) to

$$
\begin{bmatrix}
K_{tt} & K_{ts} & K_{tu} \\
K_{st} & K_{ss} & K_{su} \\
K_{ut} & K_{us} & K_{uu}
\end{bmatrix}
\begin{bmatrix}
d_t^* \\
d_s^* \\
d_u^*
\end{bmatrix} =
\begin{bmatrix}
P_t^* \\
P_s^* \\
P_u^*
\end{bmatrix}
$$

(3.210)

The middle equation merely relates the displacements and the reactive force $P_s^*$ after the condition $d_s^* = C$ was imposed. Thus it can be deleted or modified so as to represent $d_s^* = C$. Since $C$ is prescribed (known), one can now write (3.211) as

$$
\begin{bmatrix}
K_{tt} & 0 & K_{tu} \\
0 & 1 & 0 \\
K_{ut} & 0 & K_{uu}
\end{bmatrix}
\begin{bmatrix}
d_t^* \\
-d_s^* \\
d_u^*
\end{bmatrix} =
\begin{bmatrix}
P_t^* - K_{ts} C \\
C \\
P_u^* - K_{us} C
\end{bmatrix}
$$

(3.212)

This equation shows that the technique for imposing $d_s^* = C$ is similar to the technique for imposing $d_s = 0$ except for the additional operations of subtracting $K_{ts} C$ and $K_{us} C$ from the force vector.
3.10 **Diaphragm Wall with Anchors or Struts**

It has been shown in (2.41) that the element nodal displacements and the element nodal forces are related by

$$K U_n = P$$  \hspace{1cm} (3.213)

When evaluating the element stiffness matrix $K$ the forces were known and the displacements unknown. After solving the system of equations (2.55) all nodal displacements are known and one can use (3.213) to evaluate the corresponding nodal forces for every element of the structure. If this is done for all elements of the diaphragm wall one will find a self-equilibrating system of forces acting on the nodes along the exterior surfaces of the wall. Due to equality of action and reaction no forces will be found acting on the interior nodes of the wall if, as it is usual, no exterior forces had been applied to those nodes.

From these nodal forces which constitute a consistent system of loads, it is possible to check that their resultant is zero and then find the bending moments, shear forces and axial forces, considering the wall as a vertical cantilever beam. It is also possible, although slightly more difficult, to find the distributed final pressures applied by the soil to the lateral surfaces of the wall.

In (3.129) and (3.130) it was shown how the nodal values of the distributed pressure on side $\overline{12}$ of a triangular element are related to the consistent nodal forces:
These equations were derived in accordance with the node numbering system shown in Fig. 3.7 and make clear that the same method can be used to find horizontal and vertical pressures. Therefore only the horizontal pressures will be evaluated here.

Now the nodal forces $P_{ui}$ are known and the nodal values $P_{xi}$ of the distributed pressures could be found directly by inverting equations (3.214). However, for practical reasons, a formally different approach will be used.

In fact, the nodal forces are known but not the contributions of each element to those nodal forces, if they are determined simultaneously for all elements of the wall as it is convenient from a programming point of view.

Consider the new node numbering system shown in Fig. 3.12b. By comparison with the node numbering system used so far, one can now write the first of equations (3.214) as

$$\begin{bmatrix} P_{ul} \\ P_{u2} \\ P_{u3} \end{bmatrix} = \frac{\rho_y}{30} \begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix} \begin{bmatrix} P_{x1} \\ P_{x2} \\ P_{x3} \end{bmatrix}$$ (3.215)
obtained by interchanging rows and columns 2 and 3 in (3.214).
To simplify the notation $l_2$ is used rather than $l_y$ since $l_y = l_2$.

Consider the left hand side of the wall, the nodal forces $P_{ui}$ which are known at midside nodes and at the ends, and the nodal forces $P'_{ui}$ and $P''_{ui}$ which are unknown but whose sums $P'_{ui} + P''_{ui} = P_{ui}$ are known at all corner nodes.

Using (3.215), the equations for all nodes can be written as

\[
\begin{align*}
\begin{cases}
(4p_{x1} + 2p_{x2} - p_{x3}) \ l_2 / 30 = p_{u1} \\
(2p_{x1} + 16p_{x2} + 2p_{x3}) \ l_2 / 30 = p_{u2} \\
(-p_{x1} + 2p_{x2} + 4p_{x3}) \ l_2 / 30 = p'_{u3} \\
(4p_{x3} + 2p_{x4} - p_{x5}) \ l_4 / 30 = p''_{u3} \\
(2p_{x3} + 16p_{x4} + 2p_{x5}) \ l_4 / 30 = p_{u4} \\
(-p_{x3} + 2p_{x4} + 4p_{x5}) \ l_4 / 30 = p'_{u5} \\
(4p_{x5} + 2p_{x6} - p_{x7}) \ l_6 / 30 = p''_{u5} \\
(2p_{x5} + 16p_{x6} + 2p_{x7}) \ l_6 / 30 = p_{u6} \\
(-p_{x5} + 2p_{x6} + 4p_{x7}) \ l_6 / 30 = p'_{u7}
\end{cases}
\end{align*}
\]

Adding together the last equation of each set of three and the first equation of the following set, one will obtain, in matricial form,
This is a system of equations of the form

\[
\frac{1}{30} \begin{bmatrix}
4 \ell_2 & 2 \ell_2 & - \ell_2 & 0 & 0 & 0 & 0 \\
2 \ell_2 & 16 \ell_2 & 2 \ell_2 & 0 & 0 & 0 & 0 \\
- \ell_2 & 2 \ell_2 & 4(\ell_2 + \ell_4) & 2 \ell_4 & - \ell_4 & 0 & 0 \\
0 & 0 & 2 \ell_4 & 16 \ell_4 & 2 \ell_4 & 0 & 0 \\
0 & 0 & - \ell_4 & 2 \ell_4 & 4(\ell_4 + \ell_6) & 2 \ell_6 & - \ell_6 \\
0 & 0 & 0 & 2 \ell_6 & 16 \ell_6 & 2 \ell_6 & \\
\end{bmatrix} \begin{bmatrix}
\mathbf{P}_x_1 \\
\mathbf{P}_x_2 \\
\mathbf{P}_x_3 \\
\mathbf{P}_x_4 \\
\mathbf{P}_x_5 \\
\mathbf{P}_x_6 \\
\mathbf{P}_x_n \\
\end{bmatrix} = \begin{bmatrix}
\mathbf{P}_u_1 \\
\mathbf{P}_u_2 \\
\mathbf{P}_u_3 \\
\mathbf{P}_u_4 \\
\mathbf{P}_u_5 \\
\mathbf{P}_u_6 \\
\mathbf{P}_u_n \\
\end{bmatrix} 
\tag{3.217}
\]

\[
\mathbf{C} \mathbf{P}_x_n = \mathbf{P}_u_n 
\tag{3.218}
\]

where \( \mathbf{P}_x_n \) is a vector of unknowns, \( \mathbf{C} \) is a banded matrix of constants and all elements of \( \mathbf{P}_u_n \) are also known. Since the bandwidth is only two whatever the number of nodes any band solution technique is suitable to solve the system of equations.

The vertical pressures can be obtained by the same method used for both sides of the wall.

By finding the nodal forces of all elements incident on the node which is an end of a linear element (anchor or strut), the final load in the anchor and, therefore, the corresponding variation in load, are automatically determined. The result can be readily checked by another method, using the formula
\[ \Delta P = \frac{EA}{\ell} \left[ (d_{xj} - d_{xi}) \cos \theta + (d_{yi} - d_{y1}) \sin \theta \right] \]  

(3.219.)

where \( \Delta P \) is the variation of load

\( \theta \) is the inclination of the anchor or strut

\( d_{xi}, d_{xj}, d_{yi}, d_{yj} \) are the horizontal and vertical

components of the displacements of the ends of the

anchor or strut with length \( \ell \).

3.11 Description of the Program

3.11.1 Variable Definitions

The following variables are read from the first input card and used throughout the Program:

**KA** Integer number defining the space to be allocated in disc to store temporarily some arrays and also used as a pointer to find the position of every element in disc.

**PROBLEM** Two-digit number defining type of analysis and material properties, e.g. 50 means plane strain analysis of a cross-anisotropic body.

**T** Digit number equal to the integer division of **PROBLEM** by ten.

**ENP** Number of nodes of non-linear elements.

**NDF** Number of degrees of freedom per node.

**NLD** Number of load cases.

**NE** Number of non-linear elements.

**NL** Number of linear elements.

**NA** Number of anchors and struts.

**NP** Number of nodal points.
G

Number of system equations, equal to the number of columns of the stiffness matrix of the structure.

M

As input datum bandwidth of the stiffness matrix of the structure defined as the maximum nodal numbers difference for any one element of the structure (the true bandwidth is equal to M + NDF).

NB

Number of boundary conditions of the type \( d_i = 0 \) to be explicitly enforced.

DUMNB

Number of dummy nodal points, i.e. fixed nodes for which the boundary conditions are implicitly enforced by the node numbering system.

NEDISP

Number of pairs of nodal displacements \( d_i \) and \( d_j \) for which the condition \( d_i - d_j = \text{constant} \) has to be enforced (constant may be any prescribed relative movement but is usually taken as zero).

NMAT

Number of different element material types.

NCOORD

Number of nodal points that are vertices of elements.

NLUMP

Number of nodal force components given directly as input data.

NH

Number of elements under surface loading due to distributed horizontal pressure, e.g. due to excavation.

NV

Number of elements under surface loading due to distributed vertical pressure.
N1  Number of nodes on the left hand vertical surface of the diaphragm wall not in contact with soil (due to excavation).

N2  Number of nodes on the left hand vertical surface of the diaphragm wall in contact with soil.

N3  Number of nodes on the right hand vertical surface of the diaphragm wall (usually $N_1 + N_2 = N_3$ but it depends on the way the wall itself is divided into elements).

The following arrays are used at various stages of the Program to store temporarily or permanently the data indicated:

COORD  2NPxl real array (vector) of the nodal points coordinates.

CODEN  (NE+NL)xENP integer array of code numbers (element connections) in which dummy nodal points are given code numbers zero.

CODEN 2  (NE+NL)xENP integer array of code numbers differing from CODEN in that dummy nodal points are given non zero code numbers.

MATN  NExl integer array (vector) of element material types.

NEP  NMATxT real array of elastic parameters for each different material type, excluding the vertical Young's modulus.
E  \( \times 3 \) real array of vertical Young's moduli at vertices of all non-linear elements (assumed triangular).

EA  \((\text{NA+1})\times 1\) real array (vector) of Young's moduli of all anchors and struts.

ANCHOR  \((\text{NA+1})\times 6\) real array of geometric characteristics of all anchors and struts (reference number of the anchor or strut, end node numbers, cross-section, free length and inclination).

ANCHORL  \((\text{NA+1})\times \text{NLD}\) real array of axial loads of anchors and struts (for struts, the axial load is taken as negative).

TEMP  \((\text{NE+NL})\times 6\) real array of variations in temperature at the nodes of all elements.

KO  \(2\times \text{NLD}\) real array of coefficients of earth pressure at rest at all nodes of the structure.

PWP  \(2\times 1\) real array (vector) of pore water pressure at all nodes.

GAMA  \(\times 1\) real array (vector) of unit weight for all triangular elements.

B  \((\text{NDFxNP})\times \text{NLD}\) real array (load vector) of nodal components of the load system.

Bl  \((\text{NDFxNP})\times \text{NLD}\) auxiliary real array (load vector).

X  \((\text{NDFxNP})\times \text{NLD}\) real array (displacement vector) of the nodal displacements of the structure.

XX  \((\text{NDFxNP})\times \text{NLD}\) auxiliary real array to store temporarily \(X_0\) called from magnetic tape (\(X_0\) is, for instance, the solution corresponding to the previous stage of excavation).
D: 4x4 real array which is the elastic rigidity matrix, assuming vertical Young's modulus equal to unity.

KE: 12x12 real array which is the element stiffness matrix of a linear strain triangular element.

K: (M+NDF)x(M+NDF) real array storing a square portion of the banded stiffness matrix of the structure.

NFIx: (NB+1)x1 integer array (vector) of the explicitly prescribed boundary conditions (type \( d_i = 0 \)).

DUM: (DUMNB+1)x1 integer array (vector) of dummy nodal points.

EQDISP: (NEDISP+1)x3 real array storing \( i, j \) and constant, for all pairs of nodes for which the condition \( d_i - d_j = \text{constant} \) has to be enforced.

CENTEL: NEx2 real array of coordinates of the centroids of all triangular elements.

IST: NEx4 real array of initial stresses at the centroids of all triangular elements.

ST: NEx4 real array of final stresses at the centroids of all triangular elements.

PST: NEx4 real array of principal stresses and their directions at the centroids of all elements.

NOD1: (N1+1)x1 array (vector) of nodal numbers of nodes on the left hand side vertical surface of the diaphragm wall not in contact with soil (due to excavation).
NOD2  (N2+1)x1 integer array (vector) of nodal numbers of nodes on the left hand side vertical surface of the diaphragm wall in contact with soil.

NOD3  (N3+1)x1 integer array (vector) of nodal numbers of nodes on the right hand side vertical surface of the diaphragm wall.

AI    (N1+N2+N3+1)x3 auxiliary real array.

PI    (N1+N2+N3+1)xNLD real array of horizontal or vertical final pressures on the lateral surfaces of the diaphragm wall.

FI    (N1+N2+N3+3)xNLD real array of components of final nodal forces on all exterior nodes of the diaphragm wall.

The previously defined variables which are not positive by definition can be given any non-negative integer value, including zero. Obviously, the following relationships are valid:

\[ \text{NA} \leq \text{NL} \]
\[ G+NDF\times DUMNB= NDF \times NP \]  \hspace{1cm} (3.220)
\[ KA \geq (NP \times NDF) \times (M+NDF) + 3NE \times (2+4NLD) \]

The first relationship shows that, in order to reduce the bandwidth, each anchor or strut may be divided into various linear elements. The last relationship results from the fact that the stiffness matrix of the structure, array CENTEL and array IST have to be put temporarily in disc backing store.
For practical reasons, the author uses two separate Programs: one for plane stress and plane strain analysis, the other for axi-symmetric analysis. This avoids the use of many unnecessary cards, but it would be quite easy to build a single Program. The flow-charts presented in Figs. 3.14 and 3.15 refer to the program for plane stress and plane strain analysis. The differences will be noticed when describing the various routines and their use, following the flow-charts mentioned above.

3.11.2 List of Routines and Their Purposes

3.11.2.1 Routine SETMAT

After making all elements of MATN equal to -999 (to provide an easy check in case it is necessary), this routine simply reads input data to set up arrays MATN and NEP.

3.11.2.2 Routine SETNFIX

This routine simply reads the input data necessary to set up arrays DUM and NFIX which store information related to boundary conditions of the type \( d_1 = 0 \). The first check of input data is provided by the insertion of the number -999. If this number is not read, i.e. if the amount of input data is not correct, an error message is printed out and the program is stopped.

3.11.2.3 Routine SETCODENUMB

This routine reads input data to set up arrays CODEN and CODEN2 simultaneously. As input data, the code numbers are given according to the node numbering system in Fig. 3.12b but are stored according to Fig. 12a.
For the elements possessing dummy nodal points, other node numbers (all different from zero) are given and the relevant elements of array CODEN2 are modified accordingly.

Finally, the routine finds the bandwidth of every element and uses the maximum value BW of these bandwidths to check the input parameter M and modify it by setting $M \leftarrow M + NDF - 1$.

3.11.2.4 Routine SETCOORD

This routine reads the coordinates of all corner nodes, after making all elements of array COORD equal to -999, then uses array CODEN2 to evaluate the coordinates of all midside nodes by averaging the coordinates of adjacent corner nodes, thus setting up array COORD.

The total area of the structure is evaluated by adding up the areas of all triangular elements and comparing with the correct value given as input datum. If both results do not coincide within a margin of absolute error equal to 0.01, an error message is printed out and the program stopped.

3.11.2.5 Routine SETANCHOR

After making all elements of the load vector $B$ equal to zero, this routine is called only when $NA=0$. It stores in array $EA$ the Young's moduli of the anchors and struts and, given the cross-sections and end node numbers, evaluates the free-lengths and inclinations, storing all geometric information in array $ANCHOR$. It also stores in array
ANCHORL the axial loads of all anchors and struts, adding the contributions of these axial loads to the relevant elements of load vector B. The distinction between anchors and struts is made by the sign of the axial loads given as input data: positive for anchors, negative for struts.

The end of an anchor or strut not in contact with the diaphragm wall may be assumed to transfer the load to the soil through any number of specified nodes.

3.11.2.6 Routine SETYOUNGMO

After making all elements of array E equal to -999.0 it reads the input data necessary to evaluate the vertical Young's moduli at every corner node of the structure. Thus it sets up array E in which every row contains the vertical Young's moduli at the corner nodes of the corresponding triangular element.

The variation of the vertical Young's modulus with depth is given by one or more straight lines, each one defined by the coordinates of any two points as shown in Fig. 3.16.

Finally, if any Young's modulus is found to be negative an error message is printed out and the program stopped.

3.11.2.7 Routine GAPKO

After reading an auxiliary integer number depending on the particular problem, this routine is called only if that number is not zero. This routine sets up arrays GAMA, KO and PWP storing the unit weight of each triangular element, the coefficient of earth pressure at rest and the pore water pressure at every node.
The variation of $K_o$ and pore water pressure with depth is given by straight lines, each one defined by the coordinates of any two points as shown in Fig. 3.16.

3.11.2.8 Routine SETINISTRESS

This routine makes all elements of array IST equal to zero or evaluates them, if routine GAPKO has been called, using equations (3.153) referred to the centroid of each element.

3.11.2.9 Routine SETCENTEL

This routine uses the information stored in arrays COORD and CODEN2 to set up array CENTEL of coordinates of centroids of all triangular elements.

3.11.2.10 Routine SETLOAD

Any concentrated forces given as input data are read. Then this routine is called if $NH$ or $NV$ are different from zero. If the distributed surface loading over one side of an element is due only to excavation, i.e. depends on the unit weight $\gamma$, coefficient of earth pressure at rest $K_o$ and pore water pressure $u$, then the identification of the element by its number is enough for the routine to evaluate the contribution of the element to the load vector $B$. The distributed surface loading for the first stage of excavation is the initial stress given by (3.153). For the following stages of excavation, the consistent nodal forces corresponding to the initial stresses are substituted by the final nodal forces corresponding to the previous stages of excavation determined, for the layers to be excavated next, by the method outlined in 3.10.
3.11.2.11 **Routine BODYFORCE**

This routine is called only if an auxiliary integer number given as input datum is different from zero. In this case it uses (3.137) to add to the relevant elements of the load vector \( B \) the contributions of all elements resultant from a distributed body force which is the unit weight \( \gamma \).

3.11.2.12 **Routines SETMATD and SETMATD1**

When this routine is called for a particular type of element, it uses the information stored in array \( \text{NEP} \) and the parameter \( \text{PROBLEM} \) to set up array \( D \) which is the elastic rigidity matrix. Depending on the type of plane analysis and on the type of material it uses expressions (3.18), (3.20), (3.23), (3.33), (3.36) or (3.38).

Considering the case of a non-horizontal plane of isotropy it also sets up the transformation matrix \( \mathbf{T}_e \) and finds \( \mathbf{D}^* \) by using (3.46).

In axi-symmetric analysis the corresponding routine \( \text{SETMATD1} \) sets up array \( D \) by using (3.60) or (3.62) whichever is the case identified by the parameter \( \text{PROBLEM} \).

3.11.2.13 **Routine TEMPERATURE**

This routine is called only if an auxiliary integer number given as input datum is different from zero. In this case, after making equal to zero all elements of array \( \text{TEMP} \), a consistent system of nodal forces corresponding to
the initial strains due to variations in temperature is evaluated and added to the load vector $B$. The method outlined in 3.3.3 is used, assuming that the temperatures may vary quadratically over each element.

The temperature increment is defined by the general expression

$$T = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2$$

(3.221)

where the coefficients $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $a_5$ are given as input data for each set of elements identified by their numbers. When the last set has 0 elements, all temperature increments at the relevant nodal points have been evaluated by using (3.221) and stored in array TEMP. Then routines SETMATD and TEMPERATURE are used in this order, for all elements of the structure to evaluate the contributions of each element to the load vector $B$.

The load vector $B$ is printed out (previous input data are usually printed out as well) and an auxiliary number is read. If it is not equal to -999 then an error message is printed out and the Program stopped.

At this stage, arrays KO, PWP, GAMA and TEMP are deleted since they are not necessary any more. To save store during the process of assembly and solution of the system of equations, arrays CENTEL and IST are temporarily removed from core to disc backing store.
3.11.2.14 **Routines ELSTIFFMATT and ELSTIFMAT2**

The call of routine SETMATD precedes always the call of routine ELSTIFFMATT which uses information contained in arrays D, E, COORD, CODEN2 and NEP to evaluate the element stiffness matrix KE. After defining a set of variables depending on the cartesian coordinates of the corner nodes and on the material properties of the linear strain triangular element, each element of KE is given by a simple expression, function of those variables, such as (3.119) and (3.120) for KE_{1,4} in plane stress and plane strain analysis.

The same applies to routine ELSTIFMAT2 suitable for axi-symmetric analysis in which expressions of the type shown in (3.181) are used for all elements on and above the principal diagonal of KE.

3.11.2.15 **Routine ELSTIFMAT2**

This routine uses information in arrays CODEN, COORD, EA and ANCHOR to set up array KE storing the element stiffness matrix of a linear element, by using expression (3.196).

3.11.2.16 **Routine PLANTKLOWT**

This routine uses arrays CODEN and KE to add to the stiffness matrix of the structure the contributions of the element whose stiffness matrix is stored in KE. In plane stress and plane strain analysis proper advantage is taken of the fact that usually a number of elements have the same
stiffness matrix KE which is evaluated only once before assembling all elements of that same type, i.e. before adding to the stiffness matrix of the structure the contributions of each one of these elements.

In axi-symmetric analysis, due to the appearance of variable x in expression (3.164), each element has a different element stiffness matrix KE, unless the vertical Young's modulus is constant. In this case, the routines SETMATDL, ELSTIFMATT1 and PLANTKLOWT are called in this order, for every element of the structure.

If two triangular elements have their sides mutually parallel they may have the same element stiffness matrix KE, even if they have different sizes. Fig. 3.17 shows some typical cases. Elements 1 and 2 have the same KE if the Young’s modulus is constant and equal for both of them. Elements 1 and 3 have the same KE whatever the variation of the vertical Young’s modulus with depth. Elements 1 and 4 have the same KE only when they have the same constant vertical Young’s modulus and this is the only case in which two elements have the same element stiffness matrix in axi-symmetric analysis.

As an example, it is easy to show that KE*1,4 for triangular element 2 in Fig. 3.17 is equal to KE1,4 for triangular element 1. Using (3.77) one can write

\[ A^* = C^2 A \] and from (3.120) one finds \[ Q^*_{11} = C^2 Q_{11} \] and \[ Q^*_{12} = C^2 Q_{12}. \] Substituting these expressions in (3.119) one arrives at
\[ KE_{1,4}^* = (3E_1-2E_2-E_3)C^2Q_{11} + (14E_1+3E_2+3E_3)C^2Q_{12} / (60C^2A) \]

Therefore \( KE_{1,4}^* = KE_{1,4} \) if \( E_1 = E_2 = E_3 = E \).

Since the stiffness matrix of the structure is symmetric only the elements on and above the principal diagonal have to be stored. Fig. 3.18 shows how the stiffness matrix can be stored as a symmetric banded array \( K^* \) or as a rectangular array \( K^{**} \) in the usual way. For computing reasons related to the process of using disc backing store in the computer ICL 1905F of the University of Surrey, the author uses array \( K \) shown in Fig. 3.18.

Considering the actual node numbers of an element \( n \) as shown in (2.43) and using a process derived directly from (2.45), the routine PLANTKLOWT uses formula (b) of Fig. 3.18 to insert into \( K \) the contributions from an element which is stored in the element stiffness matrix \( KE \). However, the actual process is slightly complicated by the fact that only a portion of \( K \) is stored in core, the rest being kept in disc backing store.

Before assembling the element, the routine uses the information stored in CODEN to find out the size and position in disc of the portion of array \( K \) where the contributions of the element must be inserted. This portion has, at most, the dimensions \((M+1) \times (M+1)\) and has, in general, the dimensions \( \Delta_n(M+1) \), where \( \Delta_n \leq M+1 \).
Although, strictly speaking, only a portion with the dimensions $A_n \times A_n$ is necessary, depending on the bandwidth of the particular element to be assembled, it is more efficient to transfer the rectangular portion $A_n \times (M+1)$ than the square portion $A_n \times A_n$. Array $K$ is stored in disc in a single line in such a way that the elements $K_{0,1}$ to $K_{M,1}$ of the first column are followed by the elements $K_{0,2}$ to $K_{M,2}$ of the actual second column and so on. A single statement can only get from disc backing store a number of elements of $K$ which form a non-broken sequence, using variable $K_A$ to find the location of the first element. Now it is easy to understand why it would be much less efficient to use $K^{**}$ rather than $K$ (see Fig. 3.18) to transfer from disc to core (or vice-versa) the equivalent portion of the stiffness matrix of the structure.

If an element has one or more node numbers equal to zero, then routine PLANTKLOWT will simply not insert into $K$ the corresponding contributions stored in $KE$. The last operation performed by the routine is to transfer to the correct position in disc backing store the modified portion of $K$.

The same routine is used to assemble linear elements or indeed elements with any number of nodes $ENP$ and any number of degrees of freedom per node $NDF$. Obviously, the parameters $ENP$ and $NDF$ must be given the correct values in the call of the routine.
In some particular problems, a few elements can become non-existent (e.g. due to excavation) and some elements may have the same element stiffness matrix. Thus it is quite easy to make a mistake when preparing the input data for the specific purpose of assembly. For this reason, counter \( N \) is used as shown in the flow-chart of Fig. 3.14, to make sure that the correct number of triangular and linear elements have been assembled by routine PLANTKLOWT. Otherwise an error message is printed out and the Program stopped.

3.11.2.17 Routine EQUADISPLLOWT

This routine is used only if \( \text{NEDISP}=0 \), performing the operations indicated in (3.206) for all conditions of the type \( d_i - d_j = 0 \) and the additional operations indicated in (3.212) for conditions of the type \( d_i - d_j = \text{constant} = 0 \).

Fig. 3.19 shows how rows and columns \( i,j \) of array \( K^* \) are stored in \( K \) and the equivalence of row-column interchanges. The number of operations in \( K \) is greatly reduced in comparison with \( K^* \). It must be noted that the operations performed by the routine involve elements outside the band of the stiffness matrix of the structure. Therefore, when the routine EQUADISPLLOWT has to be used the parameter \( M \) must be increased, as input parameter, by the maximum nodal difference \( i-j \) corresponding to conditions \( d_i - d_j = \text{constant} \) or \( d_i - d_j = 0 \) to be imposed.

Obviously, it is necessary to transfer from disc backing store to core the relevant portion of \( K \), for each condition to be enforced, putting it back in the correct position after performing the required operations.
3.11.2.18 **Routine BOUNDCOND**

This routine is used only if NB=0 and enforces every boundary condition of the type \( d_i = 0 \) by replacing, as explained in 2.3.10, the off-diagonal elements of the \( i \)-th row and \( i \)-th column of \( K^* \) (see Fig. 3.19) and also the \( i \)-th row of the load vector \( B \) by zero. The diagonal element \( K^*_{ii} \) remains unchanged.

These operations are actually performed on a portion of array \( K \) rather than array \( K^* \). Fig. 3.19 shows how row \( i \) and column \( i \) of \( K^* \) are stored in \( K \).

3.11.2.19 **Routine EQUASOLVED**

This routine uses the well known Cholesky decomposition to solve the system of equations represented by (2.56), keeping in core only a variable portion of array \( K \) (see Fig. 3.19) with the dimensions \((M+1) \times (M+1)\). The first operation to be performed on any column \( i \) transferred from disc backing store to core is to make sure that the element \( K_{M,j} \) is different from zero. If it is zero it will be given a value equal to the nearest non-zero element \( K_{M,j} \). This enables the routine to solve the system of equations even when some elements are missing (e.g. due to excavation), i.e. when they have Young's moduli equal to zero.

The Cholesky decomposition is applicable only to symmetric and generally positive definite coefficient matrices. It consists in decomposing a matrix \( A \) into a product of a triangular matrix \( L \) by its transpose \( L^T \). In expanded form, one has
This decomposition, for a band matrix $A$, is based on the following theorem (Martin and Wilkinson, 1965): if $A$ is a positive definite matrix of band form such that

$$a_{i,j} = 0 \quad \text{for} \quad (|i-j| > M) \quad (3.223)$$

then there exists a real non-singular lower triangular matrix $L$ such that

$$LL^T = A \quad \text{where} \quad l_{i,j} = 0 \quad \text{for} \quad (i-j > M) \quad (3.224)$$

Adopting the convention that $l_{p,q}$ is to be regarded as zero if $q < 0$ or $q > p$, the algorithm for the decomposition can be represented by

$$l_{i,j} = a_{i,j} - \sum_{k=i-M}^{i-1} l_{i,k} l_{j,k} \quad (j=i-M, \ldots, i-1) \quad (3.225)$$
\[ \ell_{i,i} = \sqrt{a_{i,i} - \sum_{k=i-M}^{i-1} \ell_{i,k}^2} \]

where
\[ a_{i,j} = \sum_{k=i-M}^{i} \ell_{i,k} \ell_{j,k} \]
\[ a_{i,i} = \sum_{k=i-M}^{i} \ell_{i,k}^2 \] (3.226)

Incidentally, note that the determinant of \( A \) can be easily evaluated as follows:
\[ \det (A) = \det (L L^T) = \prod_{i=1}^{n} \ell_{i,i}^2 \] (3.227)

Once the coefficient matrix \( A \) has been decomposed as shown above, the complete solution process can be broken down into the following two steps:
(a) forward elimination by solving \( Ly = F \);
(b) backsubstitution by solving \( L^T x = y \).

The algorithm for these operations is represented by
\[ y_i = (B_i - \sum_{k=i-M}^{i-1} \ell_{i,k} y_k) / \ell_{i,i} \quad (i=1, \ldots, n) \] (3.228)
\[ x_i = (y_i - \sum_{k=i+1}^{i+M} \ell_{k,i} x_k) / \ell_{i,i} \quad (i=1, \ldots, n) \]

Approximately \( 2n(M+1) \) multiplications are involved in a solution and any number of right-hand sides (load cases) can be processed when \( L \) is known. The method outlined above can only be used when \( A \) is a positive definite symmetric band matrix and preferably only when \( M \ll n \). As \( M \) approaches \( n \) it becomes less efficient to take advantage of the band form.
All the operations are actually performed on array \( K \) shown in Fig. 3.18, keeping in core, throughout the process, only a portion with the dimensions \((M+1) \times (M+1)\). Note that \( n=G \).

The process of transferring columns of array \( K \) from core to disc backing store and vice-versa is rather slow. In order to reduce the number of transferences, the author set up the routine in such a way that a column \( j \) which is not needed is replaced, in the same position, by the next required column \( j+M+2 \). Fig. 3.20 shows how this procedure reduces the number of operations, making it unnecessary to shift to the left all columns from \( j+1 \) to \( j+M+1 \) when column \( j \) is not needed any more and column \( j+M+2 \) has to be brought into core, as in the corresponding case of \( K' \).

Routine EQUASOLVED will not solve the system of equations when the stiffness matrix of the structure appears as non positive definite due to instability (e.g. incorrect boundary conditions which do not prevent rigid body movement). The same difficulty can arise when, due to round-off errors, it is not possible to find \( \lambda_{1,1} \) (for a particular value of \( i \)) as a real number. This results from ill-conditioning (e.g. triangular elements with one dimension very different from the others, elements with very different Young's moduli, etc.)

When this situation arises, a message is printed out and the program stopped.
3.11.2.20 Routine EQUADISP

This routine is called only if NEDISP=0, i.e. if routine EQUADIPLOWT has been used. It simply transforms the displacements of the pairs of nodes whose nodal numbers are stored in array EQDISP, according to expressions (3.209) where C may be zero.

3.11.2.21 Routine ANCHORSTRESS

This routine is called only if NL#O, i.e. if there are anchors or struts, and gives their variations in load when they are assumed to be connected to the rest of the structure (soil and diaphragm wall) through only two end nodes.

Fig. 3.21 shows the displacements of the end nodes of a linear element and it is obvious that its length has changed, if the deformations are small, by an amount \( \Delta l \) given by

\[ \Delta l = (d_{jx} - d_{ix}) \cos \theta + (d_{iy} - d_{iy}) \sin \theta \]  

(3.229)

Therefore the variation in load is given by

\[ \Delta P = \frac{EA}{\ell} \Delta l \]  

(3.230)

where \( E \) is the Young's modulus stored in array EA

\( A \) is the cross-section stored in array ANCHOR

which also stores the length \( \ell \) and the inclination \( \theta \)

3.11.2.22 Routines CENTELSTRESS and CENTELSTRESS1

Routine CENTELSTRESS is used for plane strain and plane stress analysis, after calling routine SETMATD, to evaluate
the horizontal, vertical and shear stresses at the centroid of a triangular element, employing equation (3.154). All necessary information is contained in arrays CODEN2, E, D, COORD and X.

Similarly, in axi-symmetric analysis, routine CENTELSTRESS1 evaluates the stresses at the centroid of an element, after calling routine SETMATD1, using equation (3.192).

Before using these routines it may be desired to add to the displacements contained in array X other displacements previously obtained and stored in magnetic tape, e.g. those corresponding to previous stages of excavation. In this case an auxiliary array XX is used to store those displacements before adding them up to those in array X.

3.11.2.23 Routines CEPRINSTRESS and CEPRINSTRESS1

Routines PRINSTRESS and PRINSTRESS1 are used for plane analysis and axi-symmetric analysis, respectively. They find the principal stresses and their directions at the centroids of all non linear elements of the structure, using equations (3.155) and the information stored in array ST.

3.11.2.24 Routine NODEFORCE

This routine is always preceded by routines SETMATD and ELSTIFMATT (or SETMATD1 and ELSTIFMATT1, in axi-symmetric analysis) and evaluates for an element identified by its number, the consistent nodal forces after deformation.
using equation (3.213) and the information contained in arrays KE, X and CODEN2. An auxiliary matrix Bl is set to zero and then all element nodal forces are added to the corresponding elements of Bl. This can be done for three different sets of elements, each set with any number of adjacent or non-adjacent elements. In a typical case, those sets could be:

1. All elements of soil in contact with the diaphragm wall, anchors or struts (incidentally this provides another method to find the variations of loads in anchors or struts, even when the free length is different from the total length);

2. All elements of alternate layers to be excavated later. Note that, if all layers were considered simultaneously, the nodal forces at interior nodes would appear as zero in matrix Bl;

3. All elements of the diaphragm wall.

Matrix Bl is printed out for each one of these three sets.

3.11.2.25 Routine FORCES

When this routine is called, matrix Bl contains the consistent nodal forces acting on the diaphragm wall, all other elements being zero.

Routine FORCES uses arrays NOD1, NOD2, NOD3 and Bl, to check the equilibrium of all final forces acting on the diaphragm wall and to determine the magnitude and inclination of the resultant of all forces acting on the right hand side (opposite to excavation). This provides a measure of the wall friction considered in the solution.
3.11.2.26 Routine MOMENTS

This routine uses arrays NOD1, NOD2, NOD3, COORD and Bl, to evaluate, for every nodal point on the neutral axis of the diaphragm wall, the bending moment, shear force and axial force. Actually, it finds two pairs of values for each one of those nodes, therefore taking into account the possibility of existence of nodes at the same level on the vertical surfaces of the wall where there may be vertical components causing sharp variations of bending moments, shear forces and axial forces.

3.11.2.27 Routine PRESSURE

The nodal points in contact with soil, on both sides, are identified in arrays NOD2 and NOD3. Routine PRESSURE sets up, for each side, the rectangular array AI storing the coefficient matrix of equation (3.217) in a manner similar to K** in Fig. 3.18. Thus AI has only three columns.

3.11.2.28 Routine EQUASOLVE

This routine solves the equations (3.218) to determine the final pressures along the sides of the diaphragm wall. It is used four times: both sides of the wall, horizontal and vertical pressures on each side.

It differs from routine EQUASOLVED in that:
1. The matrix of coefficients is stored in a manner similar to K** in Fig. 3.18;
2. No disc backing store is used.
3.11.2.29 **Routine PRINTPI**

This routine simply prints out the horizontal and vertical pressures determined by EQUASOLVE.

3.11.3 **Graphical Display of Results**

Most of the input and output data are printed out although that was not always mentioned explicitly in the previous section.

The size of the problems solved prevented the author from using the graph-plotter through the main Program but the voluminous output produced by the analysis and the time needed to interpret it, justified the preparation of various small programs suitable to facilitate the interpretation of results by displaying them graphically.

Some of them have been used to draw a number of figures appearing in later chapters but only two will be mentioned here.

Arrays COORD, X, CENTEL and PST are stored in magnetic tape. A small Program is then used to draw the scaled principal stress vectors in appropriate directions for each element and the scaled displacement vector for each node. This can be done for the whole structure or for a particular zone where more detail is required. Only two cards with input data are needed containing the necessary parameters.

Another small program can draw up to twelve figures, each one with any number of curves or straight lines having the same or different origins, whatever the type of x and y variables.
3.12 Summary and Conclusions

A computer program has been developed based on the finite element displacement method, according to the formulation presented in chapter 2.

Although the mesh is not generated automatically, the need for input data is reduced to a convenient minimum by leaving to the computer the task of producing most of the data necessary.

The linear strain triangular element with three corner nodes and three midside nodes is used and a quadratic function (complete second-degree polynomial) is assumed for the expansions of the displacements, distributed surface and body forces, and temperature variations over the element.

The element matrices are evaluated by exact integration, using homogeneous coordinates, even when the Young's moduli and shear moduli vary linearly over the element.

The boundary conditions corresponding to fixed points can be enforced implicitly by the node numbering system. Boundary conditions of the type $d_i = 0$ or $d_i - d_j = \text{constant}$ are enforced explicitly. The possibility of enforcing conditions of the type $d_i - d_j = \text{constant}$ is useful for different purposes such as simulation of a crack in a diaphragm wall or no friction along the surface of contact between soil and structure.

The system of equations is solved by a band solution technique based on the well known Cholesky decomposition $(A = LL^T)$ of a band matrix $A$ of constants into a product of a lower triangular matrix $L$ by its transpose.
The program is specially suitable for the analysis of deformations due to excavation around long diaphragm walls with any number of anchors and struts in which case it evaluates the final (after deformation) bending moments, shear forces, axial forces, horizontal and vertical pressures along the (vertical) wall and the variations of loads in anchors and struts.

However, it can also solve, in general, plane stress, plane strain and axi-symmetric problems, in which the material is orthotropic, cross-anisotropic or isotropic, and the Young's moduli and shear moduli are constant or vary linearly over the element in such a way that $E_x/E_y$ and $G_{xy}/E_y$ are kept constant, assuming linear strain-displacement relations and linear stress-strain relations.

It was written in ALGOL and has been used in an ICL 1905F computer. Due to the use of magnetic tape and disc backing store, the capacity is not primarily dependent on the number of nodes or elements. It depends mainly on the bandwidth of the structure corresponding to the particular node numbering system used; an algorithm is presented in chapter 4 to find the best node numbering system.

Here are two typical examples of problems solved by the program using less than 81K, i.e. 81000 words:
1. Structure with a diaphragm wall, 342 triangular elements, 44 linear elements, 751 nodes, bandwidth equal to 128, one load case;

2. Structure with a diaphragm wall, 262 triangular elements, 14 linear elements, 610 nodes, bandwidth equal to 114, three load cases.
Fig. 3.1 Cylindrical Solids Considered in Plane Stress and Plane Strain Analysis
Fig. 3.2 Change of Reference Frames

Fig. 3.3 Stresses and Strains Involved in the Analysis of Axi-Symmetric Solids
Fig. 3.4 Global Cartesian Coordinates and Local Homogeneous Coordinates of a Point P

Fig. 3.5 Element of Length and Element of Area in Terms of Homogeneous Coordinates
Fig. 3.6 Recurrence Generation of Interpolation Functions for Triangular Elements

Fig. 3.7 Surface Loading on Side 12 of a Triangular Element
Fig. 3.8 Ring-Type Triangular Element

Fig. 3.9 Equivalence of a Constant Pressure $p$ Over a Circular Area to a Linearly Increasing Distributed Load over the Radius
Fig. 3.10 Linearly Increasing Distributed Loading over Side 12 of a Ring-Type Triangular Element

Fig. 3.11 Notation for End Loads and Displacements of a Linear Element
(a) general

(b) for pressures on wall

Fig. 3.12 Node Numbering Systems of a Triangular Element

Fig. 3.13 Consistent Nodal Forces (Due to Displacements) on Left Hand Side of a Diaphragm Wall
Fig. 3.14 Flow-Chart of the Program for Plane Analysis up to Evaluation of Nodal Displacements
Fig. 3.15 Flow-Chart of the Program for Plane Analysis after Evaluation of Nodal Displacements
Fig. 3.16 Variation of Vertical Young's Modulus $E_y$, Coefficient of Earth Pressure at Rest $K_o$ and Pore Water Pressure with Depth

Fig. 3.17 Different Triangular Elements Possessing the Same Element Stiffness Matrix
Transformation formulas:

(a) \[ H_{i,j} = H_{i-j+1,j} \]

(b) \[ H_{i,j} = H_{i,j+1} \]

\[
\begin{bmatrix}
H_{1,1} & H_{1,2} & H_{1,3} & H_{1,4} & H_{1,5} \\
H_{2,1} & H_{2,2} & H_{2,3} & H_{2,4} & H_{2,5} \\
& \ddots & \ddots & \ddots & \ddots \\
& & & H_{n-1,n} & H_{n,n} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
H_{1,1} & H_{1,2} & H_{1,3} & H_{1,4} & H_{1,5} \\
H_{2,1} & H_{2,2} & H_{2,3} & H_{2,4} & H_{2,5} \\
& \ddots & \ddots & \ddots & \ddots \\
& & & H_{n-1,n} & H_{n,n} \\
\end{bmatrix}
\]

Fig. 3.18 Different Arrays Storing the Same Stiffness Matrix of the Structure
Fig. 3.19 Row-Column Interchanges in Two Different Arrays Storing the Same Stiffness Matrix of the Structure
Fig. 3.20 Different Square Arrays Storing the Same Portion of the Stiffness Matrix of the Structure
Fig. 3.21 Positions of a Linear Element before and after Deformation of the Structure
Chapter 4

NODE NUMBERING OPTIMIZATION IN STRUCTURAL ANALYSIS

4.1 Introduction

4.1.1 Reasons for the Investigation

The ICL 1905F Computer of the University of Surrey is rather slow and has core available to the users of about 80K only. Therefore disc backing store had to be used in the analysis, by the finite element method, of the problems mentioned in the following chapters.

Having decided to use a "band solver" based on the Cholesky decomposition to solve the system of simultaneous linear equations, even small reductions in the bandwidth of the stiffness matrix of the structure (depending on the node numbering system) would lead to substantial savings not only in storage needs but also in computing times.

When surveying the existing methods to reduce the bandwidth of the stiffness matrix or to determine the best node numbering system, not only the author could not find published programs but also realized that he might be able to devise a better algorithm for this purpose. In addition, this somewhat side investigation would be in line with the renewed attention paid to this matter, in view of the increased usage of mini-computers and the general desire of ability of using small memories for large problems.
The author was able to program a new algorithm which can be used in small computers to find the best node numbering system for quite large problems. That explains why this chapter is almost the exact copy of a paper accepted for publication in the "Journal of the Structural Division" of the American Society of Civil Engineers, with the title "Node Numbering Optimization in Structural Analysis".

It must be emphasized that the experience of the engineer is sometimes the main factor to reduce the size of the problem from the viewpoint of using the finite element method. In some cases, the size of the system of equations needed to analyse a particular structure does not depend on the initial node numbering system because the algorithm will find the best one, anyway. In other cases, the convenient introduction of additional elements or auxiliary nodes, surprising as it may appear, will reduce much more the size of the system of equations than any other means. This may happen, for example, in the analysis of the soil movement due to excavation around a diaphragm wall with long anchors.

4.1.2 Solution of Systems of Linear Equations

Most problems in structural analysis, particularly using the finite element method, will require, at some stage, the solution of a set of linear simultaneous equations. In the execution of most structural analysis programs, the solution of equations constitutes a major factor, especially in non-linear analysis in which up to 80% of the total execution time may be spent in this way.
There is no "best equation solver" in existence, because a specific class of problems is generally characterized by a specific set of equations with its own specific optimum solution technique (Meyer, 1973). Thus there is a variety of "optimum equation solvers".

All solution methods may be grouped in two main classes: iterative methods and direct solution methods.

Among the direct solution methods, the standard Gauss algorithm is well known and most other methods are only modifications of this basic algorithm with the goal of reducing the number of calculations. Such reductions only result if advantage is properly taken of the symmetry and sparseness of the coefficient matrix of the system.

The Gauss algorithm uses the elimination technique, essentially involving two main steps: decomposition and backsubstitution. The set of equations can be written in matrix notation as

\[ \mathbf{A} \mathbf{x} = \mathbf{f} \quad (4.1) \]

where

- \( \mathbf{A} \) = the coefficient matrix;
- \( \mathbf{x} \) = the solution vector;
- \( \mathbf{f} \) = a vector of constants.

Incidentally, it is important to note that, recently, it has been shown (Klyuyev, 1965) that no method exists which requires less arithmetical operations than the Gauss elimination.

A slightly different and widely used solution technique, applicable only to symmetric and generally positive-definite
coefficient matrices, is associated with the name of Cholesky. Here the matrix $A$ is decomposed into a triangular matrix $L$ and its transpose $U = L^T$. In condensed form, one may write

$$A = LU$$  \hspace{1cm} (4.2)

where $L$ = a lower triangular matrix; 
$U$ = an upper triangular matrix.

Thus the complete solution process can be broken down into the following three steps:

1. Decomposition of coefficient matrix $A$ into lower and upper triangular matrices;
2. Forward elimination by solving $Ly = f$;
3. Backsubstitution by solving $Ux = y$.

One of the most frequently encountered sets of equations is the set of equilibrium equations of the finite element displacement method. Here the coefficient matrix $A$ is the symmetric and positive-definite stiffness matrix of the structure, the vector of unknowns $x$ stands for joint displacements and the vector of constants $f$ stands for joint forces (load vector).

A typical stiffness coefficient $a_{ij}$ is defined as the force corresponding to the $i$-th degree of freedom resulting from a unit displacement of the $j$-th degree of freedom while all other displacements are kept zero. Therefore, if the structure is stable, diagonal stiffness coefficients $a_{ii}$ are always positive. Off-diagonal elements $a_{ij}, i \neq j$ represent
the coupling effect between the i-th and the j-th degree of freedom such as the carry-over effect in a frame member. If $a_{i,j} = 0$ then no structural element exists to couple the degrees of freedom of $i$ and $j$.

4.1.3 Band Solution Techniques

Various methods have been developed to minimize the number of operations required to solve a system of linear equations. Some elimination techniques are designed to take maximum advantage of sparsity; band solutions, partitioning methods, frontal solutions, etc.

For each structural problem, there exists an optimum node numbering that minimizes the number of operations required. If nodes are numbered in such a way that the maximum nodal points difference within each element is kept small, the stiffness matrix of the structure "bands" along the diagonal.

Continuous beams, folded plate structures, building frames, bridge structures and any unbranched assembly of structural elements will produce a banded stiffness matrix. In structures with branches and generally in complex structures an unnatural numbering scheme is required if a reduced band is to be obtained.

Minimization of the bandwidth will generally result in a reduction of the number of required operations for elimination and in savings of storage demands by shifting the upper half-band into a rectangular form.
Although a minimum band does not always guarantee a minimum number of operations, the ease of assembly process and input-output handling make the band solution techniques very widely used and the most economical ones for most of structural applications.

For a coefficient matrix of a given size, the time required to solve the system of equations is directly proportional to the square of the bandwidth. Thus the reason for bandwidth reduction is obvious. For example, a reduction in bandwidth of 20% represents a reduction in solution time of 36%.

4.1.4 Existing Methods for Bandwidth Minimization

For most regular structures, the optimum numbering of nodal points is straightforward. For complex structures, however, the problem of minimizing the bandwidth by suitable node numbering becomes very difficult, even when that is done by inspection by an experienced engineer.

For this purpose, programs have been developed which automatically renumber the nodes in such a way that a minimum or at least reduced bandwidth will result.

In 1965, Alway and Martin presented a method for bandwidth reduction of a symmetric matrix. For a $N \times N$ symmetric matrix there are $N!$ ways in which the rows and columns can be arranged. Their strategy is based on a search through all possible permutations in such an order that only a limited enumeration will be considered.
Tewarson (1967) based his method on row and column permutations of a sparse matrix obtained from the initial one by replacing all non-zero elements by unity.

Rosen (1968) based his algorithm on interchanges of rows and columns (two at a time) that either reduce or leave the bandwidth unchanged.

Akyuz and Utku (1968) move rows closer to the centre of the matrix with the goal of reducing or leaving unchanged the area with non-zero elements within the band.

Cuthill and McKee (1969) reorder the rows of the connectivity matrix according to the increasing number of non-zero off-diagonal elements. The corresponding renumbering of nodes is equivalent to rearranging the rows and columns of the connectivity matrix.

The last available algorithm was present by Grooms in 1972. The basic idea is that one way to accomplish bandwidth reduction of the connectivity matrix is to move systematically closer together rows and columns that are far apart and coupled.

The writer proposes a new method using a rectangular integer array $A$ in which only non-zero off-diagonal elements of the upper band of the connectivity matrix $D$ are stored. The basic idea of the algorithm is, for any element $d_{i,j}$ causing the largest current bandwidth $BW$, to find a pair of rows and columns of $D$ that can be interchanged in order to make $d_{i,j} = 0$ without increasing the bandwidth. When it is not possible to do so for all non-zero elements causing
the same bandwidth BW, thereby reducing it, the rows and columns of $D$ are rearranged in an automatically defined way and then the first process is used again. Both the search and the operations equivalent to interchanging a pair of rows and columns of $D$ are actually carried out in array $A$.

In the next section it will be explained how to store the connectivity matrix in $A$ and how the required operations are performed by the simple routines upon which the algorithm is based.

4.2 Basic Definitions and Routines

4.2.1 Connectivity Matrix

In a finite element mesh there are nodes which are vertices of the finite elements but, in addition, there may be other nodes depending on the type of elements. Throughout this paper all of them will be referred to simply as nodes and their labels as node numbers. Similarly, the set of node numbers of any element will be referred to as code numbers of that element. In this sense, a linearly varying strain triangular element, for instance, with three vertices and three midside nodes will have six nodes, six node numbers and its connections with the rest of the structure will be represented by six code numbers.

Fig. 4.1 shows a plane pin-jointed structure and the corresponding connectivity matrix $D$ which defines its topology (Grooms, 1972). The X's represent non-zero terms while the blanks are zero terms.
All diagonal elements are non-zero and an off-diagonal non-zero element $d_{i,j}$ means that the nodes $i$ and $j$ are coupled.

The connectivity matrix is symmetric and usually bands along the diagonal. In this paper the bandwidth of the connectivity matrix is defined as the maximum nodal point difference for any one element of the corresponding structure, i.e. the maximum value of $j - i$ for any one element $d_{i,j}$ of the connectivity matrix. For the problem of Fig. 4.1, the bandwidth $5$ is caused by the element $d_{1,6}$.

The connectivity matrix is used for the nodes rather than the stiffness matrix for the degrees of freedom, thus reducing storage demands and computer time. For example, a problem with 1200 degrees of freedom (6 degrees of freedom per node) would be represented by a 200 x 200 logical or integer array instead of a 1200 x 1200 real array.

If only part of the upper triangle of the connectivity matrix stored in the symmetric array $D$ (see Fig. 4.2) is stored in a rectangular array $B$ then the interchanging of rows and columns will require less operations. $B$ has $NP$ rows and $M+1$ columns, $NP$ being the number of nodes of the structure and $M$ the original bandwidth. The diagonal elements of $D$ are stored in the column 0 of $B$ and the dotted line in Fig. 4.2 shows that this column could be deleted.

If only non-zero elements of the array $B$ are stored in a rectangular array $A$ then storage demands and computer time may be reduced even further. Array $A$ has $NP$ rows and $NC$ columns, $NC - 1$ being the maximum number of nodes coupled
with any one node. In Figs. 4.1 and 4.2, NP = 8 and NC = 5.

For every row of $A$, the value of a non-zero element is the column number of the corresponding non-zero element of $B$ in the same row. The position of any non-zero element in a row of $A$ is irrelevant: what counts is its value and the row number.

There is a simple relationship between the elements of $B$ and the corresponding elements of $D$ (considering only the upper triangle of $D$ where $j > i$):

$$d_{i,j} = b_{i,j-i}$$ (4.3)

The algorithm was programmed and tested using the connectivity matrix as stored in $B$, then it was modified in order to use the array $A$ rather than the array $B$. If this is always kept in mind, it will be easier to understand how the operations are performed on $A$ in order to reproduce what should be done on $B$ which is equivalent to $D$.

4.2.2 Description of Routines

The routine SETCODE sets up the NE x ENP rectangular array $C$ which stores the code numbers. The number of rows NE is equal to the number of elements of the structure and the number of columns ENP is equal to the maximum number of nodes in any one element: In Fig. 4.1, NE = 8 and ENP = 2.

In this manner, all elements will have the same number of code numbers even if it is greater than the actual
number of nodes. In fact, the structure may have elements of different types, for instance rectangular and linear elements. The former have, at least, three nodes while the latter only have two. In such a case, the additional code numbers for linear elements must be zero.

The routine FORMA uses the code numbers read into array $\mathcal{C}$ to set up the connectivity matrix stored in a rectangular array $\mathcal{A}$ with NP rows and NC columns (see Fig. 4.2).

The routine FINDNC uses array $\mathcal{A}$ to find for every row (and store in the vector $\mathcal{ve}$) the number of non-zero off-diagonal elements of the corresponding row of the connectivity matrix. NC will be set equal to $X + 1$ where $X$ is the greatest element of $\mathcal{ve}$. The choice of the minimum safe value for NC ensures that the number of operations subsequently performed by the algorithm is also reduced.

The information stored in the vector $\mathcal{ve}$ may also be used for an initial rearrangement of the rows and columns of array $\mathcal{A}$, i.e., a rearrangement of the initial node numbers.

The routine FINDBAND finds the element causing the bandwidth $\text{BW}$ of the connectivity matrix stored in the rectangular NP x NC array $\mathcal{A}$. Going through array $\mathcal{A}$ from row NP to row 1, it simply finds the element of row $i$ with the greatest value $j$ and then makes $\text{BW} = j$. This is the corresponding element $b_{i,j}$ of the equivalent array $\mathcal{B}$ (see Fig. 4.2).
The routine CHAROWCOL interchanges the elements \( v_i \) and \( v_j \) of vector \( v \) and performs on the rectangular array \( A \) the operations equivalent to interchanging the rows and columns \( i \) and \( j \) of the connectivity matrix stored as a symmetric array \( D \) (see Fig. 4.2). \( M + 1 \) is the width of the equivalent array \( B \) and \( y \) is the vector which stores the node numbers.

Fig. 4.2 shows how the operations for interchanging rows and columns 3 and 6 are performed for the different but equivalent arrays \( A, B, \) and \( D \). In array \( D \) it was necessary to interchange 16 pairs of elements; in array \( B \) only 7 pairs of elements needed to be interchanged; finally in array \( A \) only 3 pairs of elements had to be interchanged. This emphasises how the number of necessary operations is greatly reduced by using array \( A \) rather than array \( D \) or even \( B \).

The operations are performed according to the pattern defined by the arrows in Fig. 4.2 where \( p \leftrightarrow q \) means that the elements \( p \) and \( q \) must be interchanged. Elements with the same value need not be interchanged.

Note: As from now, for the sake of simplicity, the operations performed on the array \( A \) equivalent to interchanging the rows and columns \( i \) and \( j \) of array \( D \) will be simply referred to as interchange of rows \( i \) and \( j \). In this sense the routine CHAROWCOL interchanges one pair of rows of the array \( A \) which stores the connectivity matrix or one pair of rows of the corresponding array \( B \).
Let us consider the element $b_{i,j}$ of array $B$, say the non-zero element causing the maximum bandwidth. If one wishes to interchange a pair of rows such that $b_{i,j}$ will become zero, it can be easily seen in the example of Fig. 4.2 that there are (at most) only two possibilities, without increasing the bandwidth:

a) either rows $p$ and $i + j$ are interchanged ($i < p < i + j$) or
b) rows $i$ and $q$ are interchanged ($i < q < i + j$).

The former case is dealt with by FINDCOL and the latter is dealt with by FINDROW.

The routine FINDCOL finds (if possible) a row $p$ ($p < i + j$) that may be interchanged with the row $i + j$ satisfying all the following 4 conditions:

1. $b_{i,j}$ will become zero;
2. the bandwidth will not be increased;
3. the element $b_{i,j}$ will be moved closer to the first column as much as possible;
4. no zero element in the column $j$ will become non-zero.

This search is actually carried out on array $A$ but keeping in mind its equivalent array $B$ (Fig. 4.2). If $p = 0$ there is no row satisfying all the required conditions. In Fig. 4.2 it is $b_{i,j} = b_{1,5}$ and $p = 3$.

The routine FINDROW finds a row $q$ ($q > i$) that may be interchanged with row $i$, satisfying all the following 3 conditions:
1. $b_{i,j}$ will become zero;
2. the bandwidth will not be increased;
3. $b_{i,j}$ will be moved closer to the first column as much as possible.

This routine is used only when FINDCOL produces $p = 0$ and, similarly, if $q = 0$ there is no row $q$ satisfying all the required conditions. In Fig. 4.2 (left-hand side) for $b_{i,j} = b_{1,5}$ it would be $q = 3$.

The routine ARRANGE finds $q = i - 1$ if it is possible to reduce the bandwidth of the row $i - 1$ of the connectivity matrix by interchanging it with row $i$. $q = 0$ means that this is not possible. In Fig. 4.2 (left-hand side) for $i = 2$ it would be $q = 1$. This routine is used as a means of rearranging rows and columns of the connectivity matrix when FINDCOL and FINDROW produce successively $p = 0$ and $q = 0$.

Node numbers are stored in the vector $v$ in which a pair of elements are interchanged every time a pair of rows are interchanged in the connectivity matrix. Both operations are performed together by the routine CHAROWCOL.

The final integer values of the elements of $v$ printed versus the integers 1 to $NP$ show how the old node numbers must be changed in order to reduce the bandwidth. For the problem of Fig. 4.1, one will have (by using the algorithm):

New node numbers: 2 1 4 6 5 7 3 8
Old node numbers: 1 2 3 4 5 6 7 8

where $N1 \rightarrow N2$ means that the old node number $N1$ must be replaced by the new node number $N2$. 
Obviously the new node numbering system is not a unique solution. Even when all elements within the minimum band are non-zero there are two solutions, one being the reverse of the other.

After determining the new node numbering system, the main objective has been achieved. However, because it is a time consuming and tedious task to prepare the new code numbers, another routine (NEWCODEN) was programmed to produce the lists of new code numbers for all elements of the structure.

The old code numbers are transferred from array \( C \) to an auxiliary array \( E \). Then, for each element \( v_i \) of the vector \( v \) of new node numbers, one goes through all elements of array \( E \) and, whenever one of its elements is equal to \( v_i \), one makes the corresponding element of \( C \) equal to \( i \).

The new lists of code numbers stored in \( C \) may be printed out and also punched on a new set of cards ready to replace the old set.

The algorithm takes advantage, whenever possible, of the successive reductions in bandwidth to reduce the number of operations performed by each routine. Therefore, a final check is considered indispensable to ensure that the results are correct. The routine finds the new bandwidth of each element and stores its value in an extra column of array \( C \). The printed-out list of this array would provide an easy way of checking the results but an additional warning message is also printed out.
The initial zero code numbers are not modified by this routine and, because this is convenient, they are neither printed out.

4.3 Algorithm

The method described herein is a straightforward procedure making use of the previously described routines. A detailed flow-chart for the algorithm is shown in Fig. 4.3. For considerations of space, the flow-charts for the routines are not presented.

For the sake of simplicity, the operations performed on array A will be described as if they were performed on array B (see Fig. 4.2). The use of array A instead of array B is just a practical way of reducing storage needs, even though this is generally achieved at the expense of computing time. That depends mainly on the number of columns of array A, i.e. the value of NC which is independent of NP, and on the original bandwidth.

The algorithm consists of the following steps:

1.1 Read input parameters and code numbers, then use FORMA to set up and store the connectivity matrix in a NP x NC rectangular array A;

1.2 Call routine FINDNC to find the minimum safe value of NC to ensure that all the following operations will be properly performed;
2.1 For every non-zero element of the column \( j \) causing the largest bandwidth, call FINDCOL and, if \( p \neq 0 \), use CHAROWCOL to interchange rows \( p \) and \( i + j \); if \( p = 0 \) call FINDROW and, if \( q \neq 0 \), use CHAROWCOL to interchange rows \( i \) and \( q \);

2.2 Call FINDBAND and, if \( p \neq 0 \) or \( q \neq 0 \), go to 2.1;

3.1 If the bandwidth \( BW \) is less than its minimum recorded value, then store the current node numbers (vector \( v \)) into the vector \( ve \) of new node numbers and make the loop counter \( N \) equal to zero;

3.2 Make \( N \) equal to \( N + 1 \) and, if \( BW = \text{MINBW} \) or \( N > NT \), then go to 5;

4.1 For every row \( i \) between \( NP - 1 \) and 2, call ARRANGE and, if \( q \neq 0 \), use CHAROWCOL to interchange rows \( q \) and \( i \);

4.2 Call FINDBAND and go to 2.1;

5. Use routine NEWCODEN to find the new code numbers and the maximum node numbers difference for every element; print vector \( ve \) versus integers 1 to \( NP \), print the new code numbers and punch them on a set of cards.
Step 2 reduces the bandwidth successively until $p$ and $q$ become simultaneously zero, taking full advantage of the partial reductions in bandwidth to minimize the number of operations performed by the routines involved in the process.

Step 4 was devised to overcome the impasse when $p = 0$ and $q = 0$, i.e. it is not possible to reduce the bandwidth further by the process of step 2. Every row $i$ is interchanged with row $i+1$ if that causes a reduction in its bandwidth, keeping the bandwidth of the connectivity matrix less than or equal to $M$. If $M$ is chosen as the current bandwidth $BW$, the bandwidth will never increase during the process of rearranging the rows. However, $M = BW + 1$ seems to produce the best results and this means that the bandwidth may increase temporarily. For this reason, the current node numbers in vector $v$ are transferred to the vector $v_0$ whenever there is an additional decrease in the minimum bandwidth already achieved, after the routine ARRANGE is called for the first time.

In step 5 it is convenient to insert a warning message to be printed out whenever the bandwidth of an element exceeds the minimum bandwidth $BW$ determined by the algorithm. This is a final check of the new node and code numbers.

The search for $p$ and $q$ by the routine FINDCOL, FINDROW and ARRANGE, and the rows interchanges by the routine CHAROWCOL only involve operations on a part of array $B$ with $BW + 1$ rows, at most. Because it takes a comparatively long time to find the bandwidth, the routine FINDBAND is used as little as possible.
The information contained in vector $v_e$ immediately after using routine FINDNC could be easily used, together with CHAROWCOL, for an initial rearrangement, moving towards the middle the rows with decreasing numbers of non-zero off-diagonal elements. This is suggested by the structure of Fig. 4.1 where the original node 6 must be given the numbers 4 or 5 if the minimum bandwidth 2 is to be achieved. However, if the nodes had been originally numbered in a natural but logical manner, this would probably increase the initial bandwidth.

In addition to the code numbers (usual listings of the elements), only 6 parameters are necessary as input data: $NE$, $NP$, $ENP$, $MINBW$, $NT$ and $NC$.

The input parameter $MINBW$ is not really necessary if one is interested in finding the minimum bandwidth that can be achieved by the algorithm. If this is the case, one can use $MINBW = 1$, for instance. The input parameter $NT$ is the maximum number of times the rows will be rearranged for any partial reduction in bandwidth to take place. Usually and in economical terms, any digit number will be suitable but a two figure integer must be used if the minimum bandwidth is to be obtained in large problems. $NT = 0$ will cause the program to stop immediately before any rearrangement of rows is done, whatever the value of $MINBW$.

As input parameter, $NC$ is any value greater than or equal to the maximum number of nodes coupled together in the structure, $NC = 15$ is usually more than enough for structures with only linear elements and triangular elements of three nodes, for example. The routine FINDNC will
eventually modify NC giving it the minimum safe value which is equal to the maximum number of non-zero elements in any one row of the connectivity matrix, if it were stored in a symmetric array. For the structure of Fig. 4.1 it would be NC = 5. The use of routine FINDNC avoids finding by inspection of the structure the best value of NC and ensures that all following operations will be reduced.

One could take a small risk using NC slightly smaller than the value given by FINDNC as it stands, thus saving computer time by reducing the number of operations that would be performed by all other routines of the algorithm. Because the bandwidth is steadily reduced throughout the process, it is highly unlikely that, at any stage, any element in the last column of array A will become non-zero. This would be equivalent to a stage where all non-zero elements of one row (corresponding to the node coupled with the maximum number of other nodes in the structure) were lying on one side of the diagonal of the symmetric connectivity matrix.

Even when using the safe value of NC as given by routine FINDNC the storage needs are very much reduced in comparison with other ways of storing the connectivity matrix. For instance, in problem 10 (see Table 4.1) in which the original bandwidth was 222 and NP = 298, array A has the dimensions of 298 x 11. For array B it would be 298 x 222 and for array D 298 x 298.
4.4 Use of the Algorithm on an Example Problem

Fig. 4.4 shows the new numbering system of problem 6 (see Table 4.1) determined by the algorithm. Here are the input and output data:

<table>
<thead>
<tr>
<th>NE</th>
<th>NP</th>
<th>ENP</th>
<th>NC</th>
<th>MINBW</th>
<th>NT</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>20</td>
<td>4</td>
<td>10</td>
<td>6</td>
<td>2</td>
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</table>

<table>
<thead>
<tr>
<th>ROWS</th>
<th>ARRAY A</th>
<th>ELEMENTS</th>
<th>OLD CODE NUMBERS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 1 4 0 0 0 0 0 0 0 0 0 0</td>
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<td>2 5 3 6</td>
</tr>
<tr>
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<td>5 8 6 9</td>
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<td>4</td>
<td>11 14 12 15</td>
</tr>
<tr>
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<td>1 3 4 2 0 0 0 0 0 0 0 0 0 0 0 0</td>
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<td>1 4 2 5</td>
</tr>
<tr>
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<td>2 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0</td>
<td>6</td>
<td>4 7 5 8</td>
</tr>
<tr>
<td>7</td>
<td>1 3 4 10 0 0 0 0 0 0 0 0 0 0 0 0 0 0</td>
<td>7</td>
<td>7 10 8 11</td>
</tr>
<tr>
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<td>1 3 4 2 12 0 0 0 0 0 0 0 0 0 0 0 0 0</td>
<td>8</td>
<td>10 13 11 14</td>
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<td>11</td>
<td>8 20 0 0</td>
</tr>
<tr>
<td>12</td>
<td>2 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0</td>
<td>12</td>
<td>7 17 0 0</td>
</tr>
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<td>13</td>
<td>14 16 0 0</td>
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<td>20 16 0 0</td>
</tr>
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</tr>
<tr>
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<td>20</td>
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</tr>
</tbody>
</table>
### 4.5 Comparison with Other Available Algorithms

#### 4.5.1 Bandwidth Reduction

Table 4.1 shows a comparison of the method described herein with other available algorithms, with respect to the final bandwidths obtained for 10 example problems.

<table>
<thead>
<tr>
<th>NODE NUMBERS</th>
<th>ELEMENTS</th>
<th>NEW CODE NUMBERS</th>
<th>BANDWIDTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td>Old</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3 7 1 6</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>7 12 6 11</td>
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</tr>
<tr>
<td>2</td>
<td>3</td>
<td>12 17 11 14</td>
<td>6</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
<td>17 19 14 16</td>
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<td>19</td>
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<td>2 8 3 7</td>
<td>6</td>
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<td>8 13 7 12</td>
<td>6</td>
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<tr>
<td>5</td>
<td>7</td>
<td>13 18 12 17</td>
<td>6</td>
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<tr>
<td>4</td>
<td>8</td>
<td>18 20 17 19</td>
<td>3</td>
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<td>10</td>
<td>11 5</td>
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</tr>
<tr>
<td>9</td>
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<td>12 10</td>
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<td>8</td>
<td>12</td>
<td>13 9</td>
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<td>7</td>
<td>13</td>
<td>19 15</td>
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<td>14</td>
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<td>14</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
All values in the table, except the last two columns, were taken from the paper published by H. Grooms in which the topologies of the first 6 examples are presented. The listings of the elements (code numbers) for the last 4 larger examples have been kindly given by H. Grooms to whom the writer acknowledges his sincere gratitude.

In Table 4.1, a dash indicates that no results were available, C & M, RR, A & U and HG represent the algorithms of Cuthill and McKee, R. Rosen, Akyuz and Utku, and H. Grooms, respectively. Term JR stands for the method presented herein. The values in column (11) represent the bandwidths obtained without rearranging the rows as explained in 4.3. The values in column (12) represent the final bandwidths, using the routine ARRANGE.

Careful inspection of the first 6 example problems suggests that the final bandwidth is the minimum possible and, in examples 5 and 6, it is considerably smaller than the final bandwidth given by any other algorithm.

Considering the last 4 for examples, the results in column (12) also compare favourably with the ones corresponding to other algorithms, with the exception of example 7. However, only digit numbers have been used for parameter NT, because only limited computer facilities could be diverted from the main research project.

In fairness to A and U method, it must be noted that it minimizes the area within the band, not the band. It is this area, not the bandwidth, which is of prime importance for
certain solution techniques. However, the magnitude of this area within a given band is irrelevant when "band solution" techniques are used because zero coefficients below non-zero coefficients will become non-zero during the elimination process, thus coupling eventually all degrees of freedom within the wave front while it travels along the longitudinal axis of the structure.

4.5.2 Running Times and Storage Needs

All runs of the algorithm described herein were made on an ICL 1905F computer while an IBM 360/85 was used for other algorithms. No attempt was made to compare running times in the same computer but the writer has every reason to believe that his method will compare favourably with other algorithms.

Grooms has presented a table of running times for all other algorithms (except Cuthill and McKee), showing the advantages of his method over the others. The following considerations suggest that the writer's algorithm uses even less computer time.

The use of the connectivity matrix as a rectangular array makes it faster to perform the necessary operations, particularly interchanges of rows. In addition, if only non-zero off-diagonal elements are stored, it is possible to solve large problems in a small computer.

It appears that the bandwidth reduction is achieved with a much smaller number of rows interchanges. For instance, for the example problem of Fig. 4.1, the bandwidth was reduced from 5 to 2 by four rows interchanges while Grooms' algorithm had to modify six rows (and columns) to reduce the bandwidth from 5 to 4.
The bandwidth need not be determined before every row interchange which saves time. The use of routines FINDCOL, FINDROW and ARRANGE is comparatively little time consuming, because the search is carried out only in a small part of array A, taking into account previous reductions in bandwidth.

Fig. 4.5 shows how the bandwidth decreases as a function of the number of pairs of rows interchanged for example problems 4 to 10 using routine ARRANGE. The arrows show when this routine was called for the first time and emphasize the fact that further reductions in bandwidth take comparatively longer.

Comparison of Fig. 4.5 with Table 3 and Fig. 12 of the paper published by H. Grooms suggests that the algorithm presented herein compares favourably with other available algorithms with respect to running times, especially if it is remembered that all routines take into account previous reductions in bandwidth to reduce the number of operations performed.

It would be worthwhile to investigate the relative advantages and disadvantages of the representation of the connectivity matrix by array A over the binary representation of array B by bits. Various reasons prevented the writer from doing so but the use of the algorithm so far suggests that running times will be reduced by using this array B if the storage needs are not considerably greater than for array A. The comparative storage needs depend on the maximum number of nodes coupled together (NC), the value
of the original bandwidth and also the word length for the particular computer available. If NC is small, usual case of pin-jointed structures, for instance, there seems to be no doubt that array A is more suitable with respect to storage needs and running times.

4.5.3 Simplicity of Use

The algorithm is intended as an independent program to be used by an engineer when preparing input data to solve a structural problem rather than as a routine to be introduced in the main program. In this way the bandwidth will be reduced once and for all.

After numbering the nodes of the structure with a pencil, the code numbers defining the topology are listed and punched on a set of cards, for example. These cards together with another one containing 6 parameters constitute the only input data for the algorithm.

Lists of the original node numbers and of the corresponding new node numbers are printed out. Thus it is simple to replace the old numbers by the new ones. This is the only additional engineer's time required which is reduced to a convenient minimum.

The set of cards produced by the computer, with the new code numbers, is ready to replace the old set and the engineer can carry on with the preparation of other input data for the structural problem: nodal points coordinates, boundary conditions, etc.
4.6 Conclusion

An algorithm for reducing the bandwidth of a symmetric matrix stored as a rectangular array has been presented. If this matrix is the connectivity matrix of a structure the algorithm gives a printed out list of the old and corresponding new node numbers. It also produces the new code numbers for all elements in a set of punched cards.

The method appears to improve on the other available algorithms.
<table>
<thead>
<tr>
<th>Example No.</th>
<th>Nodes</th>
<th>Number of bars</th>
<th>Number of triangles</th>
<th>Number of quadrilaterals</th>
<th>Original Bandwidth</th>
<th>C &amp; M</th>
<th>RR</th>
<th>A &amp; U</th>
<th>HG</th>
<th>JR</th>
</tr>
</thead>
<tbody>
<tr>
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<td>16</td>
<td>0</td>
<td>0</td>
<td>15</td>
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<td>2</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>36</td>
</tr>
</tbody>
</table>
Fig. 4.1 Pin-Jointed Structure and Connectivity Matrix
Fig. 4.2 Rows Interchanged for Different Arrays Storing the Same Connectivity Matrix
Fig. 4.3 Flow-Chart of the Algorithm
Fig. 4.4 Example with Old and New Node Numbers
Fig. 4.5 Convergence for Example Problems 4 to 10.
Chapter 5
TESTS OF THE COMPUTER PROGRAM

5.1 Introduction

Each routine and the program itself have been thoroughly tested with regard both to the mathematical operations to be performed and to the physical interpretation of the results.

A number of typical examples of the problems solved will be presented in this chapter. Some are simple problems whose solution can be obtained by analytical methods. Others are more complex and no exact solution is available but the results given by the program are compared with field measurements or with the solution obtained by other authors using a constant strain triangular element and the finite element displacement method.

The main purpose of this chapter is to show that the program, developed entirely by the author, produces sufficiently accurate results and that the displacements converge towards the exact solution from below as the number of elements used for the same structural problem is increased. Simultaneously the versatility of the program for the solution of a variety of different problems will become apparent.

5.2 Plane Pin-Jointed Structures

Fig. 5.1 shows four plane pin-jointed structures solved with the specific purpose of testing routines ELSTIFMATL, PLANTKLOWT, BOUNDCOND and EQUASOLVED, i.e. the element stiffness matrix for linear elements, the assembly of the
stiffness matrix of the structure by superposing the contributions of every element, the implementation of boundary conditions of the type $d_i = 0$ and the band solution technique to solve the system of linear simultaneous equations.

Table 5.1 shows the nodal displacements and reactions evaluated by the program and the corresponding values obtained, for the first three structures, by hand calculations using the direct stiffness method. It can be seen that the computed results are virtually exact.

The first two columns show how the nodal variables (displacements and forces) in the system of equations are related to the nodal numbers which are odd numbers. Each horizontal displacement is identified by an odd number $n$ equal to the nodal point number, while the corresponding vertical displacement is identified by the even number $n+1$.

Problem C was actually solved by storing the stiffness matrix in a $4 \times 6$ rectangular array $K$ instead of a $14 \times 12$ rectangular array $K^{**}$ (see Fig. 3.18) as it would be the usual case for the node numbering system shown in Fig. 5.1. This was possible because the fixed nodal points can be disregarded since the corresponding boundary conditions are implicitly enforced by the node numbering system itself when the actual node numbers 7, 9, 11 and 13 are replaced by zero in array CODEN of code numbers (lists of connections), as explained in chapter 3.
5.3 Cantilever Beam under End Shear Load

Using the beam theory and the theorem of virtual work, the author arrived at the following expression for the deflection of a point on the neutral axis of a cantilever beam under parabolically distributed end shear, encastré at the other end and with rectangular cross-section:

\[ d_y = -\frac{2P}{EhH} \left[ (3 \ell - x) \frac{x^2}{H^2} + (1+v)x \right] \quad (5.1) \]

in which \( d_y \) is the vertical displacement
\( x \) is the abscissa
\( \ell \) is the length of the cantilever beam
\( P \) is the total end shear load
\( h \) is the thickness
\( H \) is the height
\( E \) is the Young's modulus
\( v \) is the Poisson's ratio.

When the Poisson's ratio term is disregarded, i.e. \( v = -1 \), and \( x = \ell \), the previous expression reduces to the well known formula

\[ d_y = -\frac{P\ell^3}{3EI} \quad (5.2) \]

in which \( I = \frac{hH^3}{12} \) is the moment of inertia of the cross-section \( h \times H \) of the cantilever beam.

The horizontal stresses on a vertical plane at a distance \( y_H \) from the horizontal plane containing the neutral axis are given by
where $M = -P(l-x)$ is the bending moment.

Table 5.2 shows the half distance deflection and tip deflection given by beam theory using expression (5.1) and the values computed by the program for the cantilever beam shown in Fig. 5.2, using meshes A to E and considering both plane stress and plane strain cases. The agreement is quite good, specially for the finest mesh E.

Fig. 5.3 shows the deflections of nodes on the neutral axis of the cantilever beam for total end shear loads of 24, 36 and 48 kN using meshes C, D and E, respectively, and the plane stress solution. The dots represent the computed values and the full lines represent the deflections given by the beam theory according to expression (5.1). For all nodes on the neutral axis the agreement is good and becomes better when the number of elements increases, as one should expect.

Fig. 5.4 compares the horizontal stresses computed by the program with the stresses given by the beam theory according to equation (5.3) for the centroids (of elements) along the dotted lines shown in Fig. 5.2 for meshes C, D, E. The agreement between both sets of values given by both methods is remarkably good.

The results shown in Table 5.2 for the plane stress case are represented graphically in Fig. 5.5 and it can be seen that the deflections increase, converging rapidly towards the exact solution (beam theory), when the number of unknowns, i.e. when the number of elements used to solve the same problem, increases.
The remarkably good accuracy and the convergence of results gain more significance when it is taken into account that the same problems have been solved as plane strain problems with a value of Poisson's ratio very close to the condition of incompressibility \( (\nu = 0.499999 = 0.5) \). The fact that the solution still converges in these circumstances is a good achievement by the program and is probably due to the use of exact integration in the evaluation of the element stiffness matrix. It would be interesting to investigate how close to \( 1/2 \) Poisson's ratio could be taken without losing accuracy and convergence of the solution.

The results corresponding to mesh E are also a test of routine \textsc{equadisplowt}. The same problem was solved twice both as a plane stress and as a plane strain problem, and the results were identical at least up to the sixth significant figure:

1. analysis as a single cantilever beam, in the usual manner, with 160 elements and 369 nodes;
2. analysis considering two adjacent cantilever beams, each one with 80 elements and 205 nodes, implementing the condition \( d_i - d_j = 0 \) for all adjacent nodes. These nodes lie on the neutral axis of the equivalent cantilever beam with the same length \( l \) and twice the height \( H/2 \) of each component cantilever beam.
The cantilever beam shown in Fig. 5.2 was used to test routine TEMPERATURE for a constant temperature variation $T = +100\degree C$ and a coefficient of thermal expansion $\alpha = 10^{-5} \degree C^{-1}$, considering two cases: left hand encastre' and simple support of both ends. The deformations were correct at least up to the sixth significant figure.

5.4 Plane Strain Triangle under Concentrated Load

Fig. 5.7 shows eight different triangular meshes used to analyse the deformation under a concentrated load of the triangle indicated, with two fixed nodal points.

Fig. 5.6 shows the convergence of vertical displacements for two points A and B. The exact solution is not presented but the displacements increase as the number of elements increases converging to some unknown value. The curves in Fig. 5.6 were drawn according to the results presented in Table 5.3 which contains the vertical displacements of four points and the stresses at the centroid of the triangle. The same numerical results are obtained when the problem is solved either in plane strain analysis with $E = 0.96$ and $v = 0.2$ or in plane stress analysis with $E = 1.00$ and $v = 0.25$. These two sets of parameters are related by formulae (3.40).

From the solutions obtained and considering other results not presented herein, it is of interest to note that:

1. The displacements converge from below towards some upper bound value but the convergence is not so rapid as in the case of a cantilever
beam. This is due to the fact that the strains in the cantilever beam presented in Fig. 5.2 vary linearly over the triangular element as was assumed in the formulation of the finite element method outlined in chapter 2 while, in the present case, this is not true and can only be considered approximately correct if the mesh is very fine;

2. It is known that the solution produced by any program based on the finite element method is not exact but it is not generally appreciated how different it can be from the correct solution if an appropriate number of elements is not used. Fig. 5.6 shows that the vertical displacement of point A corresponding to a mesh constituted by a single element is less than 44% of the exact value;

3. Table 5.3 and Fig. 5.6 show the vertical displacement of point A obtained by Zienkiewicz (1971) using the finite element displacement method and nine constant strain triangular elements (case K*) corresponding to 20 unknowns. The result is less accurate than the one obtained by using a single linear strain triangular element corresponding to 12 unknowns;

4. The horizontal displacements of nodal points lying on the vertical line of symmetry of the triangle are zero and the displacements of nodal points symmetric with respect to that line of symmetry are equal up to the sixth significant figure, at least, as it should be expected;
5. The stresses at the centroid of the triangle also converge towards some limit but this convergence is monotone only for a complete sequence of meshes such as A, B, D, H, which is in agreement with the theory of the finite element method. In a complete sequence, each mesh contains all nodes of all previous meshes. Finally, note that for all meshes in Fig. 5.7 there are only two different element stiffness matrices (see Fig. 3.17). All problems were actually solved by evaluating only two element stiffness matrices.

5.5 Cantilever Beam with Linear Elements

Fig. 5.8 shows a cantilever beam with one end encastre and the other end connected to an anchor, under a concentrated load \( P \) applied at point E. Using the parameters indicated in the same figure, where \( E_a \) refers to the anchor, the analysis was performed for a number of load systems and different anchors. Some typical results are presented in Table 5.4.

The purpose of this analysis was to verify the legitimacy of the method used by the author to take into account the anchors and struts connecting a diaphragm wall to the soil, even when each anchor or strut is represented by more than one linear element.

The same results are obtained independent of the number of linear elements representing each anchor or strut. This will be of interest when solving complex problems because
the decomposition of a long anchor or strut into a number of linear elements by the introduction of additional nodes, apparently unnecessary, will lead to a better node numbering system with reduced bandwidth.

Table 5.4 contains not only the displacements of point E but also the portions of the applied load supported by the cantilever beam and the anchor. These forces were evaluated by using routines ANCHORSTRESS and NODEFORCE which produce the correct answer since it was verified that the horizontal and vertical components of the resultant of forces acting on any cross-section are equal to the corresponding components $F_x$ and $F_y$ acting on the non encastré end of the cantilever beam.

All results in Table 5.4 appear to be correct as can be easily verified. For case B, for instance, one can find,

anchor: $d_y = -\Delta \rho = \frac{F_y}{E_s} S_\alpha = 18.185 \times 10^{-3} \text{m}$

cantilever beam: $d_y = \frac{F_x L^3}{3E_I} = 18.152 \times 10^{-3} \text{m}$

where $S_\alpha$ is the cross-section of the anchor.

Although the stresses are not presented herein, they also are consistent with the results shown in Table 5.4. For Case G, for instance, the horizontal stresses at the centroids of all elements of the cantilever beam not close to the ends are constant and equal to 95.175 kN/m².

It can be concluded that the method employed to consider anchors and struts yields the correct solution. However, it must be noted that, when the cross-section of the linear
elements is extremely large or extremely small compared with the cross-section of the cantilever beam, for Young's moduli of the same order, the problem may become ill-conditioned. The solution may be unreliable or even impossible to obtain by routine EQUASOLVED, since the stiffness matrix of the structure could become non-positive due to round-off errors.

5.6 Vertical Line Load on a Semi-Infinite Elastic Medium

5.6.1 The Boussinesq Problem

Lame and Clapeyron (1833) developed formulae governing the deformations of a homogeneous semi-infinite elastic medium loaded normal to its plane boundary. Boussinesq (1885) used a potential function method to develop a solution whose simplicity of form contrasted with the complexity of the formulae previously developed but difficult to apply in practice.

Many problems in stress and strain analysis, which are of practical importance, are concerned with the effect in semi-infinite media of stresses acting on their straight boundaries. The solution of such problems can be obtained by integration from the results corresponding to the solution of Boussinesq's problem.

In two-dimensional analysis, some solutions can readily be obtained through the use of an Airy stress function. These stress functions are quite valuable from a practical point of view since the integration of the results of Boussinesq's problem becomes sometimes rather tedious, although they fail to show the close relationship between the space problem and the two-dimensional problem.
Results based on the solution of Boussinesq's problem have been obtained for various related problems and are extensively used by engineers under the form of tables and charts (Scott, 1963) many of which have been photoelastically confirmed and shown to be applicable to finite bodies as long as one remains far enough from the boundaries. For example, using formulae derived for uniform stress distributed over a circular area on the plane surface of a semi-infinite solid, Newmark (1942, 1947) developed charts that can be used to compute stresses and deformations at any point of a semi-infinite isotropic elastic medium due to uniformly distributed pressures acting on areas of any shape.

Consider Fig. 5.9 where a force $P$ is acting in the $z$-direction, at the origin of coordinates, on a semi-infinite elastic medium.

The solution of Boussinesq's problem is given (Saada, 1974) by the following expressions for the displacements in cylindrical coordinates:

\[
\begin{align*}
    u_r &= \frac{P(1+\nu)}{2\pi E\rho} \left[ \frac{rz}{\rho^2} - \frac{(1-2\nu)r}{\rho+z} \right] \\
    u_\theta &= 0 \\
    u_z &= \frac{P(1+\nu)}{2\pi E\rho} \left[ 2(1-\nu) + \frac{z^2}{\rho^2} \right]
\end{align*}
\]

(5.4)

where $\rho = \sqrt{r^2 + z^2}$
In cartesian coordinates, the displacements are given by

\[
\begin{align*}
    u_x &= \frac{P(1+\nu)}{2\pi E} \left[ \frac{(1-2\nu)x}{\rho(z+\rho)} + \frac{zx}{\rho^3} \right] \\
    u_y &= \frac{P(1+\nu)}{2\pi E} \left[ \frac{(1-2\nu)y}{\rho(z+\rho)} + \frac{zy}{\rho^3} \right] \quad (5.5) \\
    u_z &= \frac{P(1+\nu)}{2\pi E} \left[ \frac{2(1-\nu)}{\rho} + \frac{z^2}{\rho^3} \right]
\end{align*}
\]

The formulae are not valid for \( \rho = 0 \).

5.6.2 Vertical Line Load on a Semi-Infinite Elastic Medium
(Plane Strain Problem)

Consider Fig. 5.10(a) where a vertical line load is acting on a semi-infinite elastic medium. The stresses can be obtained through the use of the Airy stress function (Saada, 1974)

\[
\phi = cr \theta \sin \theta 
\]

(5.6)

taking \( c = \frac{-q}{\pi} \):

\[
\begin{align*}
    \sigma_r &= -\frac{2q}{\pi r} \cos \theta \\
    \sigma_\theta &= 0 \\
    \sigma_z &= -\frac{2qv}{\pi r} \cos \theta \\
    \tau_{r\theta} &= \tau_{rz} = \tau_{\theta z} = 0
\end{align*}
\]

(5.7)

Having the stresses, the displacements can be obtained by integration, using the strain-displacement and stress-strain relations given by the linear theory of elasticity.
Obviously, it must be considered that $\epsilon_y = 0$ since it is a plane strain problem. The final expressions of the displacements will be:

\[
\begin{align*}
    u_r &= -\frac{q}{\pi E} \left[ 2(1-v^2) \cos \theta \ln r + (1-2v)(1+v) \sin \theta \right] + A \sin \theta + B \cos \theta \\
    u_\theta &= \frac{q}{\pi E} \left[ 2v(1+v) \sin \theta + 2(1-v^2) \sin \theta \ln r + (1-2v)(1+v) \left( \sin \theta - \theta \cos \theta \right) \right] \\
    u_y &= 0
\end{align*}
\]

where $A$, $B$ and $C$ are constants of integration which are to be determined from the boundary conditions.

Due to symmetry, according to Fig. 5.10(a), all points on $z$-axis have no lateral displacement, i.e. $u_r = 0$ for $\theta = 0$. This condition leads to

\[
u_\theta = A + C \ r = 0 \quad \Rightarrow \quad A = C = 0 \quad (5.9)
\]

To determine $B$ another boundary condition has to be considered and this will be done in two different ways.

(a) A point on the $z$-axis at a depth $L$ does not have vertical displacement, i.e. $u_r = 0$ for $\theta = 0$ and $r = L$. This condition leads to

\[
B = \frac{2(1-v^2)q}{\pi E} \ln L \quad (5.10)
\]

The equations (5.8) of the displacements become, in this case:
\[ u_r = \frac{(1+v)q}{\pi E} \left[ 2(1-v)\cos \theta \ln \frac{L}{r} - (1-2v)\theta \sin \theta \right] \]

\[ u_\theta = \frac{(1+v)q}{\pi E} \left[ \sin \theta - 2(1-v)\sin \theta \ln \frac{L}{r} - (1-2v)\theta \cos \theta \right] \]

\[ u_y = 0 \]

(5.11)

On the surface of the straight boundary where \( \theta = \pi/2 \) the displacements are given by

\[ u_x = u_r = -\frac{q}{2E} (1-2v)(1+v) \]

\[ u_z = u_\theta = -\frac{(1+v)q}{\pi E} \left[ 2(1-v) \ln \frac{L}{r} - 1 \right] \]

(5.12)

Along z-axis \( \theta = 0 \) and the displacements are

\[ u_x = u_\theta = 0 \]

\[ u_z = u_r = \frac{2(1-v^2)q}{\pi E} \ln \frac{L}{r} \]

(5.13)

(b) A point on the x-axis at a distance \( L \) from the origin does not have vertical displacement, i.e. \( u_\theta = 0 \) for \( \theta = \pi/2 \) and \( r = L \). This condition leads to

\[ B = \frac{(1+v)q}{\pi E} \left[ 1 + 2(1-v) \ln L \right] \]

(5.14)

The equations (5.8) of the displacements will become, in this case:

\[ u_r = \frac{(1+v)q}{\pi E} \left[ 2(1-v)\cos \theta \ln \frac{L}{r} - (1-2v)\theta \sin \theta + \cos \theta \right] \]

\[ u_\theta = \frac{(1+v)q}{\pi E} \left[ -2(1-v)\sin \theta \ln \frac{L}{r} - (1-2v)\theta \cos \theta \right] \]

\[ u_y = 0 \]

(5.15)
On the surface of the straight boundary where \( \theta = \pi/2 \) one will have

\[
\begin{align*}
    u_x &= u_r = -\frac{q}{2E} (1+\nu) (1-2\nu) \\
    -u_z &= u_\theta = -\frac{2(1-\nu^2)}{\pi E} \ln \frac{L}{r}
\end{align*}
\]  
\tag{5.16}

Along z-axis \( \theta = 0 \) and the displacements are:

\[
\begin{align*}
    u_x &= u_\theta = 0 \\
    u_z &= u_r = \frac{(1+\nu)q}{\pi E} [2(1-\nu) \ln \frac{L}{r} + 1]
\end{align*}
\]  
\tag{5.17}

In both cases (a) and (b) the horizontal displacements of points on the straight boundary and on z-axis are given by the same expressions, while the corresponding vertical displacements are given by different expressions. However, if \( L \) is very large the two expressions produce approximately the same numerical results when \( r \ll L \).

The quantity \( L \) is indeterminate and there is nothing in the analysis by which it can be found. For the reason just mentioned it is usually taken to be very large.

There is a horizontal displacement towards the origin equal for all points on the straight boundary. One can regard this as a physical possibility if a cylindrical surface of small radius around the line of application of \( q \) is removed and \( q \) substituted by an equivalent system of stresses. Actually, in this portion, the material is plastically deformed due to the stress concentration.
The solutions presented herein, like the solution of Boussinesq's problem, are subject to the restriction that their validity commences at a small distance from the points of application of the load.

5.6.3 Vertical Line Load on a Semi-Infinite Elastic Plate (Plane Stress Problem)

Consider the semi-infinite elastic plate in Fig. 5.10(b) which is assumed to have unit width so that the load is equal to \( q \). The problem is a plane stress problem and \( \sigma_y = 0 \).

Starting with the same Airy stress function (5.6), proceeding along the same lines used for plane strain but using \( \sigma_y = 0 \) instead of \( \varepsilon_y = 0 \), and assuming that a point along the \( z \)-axis and at a depth \( L \) is fixed, the equations of the displacements will be found to be:

\[
\begin{align*}
  u_x &= \frac{q}{\pi E} \left[ 2 \cos \theta \ln \frac{L}{x} - (1-\nu) \theta \sin \theta \right] \\
  u_\theta &= \frac{q}{\pi E} \left[ (1+\nu) \sin \theta - 2 \sin \theta \ln \frac{L}{x} - (1-\nu) \theta \cos \theta \right]
\end{align*}
\]

(5.18)

On the straight boundary of the plate where \( \theta = \pi/2 \) it will be:

\[
\begin{align*}
  u_x = u_z &= -\frac{(1-\nu)q}{2E} \\
  -u_z = u_\theta &= \frac{q}{\pi E} \left[ (1+\nu) - 2 \ln \frac{L}{x} \right]
\end{align*}
\]

(5.19)

On the vertical plane of symmetry \( \theta = 0 \) and the displacements are given by
\[ u_x = u_\theta = 0 \]  
\[ u_z = u_r = \frac{2q}{\pi E} \ln \frac{L}{r} \]  

(5.20)

5.6.4 Finite Element Solutions

Fig. 5.11 shows the triangular mesh used for the analysis of a semi-infinite elastic medium by the program. The distances from the origin of the corner nodes along the surface AD and the centre line AB form a geometric progression of ratio 1.4. For boundary conditions it was assumed that all points along lines BC and CD are fixed and the points along the centre line AB have no horizontal displacement.

Table 5.5 shows the parameters used, the types of analysis carried out and the maximum settlement (point A of Fig. 5.11). In cases A, B and C the vertical load \( P \) was assumed to be numerically equal to a line load distributed over a unit width while in cases A', B', and C' the same load \( P \) was assumed to be distributed over a strip of unit width. The actual loads \( P \) used in the finite element analysis were only half of the values indicated in Table 5.5 since only half of the elastic medium needs to be considered, due to symmetry.

The computed displacements for all nodes along the horizontal surface and the centre line are represented by the computer drawn graphs of Figs. 5.12, 5.13 and 5.14. The displacements corresponding to cases A, B and C are identical to those corresponding to cases A', B' and C', for points far from the loaded area, according to Saint-Venant's principle, and thus A', B' and C' will not be referred again.
The displacements corresponding to case $T_A$ were evaluated by using formulae (5.19) and (5.20) for plane stress. The displacements corresponding to cases $T_B$ and $T_C$ were evaluated by using formulae (5.12) and (5.13) for plane strain and boundary condition (a) referred above.

The horizontal displacements of points along the horizontal surface given by the program (Fig. 5.13) are similar to those predicted by Boussinesq's theory, except in the vicinity of the origin and the boundary.

For points along the centre line, the computed displacements (Fig. 5.14) are very close to those predicted by Boussinesq's theory, except in case C.

Points along the horizontal surface and far from the boundary (Fig. 5.12) have computed displacements practically identical to those predicted by Boussinesq's theory, except in case C.

Since it is reasonable, at this stage, to assume that the program produces the correct (although only approximate) solution in plane stress and plane strain analysis, even for values of Poisson's ratio close to $1/2$, it appears that the Boussinesq's solution overestimates the settlements when Poisson's ratio approaches $1/2$. This overestimation has been noticed by other authors (e.g. Burland, Sills and Gibson, 1973) in other situations and it would be interesting to investigate this matter further as it appears that the value of Poisson's ratio governs the magnitude of the difference between the computed and analytical solution.
Finally, it is interesting to note that in cases A and B the centre line settlements are identical but this does not apply to the surface settlements. Formulae (3.40) show that both problems would have the same numerical solution if \( v = 0 \) had been used in case A rather than \( v = 0.5 \). Therefore, the differences in the values of the horizontal and vertical displacements of points on the surface in cases A and B are accounted for by the Poisson's ratio alone. This is consistent with the well known fact that the settlements increase when the Poisson's ratio decreases, if all other parameters remain unchanged.

5.7 Scammonden Dam

Fig. 5.15 shows the triangular mesh with 367 elements and 804 nodes used in the analysis of Scammonden Dam which was analysed by Penman, Burland and Charles (1971), using the same finite element displacement method and the constant strain triangular element. Incidentally, note that, according to the values of NP and G shown in Fig. 5.15, this structure was analysed by solving a system of 1478 simultaneous equations instead of \( 2 \times NP = 1608 \) simultaneous equations. This results from the node numbering system used (node numbers equal to zero for fixed points, as explained in chapter 3) and leads to savings in computing time.

Scammonden Dam is of earth/rockfill construction and was described by Mitchell and Maguire (1968) and Winder (1969). Its initial behaviour was discussed by Penman and Mitchell (1970). It was fitted with very comprehensive instrumentation
and the accurate measurements of the movements of a large number of discrete points on the major section, during the construction, enables a detailed comparison to be made with the movements predicted from the properties of the fill and foundation material by an analysis using the finite element method.

A detailed analysis and comparison with field measurements has been made by the authors referred to above, assuming elastic behaviour of the soil and using a simple "equivalent compressibility" method based on the finding that, for soils in confined compression, the majority of \( \sigma_v/\varepsilon_v \) curves can be represented by equivalent constant E values with little loss in accuracy in an analysis of construction in layers.

The author does not intend to go into such detail and presents a single solution corresponding to the end of construction. Using the same soil parameters but a different mesh and a different triangular element, the purpose of the analysis is to show that the program gives approximately the correct solution for a medium size practical problem, simultaneously testing routine BODYFORCE which finds the consistent system of body forces (due to self weight of the embankment in this case).

Table 5.6 shows the soil properties used in the analysis and Fig. 5.16 the equivalent Young's modulus against the height of the dam.

Fig. 5.17 is a reproduction of Fig. 8 of the paper by Penman, Burland and Charles (1971) and shows a comparison of the observed movement with the prediction given by the finite element analysis.
Fig. 5.18 in which the same scales as in Fig. 5.17 have been used for the geometry and the displacements shows the results obtained by the author considering a single solution for the whole dam and using the material parameters presented in Table 5.6 and Fig. 5.16. It corresponds to an ideal situation: build the whole embankment and then let its self weight act on the soil structure. For this reason, the displacements of points above the first layer cannot be compared with those shown in Fig. 5.17 which do not include the portion due to the deformation of underlying layers.

By comparing Figs. 5.17 and 5.18 it can be seen that the agreement is very satisfactory and it appears that the results obtained through the use of the program developed by the author show a better agreement with the final observed movement shown in Fig. 17 (compare, for instance, the displacements of points 2, 8 and 12 at level B). Since the same material parameters and geometry have been used in both analyses and the number of elements used by the author is equivalent to a greater number of unknowns than for the finite element analysis corresponding to Fig. 17 (displacements closer to the upper bound exact solution), the better agreement is probably due to the ability of the program to consider the Young's modulus varying within the element.

5.8 Diaphragm Wall of Britannic House

The construction of Britannic House, between Ropemaker Street and Moorgate Railway Station in the City of London, commenced in 1962 and the building was occupied in 1967. The main structure is a 35-storey tower of reinforced concrete founded on a raft within the basement area.
The level of the site was first reduced from 11m above Ordnance Datum Newlyn (ODN) to about +0.7m ODN by removing the old foundations to give a clear working level. An 800mm thick reinforced concrete diaphragm wall was constructed around the site boundary using the mud slurry trenching process.

Cole and Burland (1972) have described the construction method and presented some of the field measurements. They also used the finite element method with a constant strain triangular element to find, by back analysis, the variation of stiffness of the ground with depth, showing a satisfactory agreement between the computed results and other field values. Fig. 5.19 shows the site plan, geology, water levels in piezometers and soil properties, and is a reproduction of Fig. 1 of the paper by Cole and Burland.

Fig. 5.20 shows the triangular mesh with 359 elements and 776 nodes used by the author for the analysis of the north diaphragm wall. Two solutions involving different distributions of E have been obtained, considering approximately the same geometry and material parameters as in the paper mentioned above, namely, the values shown in Fig. 5.21 and the variation of pore water pressure with depth indicated by the levels in piezometers shown in Fig. 5.19.

The displaced positions of the north wall selected for analysis correspond to the dates of 28 June 1963 (just before the raft blinding was poured) and 21 July 1963 (at the time the first struts were placed). The computed displacements
are represented by vectors in Figs. 5.23 and 5.24 using the same scales (both for the geometry and the displacements) as in the paper by Cole and Burland from which Fig. 5.22 is a reproduction.

Comparison of Figs. 5.22 and 5.23 shows that the general picture of the deformation is very similar, indicating the same pattern of displacements: inward movement towards the excavation, with vertical components positive (upwards) on the side of the excavation and negative on the other side. The movement of the wall shows an almost horizontal translation and rotation about the base towards the excavation.

The displacements obtained by the author are slightly greater which can be explained mainly by three reasons:

1. The author did not take into account the weight of raft blinding which accounts for the different displacements at the bottom of the excavation;
2. From the use of a better triangular element and a greater number of unknowns by the author, a better approximation for the displacements must be expected which will be closer to the upper bound exact solution;
3. The geometry and material parameters are not exactly the same for both analysis since not all the necessary information was numerically defined in the paper.

Nevertheless, the agreement with available field measurements is satisfactory, namely the observed movement of Ropemaker Street (Fig. 5.22).
Comparison of Figs. 5.23 and 5.24 shows a progressive reduction in the stiffness of the clay, outside the excavation, associated with the inward movement of the wall, which is greatest near the ground surface.

The horizontal components of the displacements are much greater than the vertical components which can affect adversely buildings located even far from the excavation and the performance of tied-back diaphragm walls in London Clay.

One of the purposes of the analysis of the diaphragm wall of Britannic House was to test routines SETLOAD, FORCES, MOMENTS, PRESSURE and EQUASOLVE.

The consistent nodal forces on the vertical surfaces of the wall due to initial stresses in the soil (assuming no disturbance caused by the construction of the wall) were evaluated by hand calculations using (3.153), after determining for each relevant node the coefficient of earth pressure at rest and the pore water pressure. For the finite element analysis, the top left corner of the wall was taken as origin of the system of cartesian coordinates. Routine SETLOAD produced identical results. The same routine was also used to determine the consistent system of loads applied to the structure due to the excavation.

From the consistent system of forces acting on both sides of the wall due to initial stresses in the soil the distributed horizontal pressures were again evaluated using routines PRESSURE and EQUASOLVE. As expected, the numerical values obtained were identical to the initial horizontal stresses used in the first place to find the consistent nodal forces.
The use of routines FORCES and MOMENTS obviously produced a resultant of all nodal forces acting on the wall equal to zero.

5.9 Axi-Symmetric Analysis of a Solid Cylinder

5.9.1 Introduction

All problems mentioned so far in this chapter as tests of the program refer to plane stress and plane strain analysis. However, some routines are suitable only for axi-symmetric problems, namely, SETMATD1, SETLOAD1, ELSTIFMATT1, CENTELSTRESS1 and PRINSTRESS1.

Fig. 5.25 shows the triangular meshes for the axi-symmetric analysis of a solid cylinder with various dimensions and load systems. The objective of this analysis is to show that all routines mentioned above serve their purposes properly, especially, ELSTIFMATT1 which sets up the element stiffness matrix of a ring-type triangular element, and that the program is capable of solving axi-symmetric problems with satisfactory accuracy.

5.9.2 Condition $\epsilon_x = 0$

Consider a cylindrical elastic body under axial applied pressure $q_y$ which is not allowed to deform laterally (e.g. oedometric test). The lateral strain will be $\epsilon_x = 0$ and the induced vertical stress $\sigma_y$ will be equal to the applied axial pressure $q_y$. 
Assuming cross-anisotropic material, the first two of equations (3.6) can be written as

\[ \begin{align*}
\epsilon_x &= \frac{1}{E_H} \sigma_x - \frac{v_{VH}}{E_V} \sigma_y - \frac{v_{HH}}{E_H} \sigma_x \\
\epsilon_y &= -\frac{v_{VH}}{E_V} \sigma_x + \frac{1}{E_V} \sigma_y - \frac{v_{VH}}{E_V} \sigma_x
\end{align*} \]  
(5.21)

Using (5.21) and making \( \epsilon_x = 0 \) one arrives at the following expression for the horizontal stress:

\[ \sigma_x = \frac{n}{1 - v_{HH}} \sigma_y \]  
(5.22)

Substituting (5.22) in (5.21) one finds

\[ \epsilon_y = \frac{\sigma_y}{E_V} \left( 1 - \frac{2}{1 - v_{HH}} \right) \]  
(5.23)

Assuming \( \epsilon_x = 0 \) and \( v_{HH} = v_{VH} = 0 \), one finds

\[ \begin{align*}
\sigma_x &= 0 \\
\epsilon_y &= \frac{\sigma_y}{E_V}
\end{align*} \]  
(5.24)

These values are identical to those corresponding to simple (unconfined) compression.

Assuming that the Poisson's ratios are given by expressions (3.68) corresponding to incompressibility, one arrives at the following values, whatever the value of \( n = E_H / E_V \):

\[ \begin{align*}
\sigma_x &= \sigma_y \\
\epsilon_y &= 0
\end{align*} \]  
(5.25)
This is not surprising and shows that the material behaves much like a fluid.

Table 5.7 shows the strains and stresses obtained through the use of the program for the axi-symmetric analysis of the solid cylinders represented by their meshes in Fig. 5.25. All symbols in the Table are defined in that figure or have been defined previously.

For cases A and B, the stress $\sigma_x$ and strain $\epsilon_y$ have been found to be identical to those given by equations (5.22) and (5.23), respectively, at least up to the seventh significant figure.

This shows excellent accuracy of results in this rather simple case, although all or some of the elements touch axis $y$ of symmetry (this is generally an unfavourable situation, as explained in 3.5).

For an isotropic material, equations (5.22) and (5.23) reduce to

$$\sigma_x = \frac{\sigma_y}{1 - \nu} \quad (5.26)$$

and

$$\epsilon_y = \frac{\sigma_y}{E} \left( 1 - \frac{2\nu}{1 - \nu} \right) \quad (5.27)$$

The results corresponding to cases C, D, E and F in Table 5.6 are identical to those given by equations (5.26) and (5.27), at least up to the seventh significant figure.

Considering $E = 1$ and $\sigma_y = 1$ to simplify the presentation, one can derive from (5.26) and (5.27):
It is of interest to note that:

(a) $\epsilon_y$ is a maximum in case 2 which is equivalent to unconfined compression;

(b) Case 5 corresponds to a condition of no volume change and $\sigma_x = \sigma_y$ shows that the material behaves like a fluid;

(c) In all cases $0 < \epsilon_y < 1$ and $-\frac{1}{2} < \sigma_x < 1$.

The results presented in Table 5.7 for cases A to F are valid for every element and the displacements of all points where the consistent (but different from node to node) nodal forces are applied have precisely the same displacement. This proves that routine SETLOAD1 determines correctly the consistent system of nodal forces equivalent to a constant pressure $q_y$ distributed over the circular area.

5.9.3 **Condition $\sigma_x = 0$**

When the cylindrical solid is free to deform laterally under axial pressure $q_y = \sigma_y$ then $\sigma_x = 0$ and one can derive from (5.21) the following expressions valid for cross-anisotropic material:
\[ \epsilon_x = -v \frac{\sigma_y}{E_v} \]  

(5.28)

\[ \epsilon_y = \frac{\sigma_y}{E_v} \]

From (5.28) one readily obtains the relationship

\[ v_{VH} = -\frac{\epsilon_x}{\epsilon_y} \]  

(5.29)

which provides a suggestive physical interpretation of Poisson's ratio \( v_{VH} \).

For isotropic material, equations (5.28) and (5.29) reduce to

\[ \epsilon_x = -v \frac{\sigma_y}{E} \]

(5.30)

\[ \epsilon_y = \frac{\sigma_y}{E} \]

\[ v = -\frac{\epsilon_x}{\epsilon_y} \]

This shows that always, for an isotropic elastic material,

\[ -\frac{1}{2} \leq \frac{\epsilon_x}{\epsilon_y} \leq 1 \]  

(5.31)

According to the results presented in Table 5.7 for case G, the decrease in length of the cylinder is underestimated by less than 1% and the vertical stress \( \sigma_y \) at the centroid of the element 16 identified in Fig. 5.25 is overestimated by less than 0.1%. In fact, the exact values are given by
\[ \Delta l = \frac{q_y l}{E} = -9.6 \times 10^{-3} \text{ m} \]  

(5.32)

\[ \sigma_y = q_y = -6.0 \text{ kN/m}^2 \]

5.9.4 Condition \( \sigma_y = 0 \)

When the cylindrical solid is free to deform axially under lateral constant pressure \( q_x \) then \( \sigma_y = 0 \) and one can derive from (5.21) the following expressions valid for a cross-anisotropic material:

\[ \varepsilon_x = \frac{\sigma_x}{E_H} (1 - v_{HH}) \]  

(5.33)

\[ \varepsilon_y = \frac{\sigma_x}{E_V} (-2v_{VH}) \]

Consequently,

\[ \frac{\varepsilon_y}{\varepsilon_x} = \frac{2n \ v_{VH}}{v_{HH} - 1} \]  

(5.34)

Taking the values of Poisson's ratios corresponding to incompressibility in expressions (3.68), one arrives at the relationship \( \varepsilon_y = -2 \varepsilon_x \), valid for any value of \( n = E_H/E_V \).

For an isotropic material equations (5.33) and (5.34) reduce to:

\[ \varepsilon_x = (1 - v) \frac{\sigma_x}{E} \]

\[ \varepsilon_y = -2v \frac{\sigma_x}{E} \]  

(5.35)

\[ \frac{\varepsilon_y}{\varepsilon_x} = \frac{2v}{v - 1} \]
For all values of v, always, for an elastic material

\[-2 \leq \frac{\varepsilon_y}{\varepsilon_x} \leq 1\]  

(5.36)

From the results in Table 5.7 corresponding to case H one can obtain the average strain \(\varepsilon_y = -2.70 \times 10^{-5}\) which is 10% smaller than the value given by the second of equations (5.35). The average ratio of strains is \(\varepsilon_y/\varepsilon_x = -0.51\) while the third of equations (5.35) predicts the exact value - 0.5.

However, the results are acceptable if it is taken into account that:

(a) the cylinder is quite long in comparison with the cross-section and, therefore, the case analysed is slightly different from the theoretical case;

(b) all triangular elements touch the centre line which is the most unfavourable situation for the accurate evaluation of the element stiffness matrix in axi-symmetric analysis, as explained in chapter 3.

Taking into account equations (5.28) valid for case G and equations (5.35) approximately valid for case H, one would expect the strains (and, therefore, the corresponding displacements) to be related by

\[
(\varepsilon_x)_G = (\varepsilon_x)_H
\]

\[
(\varepsilon_y)_G = 10 (\varepsilon_y)_H
\]  

(5.37)
In fact, the horizontal displacements in case G are equal, for every node of the cylinder, to the corresponding horizontal displacements in case H, up to the seventh decimal place, at least. The vertical displacements in case H are, on average, about 9% of the values for case G which is consistent with the fact that H is not exactly a plane stress problem in the sense defined in chapter 3.

5.9.5 Hollow Cylinder

Consider a hollow cylinder of length $l$, interior radius $R_i$, and exterior radius $R_e$, under axial pressure $q_y$. For elastic material of Young's modulus $E$, the decrease in length is given by

$$
\Delta l = \frac{q_y l}{E} \quad (5.38)
$$

In case I of Table 5.8 it should be $\Delta l = 1.6 \times 10^{-3}$, which is in agreement with the computed vertical displacement for point A of Fig. 5.25. Using formulae (5.30) it can be found, for the same case I, $\epsilon_x = 10^{-5}$. From the computed horizontal displacements for points E and F, identical result is obtained:

$$
\epsilon_x = \frac{\Delta u}{\Delta x} = \frac{\Delta x}{x_E - x_F} = \frac{(1.005-0.995) \times 10^{-3}}{100.5-99.5} = 10^{-5} \quad (5.39)
$$

When the thickness $R_e - R_i$ of the hollow cylinder is much smaller than the average radius $R = (R_e + R_i)/2$ it can be shown that, under radial pressure $q_x$,
which is equal to the computed values for case J, considering the values of the relevant variables in Table 5.7.

5.9.6 Conclusions

The analysis of cylinders with various geometries and elastic properties, under various load systems, shows that the program gives sufficient accuracy of results in axisymmetric analysis. For the cases analysed in this chapter it is of interest to note that:

1. In situations of confined compression (or tension) the results are virtually exact;
2. When the triangular elements do not touch the centre line and are far from it the accuracy is excellent;
3. The most unfavourable situation leading to lesser accuracy arises when all triangular elements touch the centre line.
<table>
<thead>
<tr>
<th>Nodes</th>
<th>Variables</th>
<th>Displacements evaluated by the Program</th>
<th>Exact displacements</th>
<th>Loads and Reactions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A</td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>2</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>3</td>
<td>0.00000</td>
<td>0.34314</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>4</td>
<td>-2.41421</td>
<td>-1.02943</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>5</td>
<td>2.82843</td>
<td>0.00000</td>
<td>0.00000</td>
<td>1.28989</td>
</tr>
<tr>
<td>6</td>
<td>-1.41421</td>
<td>-0.51472</td>
<td>0.00000</td>
<td>0.03048</td>
</tr>
<tr>
<td>7</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>8</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.30645</td>
</tr>
<tr>
<td>9</td>
<td>-</td>
<td>0.00000</td>
<td>0.00000</td>
<td>-0.85005</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>-0.68629</td>
<td>0.00000</td>
<td>-0.11947</td>
</tr>
<tr>
<td>11</td>
<td>-</td>
<td>-0.68629</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>12</td>
<td>-</td>
<td>-0.00000</td>
<td>0.00000</td>
<td>0.57129</td>
</tr>
<tr>
<td>13</td>
<td>-</td>
<td>-0.00000</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>14</td>
<td>-</td>
<td>-0.00000</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Mesh</td>
<td>Number of Elements</td>
<td>Number of Unknowns</td>
<td>Load kN</td>
<td>Plane Stress Deflection, mm</td>
</tr>
<tr>
<td>------</td>
<td>-------------------</td>
<td>--------------------</td>
<td>--------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Half-Distance</td>
</tr>
<tr>
<td>A</td>
<td>2</td>
<td>18</td>
<td>36</td>
<td>5.152</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>42</td>
<td>36</td>
<td>7.539</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>66</td>
<td>36</td>
<td>7.653</td>
</tr>
<tr>
<td>D</td>
<td>40</td>
<td>210</td>
<td>36</td>
<td>7.966</td>
</tr>
<tr>
<td>E</td>
<td>160</td>
<td>738</td>
<td>36</td>
<td>8.068</td>
</tr>
<tr>
<td>Beam theory (upper bound)</td>
<td></td>
<td></td>
<td>36</td>
<td>8.187</td>
</tr>
</tbody>
</table>
TABLE 5.3  Displacements and Stresses of a Plane Strain Triangle

<table>
<thead>
<tr>
<th>Case</th>
<th>Number of Elements</th>
<th>Number of Unknowns</th>
<th>Vertical Displacements</th>
<th>Stresses at C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>1</td>
<td>12</td>
<td>46.875</td>
<td>18.750</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>30</td>
<td>63.330</td>
<td>30.788</td>
</tr>
<tr>
<td>C</td>
<td>9</td>
<td>56</td>
<td>74.124</td>
<td>37.110</td>
</tr>
<tr>
<td>D</td>
<td>16</td>
<td>90</td>
<td>81.928</td>
<td>41.903</td>
</tr>
<tr>
<td>E</td>
<td>25</td>
<td>132</td>
<td>87.881</td>
<td>45.064</td>
</tr>
<tr>
<td>F</td>
<td>36</td>
<td>182</td>
<td>92.699</td>
<td>45.755</td>
</tr>
<tr>
<td>G</td>
<td>49</td>
<td>240</td>
<td>96.753</td>
<td>49.953</td>
</tr>
<tr>
<td>H</td>
<td>64</td>
<td>306</td>
<td>100.253</td>
<td>51.897</td>
</tr>
<tr>
<td>I</td>
<td>81</td>
<td>380</td>
<td>103.335</td>
<td>53.591</td>
</tr>
<tr>
<td>J</td>
<td>100</td>
<td>462</td>
<td>106.087</td>
<td>55.119</td>
</tr>
<tr>
<td>K*</td>
<td>9</td>
<td>20</td>
<td>44.473</td>
<td>-</td>
</tr>
</tbody>
</table>

Note: K* refers to constant strain triangular elements
## TABLE 5.4 Displacements of a Cantilever Beam with Linear Elements

<table>
<thead>
<tr>
<th>Case</th>
<th>Geometric Characteristics</th>
<th>Applied Load, kN</th>
<th>Variation of axial load in anchor, kN</th>
<th>End shear load of cantilever beam, kN</th>
<th>Displacements of point E, mm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cantilever Beam</td>
<td>Anchor</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>h (m)</td>
<td>H (m)</td>
<td>L (m)</td>
<td>Ends</td>
<td>$p_x$ (m)</td>
</tr>
<tr>
<td>A</td>
<td>1 1 10</td>
<td>EG</td>
<td>20 20</td>
<td>0.005</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>1 1 10</td>
<td>EG</td>
<td>0 20</td>
<td>0.005</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>1 1 10</td>
<td>EG</td>
<td>0 19.5</td>
<td>0.05</td>
<td>0</td>
</tr>
<tr>
<td>D</td>
<td>1 1 10</td>
<td>AF</td>
<td>10 1</td>
<td>0.005</td>
<td>99.504</td>
</tr>
<tr>
<td>E</td>
<td>1 1 10</td>
<td>AF</td>
<td>10 1</td>
<td>0.001</td>
<td>99.504</td>
</tr>
<tr>
<td>F</td>
<td>1 1 10</td>
<td>AF</td>
<td>10 1</td>
<td>0</td>
<td>99.504</td>
</tr>
<tr>
<td>G</td>
<td>1 1 10</td>
<td>BE</td>
<td>10 0</td>
<td>0.005</td>
<td>100</td>
</tr>
<tr>
<td>H</td>
<td>1 1 10</td>
<td>BE</td>
<td>10 0</td>
<td>0.0005</td>
<td>100</td>
</tr>
</tbody>
</table>
### TABLE 5.5 Analysis of an Elastic Medium

<table>
<thead>
<tr>
<th>Case</th>
<th>$E$ (MN/m²)</th>
<th>$\nu$</th>
<th>$P$ (MN)</th>
<th>Maximum Settlement (mm)</th>
<th>Type of Analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>50</td>
<td>0.499</td>
<td>2</td>
<td>227</td>
<td>Finite Element Method, plane stress</td>
</tr>
<tr>
<td>A'</td>
<td>50</td>
<td>0.499</td>
<td>2</td>
<td>134</td>
<td>Finite Element Method, plane strain</td>
</tr>
<tr>
<td>B</td>
<td>50</td>
<td>0</td>
<td>2</td>
<td>225</td>
<td></td>
</tr>
<tr>
<td>B'</td>
<td>50</td>
<td>0</td>
<td>2</td>
<td>140</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>50</td>
<td>0.4999</td>
<td>2</td>
<td>156</td>
<td></td>
</tr>
<tr>
<td>C'</td>
<td>50</td>
<td>0.4999</td>
<td>2</td>
<td>84</td>
<td></td>
</tr>
<tr>
<td>$T_A$</td>
<td>50</td>
<td>0.5</td>
<td>2</td>
<td>-</td>
<td>Boussinesq's theory</td>
</tr>
<tr>
<td>$T_B$</td>
<td>50</td>
<td>0</td>
<td>2</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$T_C$</td>
<td>50</td>
<td>0.5</td>
<td>2</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

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### TABLE 5.6 Soil Properties Used in Analysis of Scammondon Dam

<table>
<thead>
<tr>
<th>Material</th>
<th>Young's Modulus MN/m²</th>
<th>Poisson's Ratio</th>
<th>Bulk Density kN/m³</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rockfill (under full height of embankment, see Fig. 5.15)</td>
<td>32</td>
<td>0.25</td>
<td>21.19</td>
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<tr>
<td>Clay core</td>
<td>1.07</td>
<td>0.49</td>
<td>19.62</td>
</tr>
<tr>
<td>Upstream weight block</td>
<td>1.07</td>
<td>0.49</td>
<td>15.70</td>
</tr>
<tr>
<td>Foundations</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0-7.6m</td>
<td>138</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td>7.6-22.9m</td>
<td>690</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td>22.9-53.3m</td>
<td>1380</td>
<td>0.15</td>
<td>-</td>
</tr>
<tr>
<td>Geometry</td>
<td>Applied Pressures kN/m²</td>
<td>Elastic Parameters</td>
<td>Strains \times 10^{-6}</td>
</tr>
<tr>
<td>----------</td>
<td>--------------------------</td>
<td>-------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>Case</td>
<td>H</td>
<td>h</td>
<td>r</td>
</tr>
<tr>
<td>A</td>
<td>32</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>B</td>
<td>132</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>C</td>
<td>32</td>
<td>10</td>
<td>0.48</td>
</tr>
<tr>
<td>D</td>
<td>132</td>
<td>0.48</td>
<td>0.48</td>
</tr>
<tr>
<td>E</td>
<td>32</td>
<td>10</td>
<td>0.48</td>
</tr>
<tr>
<td>F</td>
<td>132</td>
<td>0.48</td>
<td>0.48</td>
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</tbody>
</table>
### TABLE 5.8 Displacements and Stresses of a Cylindrical Body under Axial and Radial Pressures

<table>
<thead>
<tr>
<th>Case</th>
<th>Geometry</th>
<th>Elastic Parameters</th>
<th>Applied Pressures</th>
<th>Displacements, mm</th>
<th>Stresses</th>
<th>Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>m</td>
<td>r</td>
<td>E MN/m²</td>
<td>v</td>
<td>qₓ</td>
<td>qᵧ</td>
</tr>
<tr>
<td>G</td>
<td>32</td>
<td>1</td>
<td>0</td>
<td>20</td>
<td>0.20</td>
<td>0</td>
</tr>
<tr>
<td>H</td>
<td>32</td>
<td>1</td>
<td>0</td>
<td>20</td>
<td>0.20</td>
<td>1.5</td>
</tr>
<tr>
<td>I</td>
<td>32</td>
<td>1</td>
<td>99.5</td>
<td>20</td>
<td>0.20</td>
<td>0</td>
</tr>
<tr>
<td>J</td>
<td>32</td>
<td>1</td>
<td>99.5</td>
<td>20</td>
<td>0.20</td>
<td>0.015</td>
</tr>
</tbody>
</table>
Fig. 5.1 Plane Pin-Jointed Structures A, B, C and D
Parabolically varying end shear.

Total load \( P \)

\[
E = 720 \text{ MN/m}^2 \quad \text{and} \quad v = 0.499999
\]

\( h = \text{thickness} = 1 \text{ m} \)

\( P = 36 \text{ kN} \)

\( P = 36 \text{ kN} \)

\( P = 24 \text{ kN} \)

\( P = 36 \text{ kN} \)

\( P = 48 \text{ kN} \)

Fig. 5.2 Load System and Triangular Meshes of Cantilever Beam under End Shear Load
Fig. 5.3 Deflections of a Cantilever Beam under End Shear Load
Fig. 5.4 Horizontal Stresses of a Cantilever Beam under End Shear Load
Fig. 5.5 Convergence and Accuracy of Deflections of a Cantilever Beam under End Shear Load

$$d_r = \frac{d_{\text{finite element method}}}{d_{\text{beam theory}}} \times 100$$

- tip deflection
- middle deflection

Fig. 5.6 Convergence of Displacements of a Plane Strain Triangle

Key:
- • constant strain triangular element

- [Diagram of a plane strain triangle with labels and dimensions]
Plane strain triangle
E = 0.96 & ν = 0.2
Equivalent to
Plane stress triangle
E = 1.0 & ν = 0.25

Fig. 5.7 Load System and Triangular Meshes of a Plane Strain Triangle
\( E = 2000 \text{ MN/m}^2 \)
\( E_a = 20000 \text{ MN/m}^2 \)
\( \nu = 0.2 \)
\( h = \text{thickness} = 1 \text{ m} \)

Fig. 5.8 Cantilever Beam with Linear Elements

Fig. 5.9 Boussinesq's Problem

a) Plane strain  
b) Plane stress

Fig. 5.10 Line Load on a Semi-Infinite Elastic Medium
Fig. 5.11 Triangular Mesh of Elastic Medium

NE = 156
NP = 357

SCALE: 0 10 20 30 40 50 m
Fig. 5.13 Surface Horizontal Displacements of an Elastic Medium
Fig. 5.14 Centre Line Settlements of an Elastic Medium
Fig. 5.15 Triangular Mesh of Scammonden Dam

Fig. 5.16 Equivalent Young's Modulus against the Height of the Dam
Fig. 5.17 Comparison of Observed and Predicted Movements (Penman, Burland and Charles, 1971)

Displacements:

500 mm

Fig. 5.18 Displacements Evaluated by the Program for Scammonden Dam
Fig. 5.19 Site Plane, Geology, Water Levels in Piezometers and Soil Properties Concerning Brittanic House (Cole and Burland, 1972)
Fig. 5.20 Triangular Mesh for Diaphragm Wall of Britannic House

- NE = 359
- NP = 776
- G = 1466

SCALE:
0 10 20 30 40 50 m
Fig. 5.21 Parameters Used in Analysis of Diaphragm Wall of Britannic House
Fig. 5.22 Comparison of Observed and Predicted Movements Corresponding to 21 July 1963 (Cole and Burland, 1972)

Fig. 5.23 Displacements Evaluated by the Program for Diaphragm Wall of Britannic House Corresponding to 21 July 1963
Fig. 5.24 Displacements Evaluated by the Program for Diaphragm Wall of Britannic House Corresponding to 28 June 1963
Fig. 5.25 Triangular Meshes of Solid Cylinders for Axi-Symmetric Analysis

\[ E = 20000 \text{ kN/m}^2 \]
Chapter 6
SETTLEMENT OF A CIRCULAR LOADED AREA ON THE SURFACE OF AN ELASTIC, INCOMPRESSIBLE, HETEROGENEOUS MEDIUM

6.1 Introduction
Some of the results presented later in this chapter were first intended as a test of the program on the solution of a complex axi-symmetric problem in which the material is incompressible, cross-anisotropic and heterogeneous with Young's moduli increasing linearly with depth from zero at the surface where the load is applied.

After attending The Rankine Lecture, 1974, delivered by Professor R. E. Gibson, the author became more aware of the theoretical and practical importance of this subject and eager to confirm by the finite element method the validity of the theoretical solution for the deformation of an incompressible, heterogeneous elastic layer, resting on a smooth rigid base, when subject to a uniform circular or strip loading.

The ability of the program to predict the displacements (and, therefore, the stresses) for this rather complex problem is demonstrated in this chapter and simultaneously a contribution is given to confirm some unexpected results derived by Gibson's theory.

6.2 Influence of Elastic Heterogeneity and Orthotropy on Immediate Settlements

For many years the formulae, charts and tables used by engineers for problems of surface loading based on the solution of Boussinesq's problem (stresses and displacements induced by a single point load) have assumed the semi-infinite soil medium to be isotropic and homogeneous.
The importance of anisotropy has been more widely recognized (e.g. Duncan and Seed, 1966; Mitchell, 1972) than the importance of heterogeneity (e.g. Zaretsky and Tsytovich, 1965). Recently progress has been made on the analytical solution of the problem of surface loading of a non-homogeneous elastic medium (Gibson, 1967, 1968, 1969 and 1974; Gibson and Sills, 1971; Awojobi, 1972, 1973 and 1974) and the computed solution by the finite element method (Hooper, 1974).

Following the work of Gibson (1967), Ward, Burland and Gallois (1968) have noticed discrepancies between the observed and the theoretical values (predicted by the classical elastic theory) of settlements outside a loaded area on chalk, attributing them to heterogeneity. Other field results and their comparison with the values predicted by the finite element method (Cole and Burland, 1972; Burland, Sills and Gibson, 1973) have accumulated evidence that the stiffness of London Clay increases approximately linearly with depth. The increase of stiffness with depth is generally found in real soil strata, reflecting the increasing overburden pressure (Gibson, 1974).

Thus it is of more than just academic interest to find a solution to the problem of a half-space subject to vertical loading when Young's modulus \( E \) increases linearly with depth \( z \) according to the relationship

\[
E(z) = E(0) + \lambda z
\]  
(6.1)
A solution to the problem has been first obtained by Gibson (1967) who investigated in detail the case of an isotropic, incompressible semi-infinite elastic medium where the undrained Young's modulus is given by $E_u = \lambda z$, concluding that this medium responds to any vertical surface loading exactly like a Winkler spring model, namely, that the settlement at any point of the surface is directly proportional to the intensity of vertical stress there, in conditions of plane strain or axial symmetry.

In the case of uniform surface loading, the constant settlement $w_o$ is given by

$$w_o = \frac{3q}{2\lambda} \quad \text{(within the loaded area)}$$

$$w_o = 0 \quad \text{(outside the loaded area)}$$

in which $q$ is the uniformly distributed pressure

$\lambda$ is the increase of the undrained Young's modulus per unit depth.

This shows that the surface of the medium (Gibson soil) settles under vertical pressure as a uniform bed of springs, with the difference that they are not capable of transmitting shear and horizontal stresses.

The first of equations (6.2) can be written (Gibson, 1974) more generally as

$$w = \frac{c q}{\lambda}$$

(6.3)
The value of the constant $c$ is $3/2$ for an incompressible isotropic medium ($\nu = 1/2$) and becomes infinite for $\nu \neq 1/2$.

For the corresponding cross-anisotropic medium, Gibson (1974) arrived at the following expression for the settlement $w(x, y)$ of the plane surface resulting from a general surface pressure $q(x, y)$:

$$w(x, y) = \frac{q(x, y)}{N(dE_v/dz)} \quad (6.4)$$

where $N$ is approximately given by

$$N = \frac{G_{VH}}{E_V} + \left( 4 - \frac{E_H}{E_V} \right)^{-1} \quad (6.5)$$

For an incompressible isotropic material $E_H = E_V = E$ and $G_{VH} = E/3$, and equation (6.5) reduces to $N = 2/3$ which agrees with the result $C = 3/2$ recorded above for this case. When $E_H = 4E_V$, $N$ becomes infinite and no settlement occurs.

The restrictions imposed on (6.5) by considering non-negative strain energy density ($N^{-1} < 4$) and comparison of (6.4) with (6.2) lead to the conclusion that whatever the degree of anisotropy the settlement cannot exceed that experienced by an isotropic elastic body by a factor of more than $8/3$.

It was shown in 3.1.6 that the condition of incompressibility, for a cross-anisotropic medium, reduced the five independent elastic parameters to three by the following relationships derived for the Poisson's ratios (see (3.68)):
\[ \nu_{\text{VH}} = 1/2 \tag{6.6} \]
\[ \nu_{\text{HH}} = 1 - \frac{1}{2} \frac{E_H}{E_V} \]

The other elastic parameters \( E_H, E_V \) and \( G_{\text{VH}} \) have to be determined and that can be done through undrained compression tests on samples cut with their axes inclined at different angles to the horizontal.

From the results of a number of undrained compression tests (Ward, Samuels and Butler, 1959; Ward, Marsland and Samuels, 1965), Gibson found the following average values typical for undisturbed London Clay:

\[ m = \frac{G_{\text{VH}}}{E_V} = 0.38 \]
\[ n = \frac{E_H}{E_V} = 2 \tag{6.7} \]

It is interesting to note that these values are in reasonable agreement with an expression derived analytically for the case of small deformations by Barden (1963). Using relationships (3.4), (3.5) and (3.21), Barden's expression can be written as

\[ m = \frac{G_{\text{VH}}}{E_V} = \frac{n}{1 + n + 2n \nu_{\text{VH}}} \tag{6.8} \]

In fact, for \( n = 2 \) and \( \nu_{\text{VH}} = 1/2 \), equation (6.7) gives \( m = 0.40 \) which is 5% greater than the value in (6.7).
6.3 **Comparison of Finite Element Solution with Gibson's Theory**

Fig. 6.1 shows the triangular mesh used in the finite element analysis. The actual dimensions and soil parameters for cases A to I are presented in Table 6.1 and for cases J to P in Table 6.2. Case A' differs from case A in that it was assumed that the horizontal movement of the loaded area is zero. Cases J* and K* differ from J and K, respectively, in that the rigid base of the layer was considered rough, as in cases A to I.

In all cases a uniformly distributed vertical pressure equal to 1 kN/m\(^2\) was considered to be applied over a circular area of radius \(R = 110\)m, with no horizontal movement along the centre line \(AB\), no vertical movement along the boundary \(BC\) and no movement along the boundary \(CD\) (see Fig. 6.1).

Table 6.1 shows the computed displacements for centre and edge together with the theoretical displacements obtained by Gibson (1974). The results obtained by Gibson were actually expressed in ft but they can be expressed by the same numerical value in m, for comparison with the results obtained by the author using kN and m instead of t and ft.

The triangular mesh of Fig. 6.1 is topologically identical to the mesh used in chapter seven for a different problem for which it had been specifically prepared. If this is kept in mind it is easy to understand why this mesh
is not the most suitable for some of the analyses considered in this chapter, such as cases E to I where there should be a greater number of small elements in the zone of stress concentration near the edge of the loaded area.

When comparing the results of Table 6.1 corresponding to Gibson's solution with the results corresponding to the finite element solution, it must also be kept in mind that:

(a) The elastic parameters used by the author do not correspond exactly to the condition of incompressibility although they are close to it and, for this reason alone, the computed displacements should be greater than the theoretical values;

(b) The fixed boundary \( \overline{CD} \) (Fig. 6.1) is taken at a finite distance and this factor reduces the value of the computed displacements;

(c) The finite element solution is a lower bound to the exact solution;

(d) The edge settlement shown in Tables 6.1 and 6.2 corresponds to the nodal point of the surface in the immediate vicinity of the edge, on the side of the centre line, and its accuracy would be improved by the existence of a greater number of elements in this zone.

Comparison of the theoretical and computed values in Table 6.1 shows that:
1. The agreement is satisfactory both for centre and centre-edge settlements, including the change in sign of the differential settlement in cases C and D;

2. For the homogeneous and isotropic layer, the computed displacements are slightly smaller than the theoretical values, as expected, but the converse occurs when the layer is simultaneously heterogeneous and cross-anisotropic;

3. The consideration of cross-anisotropy reduces the magnitude of the displacements;

4. Heterogeneity increases the horizontal displacements at the edge and changes the sign of the differential settlement centre-edge which increases in value when the assumed thickness of the layer is reduced;

5. The position of the boundaries not only affects substantially the magnitude of the displacements but also the magnitude and sign of the differential settlement if the stratum is heterogeneous and has a finite thickness.

From Figs. 6.2 to 6.11 it can also be noted that:

1. If the horizontal movement of the loaded area is restrained (case A') the settlement is reduced by only 10%, if the layer is isotropic and homogeneous;
2. The pattern of the displacements (shape of the curves) is not affected by cross-anisotropy but it is substantially affected by heterogeneity and by the position of the boundary conditions if the layer is heterogeneous;

3. While the horizontal displacements of points on the surface are small when the layer is homogeneous, they have a sharp peak at the edge when the layer is heterogeneous. In both cases they increase when the thickness of the layer decreases and are directed outwards, except when the layer is homogeneous and very thick;

4. Heterogeneity causes a sharp variation in surface settlement near the edge and settlement approximately equal to zero outside the loaded area, as the theory predicts, if the thickness of the layer is much greater than the radius of the circular loaded area;

5. The centre line and edge line settlements decrease more rapidly with depth when the layer is heterogeneous;

6. The horizontal displacements along the edge line decrease rapidly with depth when the layer is heterogeneous while they first increase and then decrease slowly with depth if the layer is homogeneous.
Fig. 6.12 shows the pattern of displacements within the layer corresponding to the more realistic case I. It shows that, for practical problems of this type in which the same material parameters are used, the boundary conditions must be located at a distance from the loaded area not smaller than six times the radius, if a reasonably accurate solution is to be obtained.

Fig. 6.13 shows a detail of Fig. 6.12 to illustrate the magnitude and direction of the displacements in the vicinity of the loaded area.

Fig. 6.9 shows that, in the finite element analysis of a homogeneous elastic half-space, the rigid base must be assumed at a depth greater than ten times the radius of the circular loaded area while Figs. 6.7 and 7.8 show that the horizontal boundary need not be located so far. The same figures suggest that, when the half-space is heterogeneous, the assumed depth of the rough rigid base can be comparatively smaller.

6.4 Settlement of a Layer of Gibson Soil on a Smooth Rigid Base

To illustrate how the analytical method in soil mechanics helps to distinguish between which factors are of primary significance and which are of secondary importance, Professor Gibson ended The Rankine Lecture, 1974, by presenting the problem of Fig. 6.14 and its surprising solution.

Gibson's Law states that any loaded area of an incompressible, isotropic elastic half-space in which the undrained Young's modulus varies from zero at the surface
linearly with depth will settle by an amount proportional to the pressure on the area, and this loading will in no way affect settlement of the surface outside the loaded area (see equation 6.2). Awojobi (1974) proved that the law is independent of the stratum depth when the stratum rests on a smooth rigid base. This invariance remains true so long as axial symmetry or plane strain conditions are maintained.

Fig. 6.14 shows an incompressible heterogeneous elastic layer, subject to a uniform strip loading, which rests on a smooth rigid base at depth D. The settlement \( w(x) \) of any point on the surface of the layer at a distance \( x \) from the centre line of the strip load can be expressed (Gibson, 1974) in the form

\[
w(x) = \frac{3}{4} \int_0^D \frac{(\sigma_z - \sigma_x)}{E_u(z)} \, dz \tag{6.9}
\]

in which, according to Fig. 6.14,

\[
E_u(z) = \lambda z \tag{6.10}
\]

According to Awojobi's analysis the stress difference is given by

\[
\frac{\sigma_z - \sigma_x}{q} = - \frac{z}{D} \left[ \frac{\sinh \frac{\pi(B+x)}{D}}{\cosh \frac{\pi(B+x)}{D} - \cos \frac{\pi z}{D}} \right]
\tag{6.11}
+ \frac{\sinh \frac{\pi(B-x)}{D}}{\cosh \frac{\pi(B-x)}{D} - \cos \frac{\pi z}{D}}
\]

Substituting (6.10) and (6.11) in (6.9), the surface settlement is found to be independent of the depth \( D \) and given by
\[ w(x) = \frac{3q}{2\lambda} \quad \text{within the loaded area (}|x|<B) \]
\[ = 0 \quad \text{outside the loaded area (}|x|<B) \]

(6.12)

This result is valid both for axial symmetry and plane strain conditions. Considering a circular loaded area, using the triangular mesh of Fig. 6.1 and the data in Table 6.2, the author used the program to find the values of the displacements presented in Table 6.2 and in Figs. 6.15 to 6.28, corresponding to cases J to P.

The restrictions mentioned in 6.3 to the validity of the comparison between the theoretical and the computed solutions also apply to the cases considered in this section.

The results in Table 6.2 confirm the analytical predictions in (6.2) and (6.12), showing the invariance of the surface settlement with the thickness of the heterogeneous layer resting on a smooth rigid base. It is particularly interesting to compare cases J, K and M in which the thickness of the layer is proportional to 130, 13 and 1, respectively. The computed displacements increase when the assumed thickness of the layer decreases but even this somewhat surprising feature emphasizes the invariance of the settlements predicted by Gibson's theory. In fact, this appears to be due to three factors:

1. The relative contribution of the boundary \( \overline{CD} \) (Fig. 6.1) to restrain the horizontal movement and, therefore, the settlement, increases when the thickness decreases if the distance \( \overline{AD} \) is kept constant;
2. Since the same number of elements was used in all cases the mesh is finer when the thickness of the layer is reduced which should lead to computed displacements closer to the upper bound exact solution;

3. In the formulation of the finite element displacement method presented in chapter 2 and in its application to the program in chapter 3, it was assumed that the deformations were small and the linear theory of elasticity could be applied, particular the assumption that the strain-displacement relations are linear as the change in orientation of the body due to displacement is negligible. However, when the thickness of the layer is very small as in case M for example, the deformations can hardly be considered small and this factor adds to the inaccuracy of the solution.

Equations (6.2) and (6.12) give a constant value of 165mm for the surface settlement within the loaded area. The computed values in Table 6.2 are 20% to 30% greater which could be accounted for by the fact that the parameters used do not correspond exactly to the condition of incompressibility and is consistent with the theory's prediction that they become infinite when Poisson's ratio is different from 1/2 (in the isotropic case).
From Table 6.2 it can also be concluded that:

1. Complete heterogeneity in addition to smooth rigid base is necessary for the invariance of the displacements with the thickness of the layer to occur;

2. When the boundaries are located at a great distance from the loaded area (J and J*), the displacements are identical irrespective of whether the base is rough or smooth;

3. The horizontal displacements increase substantially when the thickness of the layer decreases and seem to account for the invariance of the settlement which otherwise would be smaller for a smaller thickness;

4. The accuracy of the finite element solution is not affected by the fact that the Young's modulus is zero at points where the loads are applied (J and L), although "numerical methods abhor zeros and infinities" (Gibson, 1974), in general.

In Figs. 6.15 and 6.16 the curves identified by lower-case letters refer to the displacements along a plane at a depth of 3/113 of the total thickness of the layer, the surface settlements being identified by the corresponding block capitals also appearing in Table 6.2.

This shows that the top 2.7% of the incompressible heterogeneous layer accounts for about 20% of the centre line settlement when the thickness is a minimum, increasing to over 40% when the thickness is 130 times greater. If the layer is homogeneous the corresponding contribution is much smaller, as expected.
Outside the loaded area the surface settlements are approximately zero as predicted by the theory for the homogeneous layer, unless the boundary $CD$ is too close ($P$). However, the horizontal displacements, always directed outwards, have a considerable magnitude even at a great distance from the loaded area, especially when the thickness of the layer is very small, having a maximum at the edge which is a sharp peak unless the layer is very thin (Figs. 6.17 and 6.18).

The variations of the settlements along the centre line and the edge line (Figs. 6.19 to 6.22) follow similar patterns but the shapes of the curves depend on the thickness and on the homogeneity or heterogeneity of the layer.

The horizontal displacements along the edge line (Figs. 6.23 and 6.24) are approximately constant if the thickness of the layer is small. The rate of variation decreases with the depth and the displacements become close to zero at half-depth if the base is located very far from the surface.

Figs. 6.25 and 6.26 illustrate the variations in magnitude and direction of the displacements within the elastic layer and also show how the triangular meshes used in the typical cases $J^*$ and $P$ are actually different although topologically identical (see Fig. 6.1). It is obvious that the mesh is more suitable for case $P$. Note how the boundary $CD$ restrains the horizontal movement causing slight heave outside the loaded area, in case $P$. 
Fig. 6.27 illustrates the considerable heave appearing not far from the loaded area, in case $K^*$, caused by the roughness of the rigid base when the thickness of the layer is smaller than the radius of the loaded area. It also emphasizes the fact that the boundary $CD$ need not be located so far if the assumed thickness of the layer is comparatively small and the base is rough.

Fig. 6.28 illustrates the invariance of the horizontal displacements with depth when the layer is very thin and how their magnitude decreases slowly along the surface if the base is smooth. In this case, precisely because the thickness of the layer is very small, the boundary $CD$ must be located very far away from the loaded area, contrary to the case in which the base is rough.

6.5 Conclusions

The numerical solution obtained through the use of the program in a number of problems has been compared with the theoretical solution given by Gibson's theory. The agreement between both solutions is satisfactory and it appears that the program is capable of solving complex axi-symmetric problems, considering rough or smooth rigid base, homogeneous or heterogeneous, isotropic or cross-anisotropic material.

It also confirms some rather surprising results derived by Gibson's theory, namely the independence of the settlement of a circular loaded area from the thickness of the layer when it is heterogeneous and rests on a smooth rigid base.
In all problems solved a constant pressure was assumed over the loaded area and, in most of them, full heterogeneity of the layer.

This differs from the reality in two important points, at least:

1. Real soils do not have stiffness equal to zero at the surface although it can be very small;
2. The ground is usually loaded through a foundation possessing some flexural rigidity and therefore the distributed pressure on the ground surface may be very different from the distributed pressure on the top of the raft, depending on its rigidity.

However, it is clear from the results presented in this chapter that:

(a) a uniform surface pressure leads to a more uniform settlement profile, if the layer is thick and heterogeneous, than is predicted by the classical theory for a homogeneous layer;

(b) the shape of the deformed surface is not only very sensitive to the stiffening of the soil with depth but also depends on the depth at which the rough rigid base is located in the ground;

(c) cross-anisotropy reduces the magnitude of the displacements (for the cases analysed where $E_H > E_V$);
(d) adequate information about the soil parameters in each practical case and consideration of the interaction between soil and structure are indispensable for a correct design of a raft and the structure it supports;

(e) when solving a soil mechanics problem by the finite element method in which the fixed boundaries have to be assumed at a finite distance, one must be very careful about the choice of their location which is always a compromise between the economy of the solution and the accuracy of the results.
TABLE 6.1 Influence of Elastic Heterogeneity, Orthotropy and Position of Fixed Boundaries on Settlement of Circular Loaded Area on Surface of London Clay (q = 1 kN/m²)

<table>
<thead>
<tr>
<th>Case</th>
<th>Geometry</th>
<th>Soil Parameters</th>
<th>Horizontal Displacement</th>
<th>Settlements, mm</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>E&lt;sub&gt;H/E&lt;V&lt;/sub&gt;</td>
<td>E&lt;sub&gt;V&lt;/sub&gt;(0)</td>
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<td></td>
<td></td>
<td>kN/m²</td>
<td>kN/m²/m</td>
<td>m</td>
</tr>
<tr>
<td>A</td>
<td>110</td>
<td>1.0</td>
<td>500</td>
<td>0</td>
</tr>
<tr>
<td>A'</td>
<td>110</td>
<td>1.0</td>
<td>500</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>110</td>
<td>1.8</td>
<td>500</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
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<tr>
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<td>1.8</td>
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TABLE 6.2 Invariance of Gibson's Law for a Layer on a Smooth Rigid Base

<table>
<thead>
<tr>
<th>Case</th>
<th>Geometry</th>
<th>Soil Parameters</th>
<th>Horizontal Displacement</th>
<th>Settlement</th>
<th>Rigid Base</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>H m</td>
<td>L m</td>
<td>R m</td>
<td>t m</td>
<td>$E_H/E_V$</td>
</tr>
<tr>
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<td>1131</td>
<td>1474</td>
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<tr>
<td>J*</td>
<td>1131</td>
<td>1474</td>
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<td>290</td>
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</tr>
<tr>
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<td>1474</td>
<td>110</td>
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<td>1474</td>
<td>110</td>
<td>290</td>
<td>1.0</td>
</tr>
<tr>
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<td>1474</td>
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<td>110</td>
<td>290</td>
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<tr>
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<td>1131</td>
<td>1474</td>
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<td>290</td>
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<tr>
<td>P</td>
<td>435</td>
<td>567</td>
<td>110</td>
<td>110</td>
<td>1.0</td>
</tr>
</tbody>
</table>
FIG. 6.2 Ground Surface Settlements (A to D)

- Distance from Centre Line, m
- Settlement, mm
FIG. 6.3 Ground Surface Horizontal Displacements (A to D)
FIG. 6.4 Centre Line Settlements (A to D)
FIG. 6.5 Edge Line Settlements (A to D)
FIG. 6.6 Edge Line Horizontal Displacements (A to D)
FIG. 6.7 Ground Surface Settlements (E to I)
FIG. 6.9 Centre Line Settlements (E to I)
FIG. 6.10 Edge Line Settlements (E to I)
FIG. 6.11 Edge Line Horizontal Displacements (E to I)
FIG. 6.12 Displacement Vectors (I)
FIG. 6.13 Displacement Vectors - Detail (I)
FIG. 6.14 Which Load Settles More?
FIG. 6.16 Ground Surface Settlements (J, J*, L, O, P)
FIG. 6.17  Ground Surface Horizontal Displacements (K, K*, M, N)
FIG. 6.19 Centre Line Settlements (K, K*, M, N)
FIG. 6.20 Centre Line Settlements (J, J*, L, O, P)
FIG. 6.21 Edge Line Settlements ($K$, $K^*$, $M$, $N$)
FIG. 6.22 Edge Line Settlements (J, J*, L, O, P)
FIG. 6.23 Edge Line Horizontal Displacements (K, K*, M, N)
FIG. 6.24 Edge Line Horizontal Displacements ($J^*$, $L$, $O$, $P$)
Chapter 7

THE BEHAVIOUR OF THE FOUNDATIONS OF TWO OIL TANKS AT FAWLEY REFINERY, SOUTHAMPTON

7.1 Ground Works

The tanks, 79.2m diameter and 19.8m high, had been built originally on a piled foundation and were ready for test in 1968. During the testing process the foundation of tank 281 failed. Leggatt and Bratchell (1973a) describe the events which followed this failure and which are summarized by the author in this section. Bratchell, Leggatt and Simons (1974) also mention the same events and compare the measured settlements with the values predicted by commonly employed methods based on the use of S.P.T. N values or on the use of static cone point resistances.

Investigation showed that it would be unpractical to make any further use of the existing foundations and the feasibility of moving the tanks off the old foundations to a nearby site was studied. The generalized soil profile of this new site is shown in Fig. 7.1.

After comparative studies it was decided to remove the soft alluvium and replace it with gravel from a large pit within the refinery area. The excavation of the alluvium was carried out by suction dredger with the groundwater maintained at ground level, thus increasing the stability of the sides.

The excavation of the alluvium was carried out down to the top of the natural gravel layer under water with disposal of spoil into a nearby lagoon. After a trial excavation it
was decided to excavate to a nominal side slope of 65° to the horizontal. In the second foundation to be excavated (tank 281) the sides of the excavation stood up perfectly at that angle but in the first excavation (tank 282) the position was not so satisfactory since a major slip occurred and there were symptoms of further ground movements requiring remedial action.

The surveying and inspection of the base of excavation, to ensure a clean gravel layer, was done by sounding by lead and line and direct inspection by aqualung diving.

Floating Bailey bridge units were installed so that lorries could tip directly into the water, the bridges being floated on pontoons at one end while the land end rested on a steel skid. In addition, some filling was placed directly by conveyor belts supported on pontoons. The fill placed had up to 9 per cent of material passing the No. 200 sieve. The inclusion of these finer materials was enforced by the nature of the borrow pit (Leggatt and Bratchell, 1973b).

The fill was compacted by vibrating probes (vibroflotation), the spacing of vibration centres being defined by a grid of equilateral triangles of 2.6m sides.

Check testing by penetrometer was carried out at regular centres and standard penetration tests were also carried out in numerous boreholes over the area. Fig. 7.2 shows typical results before and after compaction of the fill. The final density achieved was approximately 70% of the in situ density at the quarry which was very dense.
When the new foundations (Fig. 7.3) were ready, the tanks were floated in turn to their new positions. The necessary waterway for the flotation and voyaging of the tanks was formed by mini-bands. In spite of the relatively thin mild steel bottom plate, the tanks just lifted off in about 0.9m of water.

7.2 Field Measurements

The major event remaining was the water tests of the tanks and their foundations. This involved filling with sea-water giving an overload on the normal oil contents of about 20 per cent. This overload was held in the tanks for ten days.

The settlement measurement devices comprised steel joists installed within the fill with plates on their lower ends so that they stood on the natural gravel layer. Readings from tape soundings through holes in the roofs of the tanks, levels on the periphery and direct levels on the external settlement devices form the basis of the results presented in Table 7.1. The observed field results, due to a distributed loading of 188 kN/m², in the Table must be reduced by 9.5 per cent when compared with the finite element analyses in which a uniform surface pressure of 170 kN/m² was considered.

After unloading, there was a small amount of recovery, averaging about 12.7mm for tanks 282 and about 31.7mm for tank 281. Due to the short duration of the hydrotests, little of the long term settlement will have taken place and the values in Table 7.1 can be considered as corresponding to immediate settlements only.
7.3 Finite Element Solutions

The program was used to solve the problem twenty times, considering different values of the soil parameters and three different types (plane strain, plane stress and axi-symmetric) of analysis. The triangular mesh of Fig. 7.4 was used in all cases.

The centre and edge displacements, at the surface and bottom of the fill, are shown in Table 7.2 in which the types of analysis are also indicated. A block capital without subscript refers to plane strain, while subscripts 1 and 3 identify axi-symmetric and plane stress analysis, respectively. Subscript 2 identifies cases of axi-symmetric analysis in which the loaded area was assumed to be fixed in the horizontal direction.

In all cases the soil was assumed to be isotropic, possessing the Young's moduli and Poisson's ratios shown in Table 7.3 according to the notations in Fig. 7.3.

The bottom plate of the tanks was assumed to have zero stiffness (uniform surface loading) and the boundary conditions were taken as follows (see Fig. 7.4): no horizontal movement along centre line $\overline{AB}$, no movement along lines $\overline{BC}$ and $\overline{CD}$. A uniform surface pressure of 170 kN/m$^2$ was considered in all cases distributed over a circular area of radius equal to 39.6m or over a strip of width equal to 39.6m, depending on the type of analysis (see Table 7.2).

The ground surface movement is presented in Figs. 7.5 to 7.8, the centre line settlements in Figs. 7.9 and 7.10, and the edge line movement in Figs. 7.11 to 7.14. The
apparent slight discrepancies between the values in Table 7.2 and 7.3 and the values in Figs. 7.11 to 7.14 are explained by the fact that, in these figures, a vertical line at a distance of 0.6m from the edge line to the side of the centre line was considered rather than the edge line itself (see mesh in Fig. 7.4).

For the thickness of the compacted gravel, a value of 10m was assumed which is approximately the average for tanks 281 and 282.

Case L differs from case C in that $v = 0.15$ was assumed for the top layer of the clay to simulate some degree of consolidation.

7.4 Analysis of the Results

7.4.1 Comparison with Field Measurements

According to Fig. 7.1 the foundations were supposed to be similar for both tanks and the same method of construction has been used for the placement and compaction of the fill. The partial settlements due to deformation of the fill alone are about the same, as expected, but the total settlements (Table 7.1) and the amounts of recovery after unloading are quite different for the foundation of each tank.

It would appear that the clay beneath tanks 282 is twice as stiff as the clay beneath tank 281, according to the field measurements in Table 7.1, but the amount of recovery is 2.5 times greater for tank 281. This seems to suggest the existence of a rigid stratum at a lower depth beneath tank 282 but the proximity of both sites makes it unlikely.
This shows how difficult it is to predict the settlements, in a practical problem, due to insufficient knowledge about the geotechnical medium. Therefore, in these circumstances, the use of a very sophisticated material model in the F.E.M. program is not justified, it increases unnecessarily the cost of the solution and can give a misleading sense of confidence in the accuracy of the results.

It must be remembered that a uniform surface pressure of 170 kN/m\(^2\) was assumed in the FEM analyses while the observed settlements in Table 7.1 correspond to a uniform surface pressure of 188 kN/m\(^2\). Therefore, for comparison, the computed values must be multiplied by 188/170.

It appears that case A\(_1\) (Tables 7.2 and 7.3) corresponds approximately to the real case. The computed values are approximately equal to the average of the field values.

Figs. 7.5 to 7.14 also show that, in case A\(_1\): 
(a) The differential settlement centre-edge is of the same order of magnitude of the edge settlement;
(b) There is a small heave on the surface far from the loaded area;
(c) The variations with depth of the settlement along the edge line and the centre line follow the same pattern, the edge line settlement being half the centre line settlement at the same depth;
(d) The surface horizontal displacements are directed towards the centre line in the vicinity of the edge at which they reach a peak and directed outwards at points of the surface far from the loaded area.
After showing that eight out of nine methods (including the usually conservative Terzaghi and Peck basic method) underestimate the settlement in the gravel fill only, Bratchell, Leggatt and Simons (1974) suggest that the deformations occurring in the underlying Barton clay may have allowed greater settlements in the granular fill than otherwise would have been the case, if the fill had rested on a more rigid stratum. The surrounding soft alluvium may also have influenced the observed settlements for the fill only.

Simons, Rodrigues and Hornsby (1974) present the results of three finite element analyses to show that the presence of the alluvium and the Barton clay only increases the settlement of the fill by 5 or 6 per cent, depending on the soil parameters assumed.

The values in Table 7.2 corresponding to a variety of FEM analyses confirm that the presence of the alluvium and the Barton clay only increases the settlement of the fill by up to 9 per cent, the actual increase depending on the soil parameters assumed. Therefore, this cannot account for the fact that the observed settlements are very much larger than those predicted by the standard Terzaghi and Peck procedure (55%), in the present case.

7.4.2 Influence of Restraining Horizontal Movement of the Loaded Area

Comparison of cases A₁, B₁ and C₁ with cases A₂, B₂ and C₂, respectively, shows that:
(a) The influence of restraining horizontal movement of the loaded area is almost negligible;
(b) The centre settlements are reduced by less than eight per cent while the edge settlements are increased by less than five per cent. There is, therefore, a slight decrease in the differential settlement centre-edge;
(c) The displacements tend to be the same as the distance from the loaded area increases, if the points are not close to the centre line.

7.4.3 Influence of Type of Analysis

In the present case, the most suitable type of analysis (axi-symmetric) is automatically defined by the conditions of the problem (deformation of half-space under circular loaded area). In other cases the choice is not so obvious when a program for three-dimensional analysis is not available or one is not willing to use it due to greater storage needs and running times or when the problem does not fall into one of the standard types of analysis. For example, the program developed by the author could not solve, at this stage, with the same degree of accuracy, the problem of deformation of the same medium under distributed loading over a rectangular area of finite dimensions. Therefore, it is of interest to investigate the influence of the type of analysis, even though the conclusions from this particular problem cannot be readily generalized.
Other variables being kept constant, plane stress analysis \((A_3)\) yields much greater settlements than plane strain or axi-symmetric analysis, with very slight heave of the surface far away from the loaded area. The horizontal displacements are also greater in the vicinity of the edge.

Comparison of the results for plane strain \((A, B\text{ and } C)\) with the corresponding results for axi-symmetric analysis \((A_1, B_1\text{ and } C_1)\) shows that:

(a) Both centre settlements and edge settlements can be greater for plane strain analysis than for axi-symmetric analysis or vice-versa, depending on the values of the soil parameters;

(b) The partial settlement due to deformation of the fill is always about 15 per cent smaller in axi-symmetric analysis and the differential settlement centre-edge is also smaller;

(c) Plane strain analysis yields greater heave at a shorter distance from the loaded area, especially if Poisson's ratio is close to 1/2;

(d) Plane strain analysis yields greater horizontal displacements outside the loaded area, the converse being true in the vicinity of the edge;

(e) When there are several strata with different properties, the pattern of variation of settlements with depth is different.
7.4.4 Influence of Poisson's Ratio

If Poisson's ratio is decreased, other variables being kept constant, the effect will be, in the present problem:

(a) The settlement increases, simulating drainage;
(b) The heave outside the loaded area decreases or even disappears;
(c) The surface horizontal displacement decreases and may change sign increasing in magnitude when Poisson's ratio is further decreased. The horizontal displacements are directed outwards when $\nu = 0.5$ and towards the centre line when $\nu = 0.15$;
(d) The differential centre-edge settlement becomes comparatively smaller (compare A₁ and B with G₁ and J, respectively).

7.4.5 Influence of Heterogeneity

Some of the effects of heterogeneity can be noticed by comparing cases B₁ and J with E₁ and H, respectively, and summarized as follows:

(a) Heterogeneity decreases the differential settlement of the loaded area and the heave outside this area;
(b) The curve representing the surface settlements becomes steeper at the edge when the degree of heterogeneity increases, obviously producing stress concentration in this zone;
(c) If the soil is heterogeneous, the surface horizontal displacements have a maximum at the edge whatever the value of Poisson's ratio which determines the direction of the displacements; 

(d) If the soil is heterogeneous both the settlements and the horizontal displacements decrease more rapidly with depth, i.e. the contribution of the top layers for the total displacement is comparatively greater, as expected.

7.5 Conclusions

The program was used to study the behaviour of the foundations of two 100000 ton oil tanks and the results compared with field measurements. Then it was used for a comparative study of the influence of various factors on the values of the displacements.

From this study some conclusions can be drawn, such as:

(a) Although the foundations were supposed to be similar, the observed total settlements were quite different in each case;

(b) When the properties of engineering media, as in the present problem, are not known to a degree of great accuracy, the use of a too sophisticated material model in the FEM analysis is not justified;

(c) The computed settlements were approximately equal to the average of the observed displacements and showed that the settlement of the fill alone is only slightly increased by the presence of alluvium and Barton clay;
(d) The influence of restraining horizontal movement of the circular loaded area is to reduce slightly the differential centre-edge settlement, by increasing the edge settlement and reducing the centre settlement;

(e) When the same problem is analysed by different types of FEM analysis, the displacements yielded by plane stress analysis are much greater than those produced by plane strain and axi-symmetric analysis which are fairly similar;

(f) A reduction in the value of Poisson's ratio increases the settlements of the loaded area, reduces heave outside the loaded area and may change the direction, in addition to the magnitude, of the surface horizontal displacements;

(g) Heterogeneity of the soil reduces the differential centre-edge settlement but increases the curvature of the curve representing the surface settlements in the immediate vicinity of the edge which should be taken into account in the design of such structures as a raft, for instance.
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<tr>
<th>Description</th>
<th>Settlement, mm</th>
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<td>Perimeter</td>
<td></td>
</tr>
<tr>
<td><strong>Tank 281</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Fill</td>
<td>63*</td>
<td>32</td>
<td></td>
</tr>
<tr>
<td>Clay</td>
<td>184*</td>
<td>76</td>
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<td></td>
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<td><strong>Tank 282</strong></td>
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<tr>
<td>Fill</td>
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<td>Clay</td>
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<td>Total</td>
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</tbody>
</table>

* Deduced values
### TABLE 7.2 Displacements and Types of Finite Element Analyses (Applied pressure 170 kN/m²)

<table>
<thead>
<tr>
<th>Case</th>
<th>Settlement, mm</th>
<th>Horizontal Displacement, mm</th>
<th>Type of F.E.M. Analysis</th>
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</tr>
<tr>
<td>A₁</td>
<td>59</td>
<td>120</td>
<td>179</td>
</tr>
<tr>
<td>A₂</td>
<td>59</td>
<td>117</td>
<td>176</td>
</tr>
<tr>
<td>A₃</td>
<td>66</td>
<td>225</td>
<td>291</td>
</tr>
<tr>
<td>B</td>
<td>64</td>
<td>117</td>
<td>181</td>
</tr>
<tr>
<td>B₁</td>
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<td>B₂</td>
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<td>82</td>
<td>139</td>
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<tr>
<td>C</td>
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<td>246</td>
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<td>29</td>
<td>223</td>
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<td>C₂</td>
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<td>D₁</td>
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<td>G₁</td>
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<td>56</td>
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</tr>
<tr>
<td>H</td>
<td>11</td>
<td>51</td>
<td>62</td>
</tr>
<tr>
<td>I</td>
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<td>222</td>
<td>244</td>
</tr>
<tr>
<td>J</td>
<td>32</td>
<td>281</td>
<td>313</td>
</tr>
<tr>
<td>K</td>
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<td>50</td>
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* Loaded area was assumed fixed in the horizontal direction
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<th>YOUNG'S MODULUS, MN/m²</th>
<th>SETTLEMENT, mm</th>
<th>HORIZONTAL DISPLACEMENT, mm</th>
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<tr>
<td>A₁</td>
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<td>0.490</td>
<td>16</td>
<td>320</td>
</tr>
<tr>
<td>A₂</td>
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<td>0.490</td>
<td>16</td>
<td>320</td>
</tr>
<tr>
<td>A₃</td>
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<td>16</td>
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</tr>
<tr>
<td>B</td>
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</tr>
<tr>
<td>B₁</td>
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<td>0.150</td>
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<td>0.150</td>
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</tr>
<tr>
<td>C</td>
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<td>0.499</td>
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<td>640</td>
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<tr>
<td>C₁</td>
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<td>0.490</td>
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<tr>
<td>C₂</td>
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<tr>
<td>D₁</td>
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Typical Boreholes

<table>
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<th>Task 282</th>
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<tbody>
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<tr>
<td>14</td>
<td></td>
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</tr>
</tbody>
</table>

**Soil Layers:**

- **Fill:** Sand and gravel
- **Soft Alluvium:** Mostly grey silty clay. Very soft. Undrained shear strength 12-19 kN/m².
- **Gravel:** Medium dense sandy gravel
- **Barton Clay:** Undrained shear strength 48-96 kN/m²

**Peat Layers:**

**Groundwater Level:**

**Figure 7.4 Generalized Soil Profile**
FIG. 7.2 TYPICAL PENETRATION TESTS
Fig. 7.3 Cross-section of tank foundation

Assumed all gravel or clay
FIG. 7.7  GROUND LEVEL HORIZONTAL DISPLACEMENTS (A to C2)
FIG. 7.9 CENTRE LINE SETTLEMENTS (A to C_2)
FIG. 7.10 CENTRE LINE SETTLEMENTS (D₁ to M)
FIG. 7.11 EDGE LINE SETTLEMENTS (A to C_2)
FIG. 7.12  EDGE LINE SETTLEMENTS (D₁ to M)
FIG. 7.13  EDGE LINE HORIZONTAL DISPLACEMENTS (A to C2)
FIG. 7.14  EDGE LINE HORIZONTAL DISPLACEMENTS (D₁ to M)
8.1 Introduction

The full depth of the underpass taking Neasden Lane underneath the North Circular Road, in London, is formed by using a concrete diaphragm wall, with a nominal thickness of 0.61m and 13m deep, the base of excavation being about 8.5m below ground surface. Fig. 8.1 shows the site plan and also the position of inclinometers and reference points.

Fig. 8.2 shows a cross-section of the wall, the position of the instrumentation and four rows of ground anchors. Fig. 8.3 shows an elevation of the test panel with the positions of the anchors and their geometric and elastic characteristics. Fig. 8.4 shows the soil profile and the results of triaxial undrained compression tests on samples taken at various depths. London Clay extends to a depth of 28m, followed by the Woolwich and Reading beds.

All field measurements and all information about instrumentation included in this chapter have been kindly given to the author and his supervisor by Dr. J. B. Burland and Dr. G. C. Sills of the Building Research Station.

8.2 Instrumentation

8.2.1 Inclinometers

The type of inclinometer used was supplied by Soil Instruments Ltd., and is operated by a pendulum hanging freely inside the metre-long torpedo and a strain gauge measures the
angle between the pendulum and the fixed wheels of the torpedo. The instrument is lowered down an aluminium tube 13m deep and the horizontal movements of the top relative to the bottom were obtained with an accuracy of the order of 3mm.

A weak grout was used around the tube to reduce the influence of the installation of the tube on the movement recorded by the inclinometer.

Three inclinometers were installed in a plane approximately perpendicular to the diaphragm wall to measure horizontal deformation, one of them in the wall itself (Figs. 8.1 and 8.2).

8.2.2 Magnet Extensometers

The magnet extensometer used has been developed at the Building Research Station and was described by Burland, Moore and Smith (1972). Ring magnets are positioned in (horizontal or vertical) boreholes with a 76mm diameter PVC guide tube passing through them. A reed switch is introduced into the guide tube and, as it is moved into the field of the magnet, the switch is closed, operating a light or buzzer. An accuracy of 0.2mm in the location of the magnet can be obtained.

Three vertical magnet extensometers were installed to measure upward movement (Fig. 8.2).

8.2.3 Load Cells

These were standard vibrating wire load cells supplied by Soil Instruments Ltd. Three wires are placed in tension symmetrically in a hollow cylinder and the frequency of the wires changes when the loads applied to the ends of the cylinder change. Each load cell is calibrated against known
loads, before installation.

A load cell was placed on each of the eight anchors terminating in the test panel (Fig. 8.3).

8.2.4 Piezometers

Four pneumatic piezometers were placed in each of the three boreholes shown in Fig. 8.2, with an extra one as a check. The piezometers have a hinged valve that closes when the air pressure applied on one side is equal to the pore water pressure. A certain amount of trouble was experienced with valves becoming sticky and not closing clearly.

8.2.5 Survey

A precise survey was also carried out periodically by the Central Unit, Department of the Environment, using reference points. The farthest is located at about 60m behind the diaphragm wall (Fig. 8.1).

8.3 Field Measurements

Fig. 8.5 shows the horizontal movements of the tops of the inclinometers and of the reference points determined by the survey on the dates indicated in the same figure from November 1971 to April 1973. Fig. 8.6 shows separately the three components of the movements of the reference points, during the same period, and Fig. 8.7 shows similar results for the tops of the inclinometers.

The horizontal displacements, relative to the base assumed to be fixed, of points at one-metre intervals given by inclinometers I₁, I₂ and I₃, are presented in Figs. 8.8,
8.9 and 8.10, respectively. Some of these values are used to show typical horizontal deformations of the axes of the tubes of inclinometers I_1, I_2 and I_3 which are presented in Figs. 8.11, 8.12 and 8.13, respectively, for the same dates during the period from February 1972 to June 1973.

Fig. 8.14 shows the variations of loads in anchors during the first three months after installation. The behaviour of the bottom row of anchors after installation is not presented due to lack of available data. However, it is known that this pair of anchors was unloaded some time after installation on 18 October 1972 and reloaded on 6 December 1972.

8.4 Finite Element Solutions

8.4.1 Simulation of Sequential Excavation

Fig. 8.15 shows the triangular mesh used in the FEM analysis and Fig. 8.16 is a detail of the zone around the diaphragm wall with some relevant information which will be used here to explain how the program handles this specific type of problem.

Incidentally, note that each one of the four anchors has several linear elements and transmits its axial load to the soil through seven different nodes while, at the left end, the axial load is transmitted to the wall through a single node. An unnatural node numbering system has been used and the zone to be excavated has been divided into triangular elements in such a way that four levels of excavation (L_1, L_2, L_3, L_4 in Fig. 8.17) can be considered to simulate the progress of excavation.
Fig. 8.18a shows schematically how the sequential construction of embankments or excavations can be simulated in the FEM analysis (Desai and Abel, 1972).

For embankments, the loading for the first solution (first lift) is due to the self-weight of the first layer of embankment placed. For the \( i \)-th lift the loading is due to the self-weight of the \( i \)-th layer placed. The actual structure becomes different from lift to lift (as in the case of excavation) because, in addition to the common foundation, it includes all previous layers placed. The stiffness of the underlying layers may increase considerably during construction due to increased overburden pressure as successive overlying layers are placed.

The final displacements \( U \) and stresses \( \sigma \) corresponding to the end of the construction are given by

\[
\begin{align*}
U &= U_0 + \sum_{i=1}^{n} \Delta U_i \\
\sigma &= \sigma_0 + \sum_{i=1}^{n} \Delta \sigma_i
\end{align*}
\]

where \( \Delta U_i \) and \( \Delta \sigma_i \) are the incremental displacements and stresses, respectively, due to each successive lift of the embankment and \( U_0 \) is taken equal to zero if strains not associated with stresses (e.g. temperature) are disregarded.

The simulation of sequential excavation (Fig. 8.18b) is comparatively more difficult because the loads for each lift cannot be evaluated in so simple a manner as in the case of an embankment (loads due to self-weight are known with
reasonable accuracy). There are additional difficulties concerning the automation of the computer operations, forcing the increased use of the human factor, as will be apparent later. The main factor, however, is that the value of the coefficient of earth pressure at rest $K_0$ is not known with sufficient accuracy.

The loads to simulate sequential excavation are derived from the initial stresses within the soil medium and their changes due to removal of the previous layers. The initial total stresses are given by equations (3.153) which can be written as

$$
\bar{\sigma}_0 = \begin{bmatrix}
\sigma_{ox} \\
\sigma_{oy} \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
K_0(-\gamma y - u) + u \\
-\gamma y \\
0
\end{bmatrix}
$$

(8.2)

in which $\gamma$ is the bulk unit weight
$u$ is the pore water pressure
$-y$ is the depth (Fig. 8.16)

$$
K_0 = \frac{\sigma_H'}{\sigma_V'}
$$

is the coefficient of earth pressure at rest, $\sigma_H'$ and $\sigma_V'$ being the horizontal and vertical effective stresses, respectively.

The stresses given by (8.1) and (8.2) at the boundaries of a layer to be excavated are usually employed to find the loading for the FEM analysis corresponding to the excavation of that layer. The author uses a slightly different approach, avoiding the use of the stresses, although they are always evaluated and printed out. The main reason is that the
stresses are less accurate than the displacements when the formulation of the program is based on the finite element displacement method, especially at the points of contact of two materials with very different stiffnresses (e.g. concrete diaphragm wall in London Clay). The stresses at a node derived from the strains of the various elements incident on that node frequently have quite different values and the average is usually considered as the stress at that node (e.g. Hooper, 1974).

The method used by the author will be explained with an example (mentioned later in this chapter).

The loads corresponding to the excavation of the first layer are easy to evaluate, using (8.2), provided the variation of $K_o$ over the soil mass is known with reasonable accuracy. The consistent system of nodal forces used for the FEM analysis corresponding to the excavation of the first layer (1st lift) and stressing of the first anchor ($A_1$), according to Fig. 8.16, are shown in Table 8.1. It has been assumed that $K_o$ varied with depth as shown in Fig. 5.21, that $u = 0$ everywhere (due to the absence of reliable field data) and that the bulk unit weight was $\gamma = 20$ kN/m$^3$.

The self-equilibrating system of forces due to axial load in anchor $A_1$ was evaluated by routine SETANCHOR and the consistent system of forces due to excavation of the first layer was evaluated by routine SETLOAD. For practical reasons related to the computation of pressures applied by the soil mass to the wall, at any stage of excavation, a self-equilibrating
system of horizontal forces (corresponding to the initial stresses and assuming no disturbance caused by construction of the wall) is also applied to the exterior nodes of the diaphragm wall, after evaluation by routine SETLOAD using equations (8.2).

As a check of the values in Table 8.1 it can be verified that:

1. The vertical component of the resultant is equal to the self-weight of the first layer:
   \[ W_1 = 20 \times 2.85 \times 18 = 1026 \text{ kN} \] (see Fig. 8.16);

2. The horizontal component of the resultant is equal to the effect of releasing the surface distributed pressures on the sides of elements 2 and 4 of the wall.

Table 8.2 (column 6) shows the vertical components of the consistent system of nodal forces used in the FEM analysis corresponding to the excavation of the second layer and stressing of the second anchor, and indicates how these forces are obtained by adding up the values in columns 2 to 5. Since the procedure for the horizontal components is basically the same, only the vertical components will be dealt with in some detail.

The values in column 2 are derived by routine SETLOAD from the total stresses at the bottom of the second layer and are the vertical forces which should be applied to the remaining soil mass to simulate the excavation of the first two layers. The resultant is equal in magnitude to the self-weight of these layers (Fig. 8.16):

\[ W = 20 \times 4.85 \times 18 = +1746 \text{ kN}. \]
The values in columns 3 and 4 correspond to the partial release of initial stresses caused by the excavation of the previously overlying layers (only the first layer, in this case). The resultant is equal in magnitude to the self-weight of the first layer: \( W_1 = -20 \times 2.85 \times 18 = -1026 \) kN. This resultant is taken as negative because it balances, partially, \( W = +1746 \) kN.

Column 5 of Table 8.2 shows the self-equilibrating system of nodal forces due to stressing of anchor A2 evaluated by routine SETANCHOR. The sum of the values in column 6 is equal, in magnitude, to the self-weight of the (second) layer whose excavation will be simulated:
\[
W_2 = 20 \times 2 \times 18 = +720 \text{ kN.}
\]

The reason for the inclusion of column 4 is better understood by considering Fig. 8.19. After deformation due to loads shown in Table 8.1 (the vertical components applied to the top of the second layer are represented by scaled vectors in Fig. 8.19a), the second layer will be in equilibrium under two systems of consistent nodal forces: directly applied loads simulating the excavation of the first layer and corresponding reactions from the soil mass and the wall. These reactions are evaluated by routine NODEFORCE after the FEM analysis and the vertical components, in the present case, are represented by scaled vectors in Fig. 8.19b.

The value of the reaction found by the routine for a node, such as node 131, where some load (action) had been applied does not include the reaction to that nodal load.
Therefore, the true reaction at such a node is equal to the value determined by routine NODEFORCE plus the load (if any) applied to that node, in the same FEM analysis, after changing its sign (reaction = -action). In this case, at node 131, the true reaction is

$$Q_{131} = -38.188 + (-19) = -57.188 \text{ kN}.$$ 

Incidentally, note that the values in Table 8.2 are correct at least up to the third decimal place which would be impossible if all consistent loads had been derived directly from the stresses.

Summarizing, the excavation of a layer \( i \) is simulated by applying to its boundaries in contact with the remaining mass, the consistent forces corresponding to the initial stresses at those boundaries (evaluated by routine SETLOAD), after adding to their values the effect (evaluated by routine NODEFORCE) of partial release of initial stresses due to excavation of the previous \( i-1 \) layers.

8.4.2 Diagrams of Bending Moments and Shear Forces

The program, like any other formulation based on the finite element method, can only handle distributed forces by replacing them with the corresponding consistent system of nodal forces. Therefore, consistent nodal forces corresponding to the deformation are used to evaluate the bending moments, shear forces, axial forces and distributed pressures along the diaphragm wall. Thus it is of interest to see how these values, especially for bending moments and shear forces, compare with the exact values corresponding
to the actual distributed forces, in cases in which the exact solutions are known.

Fig. 8.20a shows a simply supported overhanging beam in equilibrium under a consistent system of nodal forces corresponding to the uniformly distributed loading also indicated. Fig. 8.20b shows that the nodal values (represented by *) of the bending moments lead, in practical terms, to the same diagram as the distributed loads. This is also true for the diagram of shear forces shown in Fig. 8.20c in which the dashed line corresponds to the consistent forces, if the average values are considered at the nodes. However, this method of averaging cannot be used for the end nodes and nodes corresponding to the reactions at which there is a sharp variation in shear force. A similar situation arises at the nodes of a diaphragm wall to which loaded anchors are connected and at the nodes at the same level as the base of excavation. It is, therefore, a matter of judgement how to draw the diagrams corresponding to the pairs of nodal values produced by the program.

Due to wall friction, when excavation is carried out on one side of a diaphragm wall, surface forces will appear acting along the surfaces of contact between soil and wall. These forces are directed upwards on the side of excavation and downwards on the opposite side. Thus there will be bending moments on the wall created by these excentric loads (about the neutral axis of the wall).
Fig. 8.21a shows a beam in equilibrium under a self-equilibrating arbitrary system of distributed horizontal and vertical surface forces. Fig. 8.21b shows the same cantilever beam in equilibrium under the corresponding system of consistent nodal forces.

Fig. 8.22 shows the diagrams of bending moments and shear forces for both systems of forces indicated in Fig. 8.21. The dashed lines correspond to the consistent nodal forces and the full lines (defined by the equations indicated) to the distributed forces. Fig. 8.22 shows that the diagrams which could be drawn by using the average of each pair of nodal values, as evaluated by the program, would lead to diagrams identical to those corresponding to the actual distributed surface forces.

From all previous results it can be concluded that the pairs of nodal values produced by the program for the bending moments and shear forces corresponding to consistent nodal forces (equivalent to distributed forces) can be used to draw the diagrams with a good degree of accuracy, provided that:

1. The average of the pairs of values produced by the computer for a node is taken as the bending moment at that node;
2. The average is also taken for the shear forces except where the values for the same node are very different in which case both values should be used to draw the diagram;
3. The end values must be taken always as zero for the bending moments diagram and for the shear forces diagram at the top of the wall (at the bottom there may be a shear component which should not be disregarded).

The previous considerations will be taken into account later when presenting diagrams of bending moments, shear forces and also axial forces.

8.4.3 Description of FEM Analyses

The study of the diaphragm wall of Neasden Lane was first intended as a search, by back analysis through the use of the program, for the soil parameters which would best fit the field measurements. Hopefully, the results of this study would help not only to understand the behaviour of this wall and the soil mass but also to provide suitable parameters which could be used for the prediction of movements in similar constructions in London Clay.

However, lack of time and the relative low speed of the computer ICL 1905F of the University of Surrey forced the author to reduce his scope and, therefore, the results presented herein are only a preliminary study which would be of interest to prosecute further, considering the amount of field measurements available for comparison with the computed solutions.

When analysing by the FEM method the deformations of media due to excavation, it is usually assumed that the final displacements and stresses do not depend on the number
of steps (or their order) considered in the excavation. Using the theorem of virtual work and various stress-strain relationships for the material, Ishihara (1970) has shown that the uniqueness of the solution depends highly on the material properties and requires, as a sufficient condition, that the material is linear, time-independent and elastic throughout the excavation. Therefore, the multiple-stage cutting process can be simulated by a single-step cutting if the medium is linear and elastic and time-independent, although the execution of excavation involves the change in the geometrical shape of the medium. This leads to considerable economy in the solution and the author's program is particularly suitable for this type of problem as routine SETLOAD evaluates automatically all loads to be used.

However, the existence of anchors in the present case, makes the solution dependent not only on the steps considered in the excavation but also on the time when the anchors are installed and loaded. Since there are four rows of anchors, even assuming that all material parameters are known and each anchor is stressed immediately after each layer is excavated, at least eight FEM analyses have to be carried out to arrive at the final states of stress and strain. This number would be increased if a nonlinear or time dependent constitutive material model was used.

If the material parameters are to be determined by back analysis, the number of FEM solutions mentioned above will be multiplied by the different sets of material parameters considered.
The author considered only three different sets of material parameters (Fig. 8.23) and used them in ten FEM analyses (Table 8.3). Values of Poisson's ratios close to the condition of no volume change and undrained Young's modulus increasing linearly with depth have been used. The most difficult assumption to make was the variation of $K_0$ with depth (Fig. 8.23).

It is difficult to measure $K_0$ in the field directly and an ingenious indirect method was developed by Skempton (1961) and used to estimate the in-situ stresses in the London Clay at Bradwell. This method was also applied by Bishop, Webb and Lewin (1965) at the site of the Ashford Common site. The author followed Cole and Burland (1972) in considering a linear decrease of $K_0$ with depth which lies approximately mid-way between the two sets of results found by the other authors mentioned above.

Since the geometry of the medium is the same when simulating the excavation of the first layer and the stressing of the first anchor, a single FEM analysis is enough to obtain the corresponding deformations (case A in Table 8.3). The same applies to the following stages of excavation. The values of displacements and stresses for case D, for example, were obtained by adding together the results of four different FEM analyses. The consistent system of loads used in the first of these analyses (case A) was presented in Table 8.1.
Cases E and F differ from B and D, respectively, in that a single-step excavation was considered. The horizontal and vertical displacements of the wall (Fig. 8.24) show that, in this case, the results are not very different. Thus it seems reasonable to consider the whole excavation and stressing of anchors to be executed simultaneously in a single stage, at least for these two particular aspects of the study:

1. To find by back analysis, the approximate values of material parameters which lead to results comparable with field measurements;

2. To investigate the relative influence of some relevant factors.

Cases G to J have been considered to investigate the influence of cross-anisotropy, the increase of the depth of excavation of the wall and the cross-section of the anchors. An additional purpose of these analyses was to test the use of the program on this type of problem.

After deformation, the diaphragm wall is in equilibrium under a system of consistent nodal forces applied by the anchors and the soil mass. Table 8.4 shows the accuracy of the values evaluated by the program for those forces and justifies their use to find the bending moments, shear forces, axial forces and distributed pressures, along the wall. The values of the resultants of the reaction forces applied to each lateral surface of the wall include some contribution from the distributed forces over the bottom surface. This contribution is approximately equal to the
value quoted for the mid-side node of the bottom in Table 8.4. The values of \( \theta \) give a measure of the friction assumed to be mobilized along the surface of contact between soil and wall, on the side opposite to the excavation.

It is of interest to note that, in spite of the limitations to the validity of the FEM analysis, it is clear that considerable friction forces are mobilized along the surfaces of contact between soil and wall. The soil tends to move upwards on the side of the excavation and downwards on the opposite side. The net effect is to create compression of the wall, in addition to bending moments.

No sliding was assumed of the soil along the surfaces of the wall but this is likely to occur near the free surface. For this reason, and others previously mentioned, the results presented in Table 8.4 and elsewhere in this chapter must be regarded as having primarily a qualitative importance.

Fig. 8.25 shows the variation of horizontal stresses along the surfaces of the wall in four of the cases described in Table 8.3. The initial stresses given by routine SETLOAD increase parabolically with depth due to the assumed linear variation of \( K_0 \). The curves corresponding to the four FEM analyses show some waving partly due to the actual variations corresponding to the deformation, partly due to the limitations of the assumption, in the program, that the distributed pressures vary.
quadratically over the sides of each triangular element. This assumption loses validity specially when there are sharp variations in pressures.

The curves also show some tension in the soil near the top of the wall which, in the field, could produce failure. This is a consequence of using a constitutive linear elastic material model.

In spite of the limitations mentioned previously, the curves in Fig. 8.25 show that:

1. The horizontal stresses in the soil behind the wall decrease when the depth of excavation increases, particularly for large depths of excavation but even so the horizontal pressures on the active force are fairly high;

2. The horizontal stresses on the passive side of the excavation are greater than the initial stresses immediately below the level of excavation;

3. Cross-anisotropy has very slight influence on the horizontal pressures (the curves for case H were not presented because they are almost coincident with those corresponding to case G).

The diagrams of bending moments, shear forces and axial forces along the wall (Fig. 8.26), for cases F, I and J, show that:

1. If the cross-section of the anchors is small (247mm²) the results of the FEM analysis are approximately equal to those obtained considering
the cross-section equal to zero (anchors with zero stiffness) as in case I. In computational terms this is very important. In fact, very small cross-section can lead to round off errors (due to ill-conditioning) such that it is impossible to solve the system of equations. When that occurs, one can make the cross-section equal to zero and obtain a fairly accurate solution. In these circumstances, the computed deformation of the wall will be slightly greater (Table 8.3), as expected;

2. The diaphragm wall is subject to considerable compression forces which are a maximum immediately above the level of excavation. These forces are partly due to the anchors and partly due to the friction along the surfaces of contact between soil and wall. The bottom of the wall is also under compression which is greater when the depth of excavation is greater;

3. The anchors reduce substantially the shear forces and bending moments on the wall but the diagrams differ from the usual shape assumed in simplified methods to design braced excavations.

The displacements and the principal stresses corresponding to case A are represented by scaled vectors in Figs. 8.27 and 8.28. It should be noted that the principal stresses
(evaluated as explained in chapter 3) are always changed in sign before being printed out and plotted. This is due to the convenience in representing the tension stresses (exceptional case in soils) by negative values.

It is interesting to notice how the stressing of a single anchor \( A_1 \) affects the distribution of stresses in the soil mass and causes the heave of the soil surface immediately behind the wall.

Figs. 8.29 to 8.36 show the displacements and principal stresses for cases E, F, G and H. From those figures it can be concluded that:

1. The soil movement towards the excavation has a horizontal component of considerable magnitude even far behind the wall;
2. If the depth of excavation is small, there is heave of the soil surface behind the wall for a certain distance but the heave disappears and the soil thus tends to move downwards when the depth of excavation increases. The friction caused by this movement (or, rather, this tendency to slide along the wall) gives a barrel-like shape to the deformed wall;
3. The ratio of the horizontal displacement of the wall to the average heave of the excavation increases as the excavation progresses;
4. The soil behind the wall tends to move horizontally, almost as a block up to the ends of the anchors, as the depth of excavation increases;

5. Cross-anisotropy, when $E_H > E_V$, reduces the magnitude of all displacements and also the orientation of the principal stresses, except near the base of excavation.

8.5 **Comparison with Field Measurements**

In the FEM analysis, only immediate settlements have been considered due to the choice of values for Poisson's ratio close to the condition of incompressibility. However, in the field there have been considerable long term movements during and after construction. Fig. 8.5, for instance, shows that the absolute movement of the top of the wall ($I_1$) eight months after construction (point 10) is about twice the movement which occurred during construction (point 7), and this movement itself includes some long term displacements.

The relative movements of the wall indicated by inclinometer $I_1$ (Fig. 8.11) confirm the indications given by the FEM analysis, namely that the top of the wall tends to be pulled away from the excavation causing bulging. This cannot be due exclusively to the stressing of the anchors since the deformations corresponding to dates E, F and G (Fig. 8.11) took place before any anchor was stressed. No field measurements disprove the author's
conviction that this is due to wall friction when the soil tends to move downwards on the active side of the excavation.

The upward vertical movement of the top of the wall \( I_1 \) during the early stages of the excavation also confirms the FEM analysis predictions. The considerable settlement of the top of the wall during the last stages of excavation and after construction is due to drainage of the water from the soil and long term movement. This movement accelerated during the summer of 1972 which is known to have been dry. Fig. 8.6 also confirms this point showing not only slowing down but even reversing of the vertical movement after the end of the summer.

The anchors were very effective in reducing the lateral movement (at least the short term movement) of the wall by pushing it into the soil mass, away from the excavation. This is shown by Fig. 8.11 (e.g. H and I, before and after stressing the second row of anchors) and Fig. 8.8 (by the end of November 1972 measurements were taken, on the same day, before and after restressing the fourth row of anchors).

Figs. 8.10 and 8.13 show how the wall is pushed by the anchors into the soil mass, especially by restressing anchors D in November 1972. This action generates horizontal movement of the soil far behind the wall in the opposite direction, showing horizontal compression of the soil mass in the zone of the anchors.
The FEM analyses show consistently an increase in the anchor loads as the excavation depth increased (although the results are not presented herein) except when the anchors had been assumed to have no stiffness (cross-section equal to zero). Fig. 8.14 shows that, in the field, there is a drop in the level of stress in the anchors which then increases soon after and then declines progressively. It appears that the loads in the anchors increase as the excavation progresses, until creep and other long term effect reverse the effect of the excavation.

Figs. 8.5 to 8.7 show a greater horizontal movement of the wall ($I_1$) and the soil immediately behind it ($I_2$) than the soil mass further away from the wall. Therefore, although the immediate effect of stressing the anchors is to create compression behind the wall, the long term net effect of all factors is to generate tension (or reduced compression) in that zone.

Taking into account all limitations of the simplified FEM analyses carried out in the present study, it appears that cases D and F (Table 8.3) are reasonably in agreement with field measurements corresponding to the end of the construction, except with respect to vertical displacements of the wall itself.

The horizontal movements of the top of the wall (cases D and F, in Table 8.3) are slightly smaller than the actual movements shown in Fig. 8.5 ($I_1$ and $I_2$). The relative
movement of the top with respect to the bottom (D and F in Fig. 8.24) is of the same order of the actual relative movement (case K in Fig. 8.11) although slightly smaller. The shape of the curve (K) corresponding to field measurements is comparable, especially if it is remembered that some long term movement has displaced the upper part of the wall towards the excavation.

Thus the displacements and stresses corresponding to the FEM analysis (case F) represented by scaled vectors in Figs. 8.31 and 8.32 give a reasonably good picture of the behaviour of the wall and the soil mass. The stresses in the wall are a maximum immediately above the level of excavation, the soil behind and beneath the wall moves horizontally towards the excavation, there is some surface settlement behind the wall for a considerable distance from the top of the wall, the horizontal component having about three times the magnitude of the vertical component, and the wall bulges above the base of excavation.

8.6 **Conclusions**

Field measurements obtained during and after the construction of the diaphragm wall of Neasden Lane have been presented. A limited number of FEM analyses has been carried out, some of them simulating sequential excavation, to compare with the field measurements and to investigate the influence of some factors on the behaviour of the soil mass and the wall itself.

Although the contents of this chapter are only a preliminary study, some conclusions can be drawn and summarized as follows:
(a) The program can handle this type of problem giving an accuracy of solution compatible with the degree of accuracy in the knowledge of the material parameters;

(b) At least during the first stages of the back analysis of a case history of braced excavation, it is not necessary to consider sequential construction and, therefore, the FEM solutions become simpler and more economical;

(c) If the cross-section of the anchors is so small that difficulties arise in the solution of the system equations (non positive definite stiffness matrix due to round off errors), the solution can be obtained assuming that the cross-section of the anchors is zero as the influence of this factor is negligible in such a case;

(d) The assumed vertical boundary behind the wall must be located at a considerable distance;

(e) The program can be used to draw reliable diagrams of bending moments, shear forces, axial forces and pressures along the wall;

(f) Cross-anisotropy ($E_H > E_V$) only reduces the magnitude of the displacements and changes the orientation of the principal stresses;

(g) The anchors reduce considerably the bending moments on the wall but not as much as assumed in simplified methods used in the design of braced walls;
(h) The wall bulges towards the excavation due to the action of the top anchors and to the friction along the surfaces of contact between the soil and the wall, causing a net compression;

(i) The four rows of anchors used for the diaphragm wall of Neasden Lane have proved very effective in reducing the short term movements;

(j) The ground behind the wall appears to have moved almost as a block in the zone of anchors, carrying movements back a long way behind the wall. The movements are perhaps greater than one would expect for an excavation of only 13m supported by a wall with four rows of anchors;

(k) The ground movements shown by the instrumentation and the FEM analyses suggest that some instrumentation should also have been placed outside the volume reinforced by the anchors to show how the displacements change outside the immediate vicinity of the wall and the anchors;

(l) The validity of these conclusions is obviously restricted by the values of material parameters used herein, namely fairly rigid concrete wall and heterogeneous soil with Young's modulus increasing linearly with depth and in a condition of near incompressibility;
(m) The compressive forces in the wall, due to wall friction and the vertical component of the anchor forces, reduce the tensile stresses on the passive face resulting from the bending moments. This could have important consequences in the design of the reinforcement.
TABLE 8.1 Loading for the First Lift

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<td>-653.080</td>
<td>0</td>
<td>411</td>
</tr>
</tbody>
</table>

* These values include the nodal forces due to axial force 199.280 kN in anchor \( A_1 \) \( (P_H = 187.267 \text{ kN}; P_V = -68.148 \text{ kN}) \).  

**HORIZONTAL COMPONENT OF THE RESULTANT** = -283.511 kN  
**VERTICAL COMPONENT OF THE RESULTANT** = +1026.000 kN
### TABLE 8.2 Evaluation of Vertical Components of Loading for Second Lift, Including Stressing of Second Anchor (A₂).

<table>
<thead>
<tr>
<th>Nodes (1)</th>
<th>VERTICAL COMPONENTS, kN</th>
<th> </th>
<th> </th>
<th> </th>
<th> </th>
<th> </th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Corresponding to Release of Initial Stresses (2)</td>
<td>Due to Excavation of Overlying Layers (3)</td>
<td>Derived from Loads Used for Previous Lift (4)</td>
<td>Due to Stressing of Second Anchor (5)</td>
<td>Total (6)</td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>32.333</td>
<td>-18.980</td>
<td>0</td>
<td>-</td>
<td>13.353</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>129.333</td>
<td>-75.947</td>
<td>0</td>
<td>-</td>
<td>53.386</td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>97.000</td>
<td>-56.854</td>
<td>0</td>
<td>-</td>
<td>40.146</td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>258.667</td>
<td>-151.842</td>
<td>0</td>
<td>-</td>
<td>106.825</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>129.333</td>
<td>-75.699</td>
<td>0</td>
<td>-</td>
<td>53.634</td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>258.667</td>
<td>-151.752</td>
<td>0</td>
<td>-</td>
<td>106.915</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>129.333</td>
<td>-74.382</td>
<td>0</td>
<td>-</td>
<td>54.451</td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>258.667</td>
<td>-150.127</td>
<td>0</td>
<td>-</td>
<td>108.550</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>97.000</td>
<td>-52.309</td>
<td>0</td>
<td>-</td>
<td>44.691</td>
<td></td>
</tr>
<tr>
<td>53</td>
<td>129.333</td>
<td>-66.499</td>
<td>0</td>
<td>-</td>
<td>62.834</td>
<td></td>
</tr>
<tr>
<td>77</td>
<td>64.667</td>
<td>-27.670</td>
<td>0</td>
<td>-</td>
<td>36.997</td>
<td></td>
</tr>
<tr>
<td>103</td>
<td>129.333</td>
<td>-41.754</td>
<td>0</td>
<td>-</td>
<td>87.579</td>
<td></td>
</tr>
<tr>
<td>135</td>
<td>32.334</td>
<td>-25.572</td>
<td>0</td>
<td>-</td>
<td>6.762</td>
<td></td>
</tr>
<tr>
<td>179</td>
<td>-</td>
<td>1.065</td>
<td>0</td>
<td>-79.884</td>
<td>-78.819</td>
<td></td>
</tr>
<tr>
<td>131</td>
<td>-</td>
<td>-38.188</td>
<td>-18</td>
<td>-</td>
<td>-57.188</td>
<td></td>
</tr>
<tr>
<td>763</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>6.657</td>
<td>6.657</td>
<td></td>
</tr>
<tr>
<td>783</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>13.314</td>
<td>13.174</td>
<td></td>
</tr>
<tr>
<td>827</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>13.314</td>
<td>13.314</td>
<td></td>
</tr>
<tr>
<td>877</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>13.314</td>
<td>13.314</td>
<td></td>
</tr>
<tr>
<td>929</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>13.314</td>
<td>13.314</td>
<td></td>
</tr>
<tr>
<td>987</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>13.314</td>
<td>13.314</td>
<td></td>
</tr>
<tr>
<td>1051</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>6.657</td>
<td>6.657</td>
<td></td>
</tr>
<tr>
<td>$E_P$</td>
<td>+1746.00</td>
<td>-1007.00</td>
<td>-19.000</td>
<td>0.000</td>
<td>729.000</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE 8.3 Parameters Used in FEM Analyses and Computed Movement of Top of Diaphragm Wall

<table>
<thead>
<tr>
<th>Case</th>
<th>Excavation m</th>
<th>Type of Soil Properties</th>
<th>Anchors</th>
<th>Movement of Top of Wall, mm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cross-Section, mm²</td>
<td>Axial Load, kN</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>A₁ A₂ A₃ A₄</td>
<td>P₁ P₂ P₃ P₄</td>
</tr>
<tr>
<td>A</td>
<td>0-2.85</td>
<td>T1</td>
<td>0 0 0 0</td>
<td>199 0 0 0</td>
</tr>
<tr>
<td>B</td>
<td>0-4.85</td>
<td>T1</td>
<td>247 0 0 0</td>
<td>199 234 0 0</td>
</tr>
<tr>
<td>C</td>
<td>0-6.85</td>
<td>T1</td>
<td>247 247 0 0</td>
<td>199 234 215 0</td>
</tr>
<tr>
<td>D*</td>
<td>0-9.00</td>
<td>T1</td>
<td>247 247 247 0</td>
<td>199 234 215 201</td>
</tr>
<tr>
<td>E</td>
<td>0-4.85</td>
<td>T1</td>
<td>247 0 0 0</td>
<td>199 234 0 0</td>
</tr>
<tr>
<td>F</td>
<td>0-9.00</td>
<td>T1</td>
<td>247 247 247 0</td>
<td>199 234 215 201</td>
</tr>
<tr>
<td>G</td>
<td>0-11.50</td>
<td>T1</td>
<td>247 247 247 247</td>
<td>199 234 215 201</td>
</tr>
<tr>
<td>H</td>
<td>0-11.50</td>
<td>T3</td>
<td>247 247 247 247</td>
<td>199 234 215 201</td>
</tr>
<tr>
<td>I</td>
<td>0-9.00</td>
<td>T1</td>
<td>0 0 0 0</td>
<td>199 234 215 201</td>
</tr>
<tr>
<td>J</td>
<td>0-4.85</td>
<td>T2</td>
<td>247 0 0 0</td>
<td>199 0 0 0</td>
</tr>
</tbody>
</table>

* These analyses were carried out by considering sequential excavation.
### TABLE 8.4 Self-Equilibrating System of Consistent Nodal Forces Applied to the Diaphragm Wall after Deformation

<table>
<thead>
<tr>
<th>Case</th>
<th>$F_x$ kN</th>
<th>$F_x'$ kN</th>
<th>$F_x''$ kN</th>
<th>$F_y$ kN</th>
<th>$F_y'$ kN</th>
<th>$F_y''$ kN</th>
<th>$\theta$ degrees</th>
<th>$\sum F_x$ kN</th>
<th>$\sum F_y$ kN</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-4848.354</td>
<td>4847.630</td>
<td>10.724</td>
<td>-176.064</td>
<td>181.012</td>
<td>-4.948</td>
<td>2</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>B</td>
<td>-4673.725</td>
<td>4665.110</td>
<td>8.615</td>
<td>-351.551</td>
<td>356.483</td>
<td>-4.932</td>
<td>4</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>C</td>
<td>-4298.686</td>
<td>4287.400</td>
<td>11.287</td>
<td>-614.119</td>
<td>618.214</td>
<td>-4.094</td>
<td>8</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>D</td>
<td>-3811.312</td>
<td>3797.457</td>
<td>13.856</td>
<td>-866.037</td>
<td>860.246</td>
<td>5.791</td>
<td>13</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>E</td>
<td>-4708.991</td>
<td>4685.327</td>
<td>23.664</td>
<td>-305.225</td>
<td>305.483</td>
<td>-0.258</td>
<td>4</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>F</td>
<td>-3928.220</td>
<td>3845.511</td>
<td>82.710</td>
<td>-649.652</td>
<td>593.915</td>
<td>55.736</td>
<td>9</td>
<td>0.001</td>
<td>-0.001</td>
</tr>
<tr>
<td>G</td>
<td>-2806.780</td>
<td>2582.612</td>
<td>224.168</td>
<td>-956.172</td>
<td>781.932</td>
<td>174.239</td>
<td>19</td>
<td>0.000</td>
<td>-0.001</td>
</tr>
<tr>
<td>H</td>
<td>-2873.090</td>
<td>2678.023</td>
<td>195.068</td>
<td>-985.581</td>
<td>818.460</td>
<td>167.121</td>
<td>19</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>I</td>
<td>-3824.067</td>
<td>3714.100</td>
<td>109.968</td>
<td>-689.483</td>
<td>619.727</td>
<td>69.756</td>
<td>10</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td>J</td>
<td>-3931.620</td>
<td>3847.135</td>
<td>84.485</td>
<td>-629.974</td>
<td>569.425</td>
<td>60.548</td>
<td>9</td>
<td>0.000</td>
<td>-0.001</td>
</tr>
</tbody>
</table>
FIG. 3.1 SITE PLAN WITH LOCATION OF REFERENCE POINTS AND INCLINOMETERS
Fig. 8.2 Cross-section of diaphragm wall showing the position of anchors and instrumentation.
ANCHORS:
Angle of inclination - 20°
Fixed length - 8.00m
Free length
A & A' - 11.00m
B & B' - 9.75m
C & C' - 8.75m
D & D' - 8.75m

ANCHOR TENDONS:
4 Low Relaxation Bridon strands per anchor
nominal diameter - 15.2mm
nominal area - 138.7mm²
breaking load - 247 kN
\( E_s = 197.9 \text{ kN/mm}^2 \)

FIG. 8.3 ELEVATION OF THE TEST PANEL WITH ANCHORS
<table>
<thead>
<tr>
<th>Depth, m</th>
<th>GROUND LEVEL</th>
<th>Stiff brown fissured silty clay with silt lenses; occasional white crystals and veins of grey clay (London Clay)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Su</td>
<td>x</td>
</tr>
<tr>
<td>1.85</td>
<td>20°</td>
<td>76</td>
</tr>
<tr>
<td>3.85</td>
<td>20°</td>
<td>x</td>
</tr>
<tr>
<td>5.85</td>
<td>20°</td>
<td>158</td>
</tr>
<tr>
<td>7.85</td>
<td>20°</td>
<td>172</td>
</tr>
<tr>
<td>9.00</td>
<td>BASE OF EXCAVATION</td>
<td>Stiff grey-blue fissured silty clay with occasional dustings of silt in fissures (London Clay)</td>
</tr>
<tr>
<td>13.00</td>
<td>0.61m</td>
<td>138</td>
</tr>
<tr>
<td></td>
<td>Triaxial compression test:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\phi_u = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$S_u$ in kN/m²</td>
<td></td>
</tr>
</tbody>
</table>

FIG. 8.4 SOIL PROFILE AND CROSS-SECTION OF WALL WITH ANCHORS
FIG. 8.6 MOVEMENT OF REFERENCE POINTS
FIG. 8.7 MOVEMENT OF TOPS OF INCLINOMETERS
Fig. 8.9 Relative horizontal displacements - inclinometer 12
Fig. 8.10 Relative Horizontal Displacements - Inclinometer 1
Figure 8.11 Relative Horizontal Deformation - Inclinometer I

- A, B, C, D - Anchors
- E = 16, 3.72
- F = 14, 4.72
- G = 13, 6.72
- H = 13, 6.72
- I = 14, 7.72
- J = 14, 7.72
- K = 15, 8.72
- L = 15, 8.72
- M = 16, 11.72
- N = 17, 11.72
- P = 18, 6.73

* after stressing anchors P
FIG. 8.13 RELATIVE HORIZONTAL DEFORMATION - INCLINOMETER I,3

- after stressing

anchors D

E = 16.3, 72
F = 14.9, 72
G = 13.5, 72
H = 13.6, 72
I = 14.3, 72
J = 14.9, 72
K = 15.2, 72
L = 28.11, 72
M = 15.12, 72
N = 18.6, 72
P = 18.6, 72

Direction of excavation

HORIZONTAL DISPLACEMENT, mm

DEPTH

0
1
2
3
4
5
6
7
8
9
10
11
12
13

I

P

F

G

O

M

K

E

P

N

anchor D

13.5

10

5

0

-5

-10
FIG. 8.17 PROGRESS OF EXCAVATION DURING 1972

A, B, C, D - ANCHORS FIRST STRESSED IN 1.6.72, 23.6.72, 24.7.72, 18.10.72, RESPECTIVELY.

L₁, L₂, L₃, L₄ - STAGES OF EXCAVATION CONSIDERED IN FINITE ELEMENT ANALYSIS.
\[ \sigma_1 = \sigma_0 + \Delta \sigma_1 \]

**lift 1**

**stress-free surface**

\[ \sigma_1 = \sigma_0 + \Delta \sigma_1 \]

**lift i**

**stress-free surface**

\[ \sigma_i = \sigma_0 + \sum \Delta \sigma_1 \]

**lift i**

**a) Embankment**

**b) Excavation**

**FIG. 8.18 Analytic Simulation of Sequential Construction**
(a) Loads Applied to Top of 2nd Layer in the First FEM Analysis (A)

(b) Vertical Reactions of Soil Mass and Wall to Deformation of 2nd Layer under Loads Considered in the First FEM Analysis (A)

FIG. 8.19 Vertical Components of Consistent Forces Corresponding to Release of Stresses Due to Excavation of the First Layer
Fig. 8.20 Diagrams of Bending Moments and Shear Forces for a Simply Supported Overhanging Beam.
a) Distributed forces

b) Consistent System of Nodal Forces

FIG. 8.21 Beam in Equilibrium under Two Equivalent Load Systems
FIG. 8.22 Diagrams of Bending Moments and Shear Forces for a Beam

a) Bending Moments

\[ M = -3x(x-5)(x-10) \]

b) Shear Forces

\[ T = 9x(x-10) \]
FIG. 8.23 Material Parameters Used in FEM Analysis
FIG. 8.24 Movement of the Wall (A to F)
FIG. 8.25 Horizontal Pressures on the Wall
FIG. 8.26 Diagrams of Bending Moments, Shear Forces and Axial Forces along the Diaphragm Wall
FIG. 8.28 Scaled Vectors of Principal Stresses (A)
FIG. 8.30 Scaled Vectors of Principal Stresses (E)
FIG. 8.34 Scaled Vectors of Principal Stresses (C)
Chapter 9

PRELIMINARY PARAMETRIC STUDY OF A DIAPHRAGM WALL

9.1 Introduction

Diaphragm walls, especially tied-back walls, have become popular as a support system for major urban excavations. They have also been used as load bearing foundations, for embankment dams, etc. A conference on "Diaphragm Walls and Anchorages" held recently in London (September 1974) has shown, by the number of participants and papers presented, the current importance of this subject in geotechnical engineering.

The case of a simple retaining wall and its interaction with a frictional material was treated by Coulomb (1776) and Rankine (1857). The theories developed have continued to be used with remarkable success up to the present time (1975) as the most common basis for design of retaining walls.

Empirical and semi-empirical techniques are generally used in the design of most of earth support systems other than retaining walls. Some of the first significant contributions to the analysis of flexible retaining structures were given by Terzaghi (1943, 1954).

Triangular and trapezoidal distributions of earth pressure are usually assumed to design a strutted or tied-back wall considered as an equivalent beam or as an elastic line (Peck, 1969). Various simplified methods have been proposed for the design of walls (Richart, 1960; Haliburton, 1968; Littlejohn et al., 1971; James and Jack, 1974) and, recently, the finite element method has been
used to analyse practical cases (Clough, Webber and Lamont, 1972; Barla and Mascardi, 1974).

The process of anchor installation and prestressing makes it difficult to predict from the classical theory the earth pressures acting on a tied-back wall and few data are available on the movements to be expected. However, the FEM can provide an analytical solution to this problem, if the construction process is properly simulated.

The FEM can also be of value in a parametric study to investigate the relative influence of relevant factors on the solution. The results presented in this chapter are a first step for such a parametric study. Since it is convenient to investigate the influence of other factors first, the existence of anchors is not considered herein. Sequential construction is also not considered and the pore water pressure is assumed to be zero, unless specifically stated otherwise.

9.2 Influence of Some Factors on the FEM Solution

9.2.1 Location of the Boundaries

Fig. 9.1 shows the triangular mesh $M_1$ with 379 elements and 822 nodes, Fig. 9.2 shows the material properties assumed (including pore water pressure) and Table 9.1 describes the FEM analyses carried out (A to E) to investigate the influence of the position of the assumed boundary conditions. In all five cases no horizontal movement was assumed along the centre line AB (Fig. 9.1) and no movement.
along the boundaries BC and CD. The soil was assumed to be heterogeneous and cross-anisotropic (Table 9.1). Plane strain solutions obtained for different positions of the boundaries, keeping all other factors unchanged were obtained and the corresponding displacements are presented in Figs. 9.3 to 9.7.

Comparison of cases A and D shows that the greater the assumed depth, the greater the movement, the ratio between them being of the order of two or more in some zones. However, the horizontal displacements of the upper part of the wall itself are only slightly different. It is of interest to note that the movement of the ground surface is considerable (Figs. 9.3 and 9.4) even at a distance in excess of ten times the depth of excavation. It should be noted, however, that the situation considered is rather extreme, i.e. deep excavation supported by a simple diaphragm wall. Nevertheless, the results make it clear that, in a practical problem, care must be exercised in assuming the position of the boundaries for the FEM analysis so that it is far enough away from the wall.

Comparison of cases A and B with E and C, respectively, shows that the movement, in general, increases substantially when the width of the excavation increases, even on the active side of the excavation at a great distance from the wall. However, the upward vertical displacement of the wall itself is smaller when the width of the excavation is greater which appears to be due to the compression of the wall by the settlement of the soil behind it (see Fig. 9.4).
Comparison of cases A and B shows that, if the position of the vertical boundary on the active side is located at a distance greater than about ten times the depth of excavation, the displacements in the vicinity of the wall are not much affected by the actual position of the boundary. When the boundary is located further away from the wall, the computed displacements are generally greater, as expected, but the settlements on the active side are smaller up to a certain distance. For the point on the ground surface whose displacements were assumed to be zero in case B, the computed movement in case A is about 20% of the maximum surface movement. This shows that the vertical boundary on the active side must be assumed, in the FEM analysis, at a distance greater than ten times the depth of excavation.

9.2.2 Poisson's Ratio

Using the triangular mesh shown in Fig. 9.1, the material parameters indicated in Fig. 9.2 and the geometry described in Table 9.1, four FEM analyses (F, G, H and I) were carried out to investigate the influence of Poisson's ratio. The computed displacements are displayed in Figs. 9.8 to 9.12. In all cases heterogeneous isotropic material was assumed, no horizontal movement along centre line AB, no movement along boundaries BC and CD, and plane strain analysis was considered.

The horizontal displacements decrease when Poisson's ratio decreases. When it becomes zero, the ground surface has no horizontal movement at a relatively short distance.
from the wall. The heave of the base of the excavation also increases when Poisson's ratio decreases but the settlement behind the wall decreases first and then changes sign increasing in magnitude. The deformed shape of the ground surface is determined by the value of Poisson's ratio assumed in the FEM analysis.

The wall itself rises unless Poisson's ratio is close to the condition of incompressibility. It appears that, because there can be no volume change, the soil behind the wall pushes it downwards by wall friction more than balancing the effect of the upward movement on the passive side.

9.2.3 Type of Analysis

Plane stress analysis was used in case F₁ (Table 9.1) to compare the results with the solution obtained for case F (plane strain), keeping all other factors unchanged.

The horizontal movement is considerably greater for the plane strain analysis while the vertical displacements are smaller on the passive side but greater on the active side. The shapes of the deformed surfaces are very similar, except near the top of the wall where the vertical displacements have opposite sign. The horizontal displacement of the bottom of the wall is greater, for the plane strain analysis, but the converse occurs at the top.

9.2.4 Number of Elements

Fig. 9.13 shows the triangular mesh M₂ used to investigate the influence of the number of elements (and, therefore, unknowns) on the accuracy of the computed solution.
Cases F, H and I (Table 9.1) were analysed, keeping all factors unchanged but the mesh.

The displacements of the wall corresponding to these three pairs of plane strain analyses are presented in Fig. 9.14 and are only very slightly different. The values obtained using mesh M2 (152 elements) are not always smaller than the displacements corresponding to mesh M1 (379 elements). However, this is no way in contradiction with the usual increase of the displacements when the number of unknowns increases. This is valid for every point only when there is a single force acting on the body, as mentioned in chapter 3.

9.2.5 Wall Friction

The friction along the surfaces of contact between the soil and the wall seems to play an important role in the deformations and stresses of a diaphragm wall, resulting from excavation. Since the program can solve this problem assuming either no friction or no sliding, it was considered worthwhile to solve the same problem keeping all factors unchanged but for the assumption concerning wall friction.

For this specific purpose mesh M3 was used, shown in Fig. 9.15, with two different node numbering systems: the former had single nodes on the surfaces of contact between the soil and wall, the latter had those nodes replaced by pairs of adjacent nodes, one belonging to the wall and the other to the soil. The first node numbering system (812 nodes in Fig. 9.15) can only be used if wall...
friction is implicitly assumed. The second node numbering system can be used to simulate both situations by calling routines EQUADISPLLOWT and EQUADISP. The imposition of the condition \( d_{px} - d_{qx} = 0 \) to the horizontal displacements of all pairs of adjacent nodes \( p \) and \( q \) will simulate the situation of no wall friction when the lateral surfaces of the wall are vertical. If the condition \( d_{py} - d_{qy} = 0 \) is also imposed, all pairs of adjacent points will have zero relative movement and no sliding can occur.

Table 9.2 describes the plane strain analyses A, B and C carried out to investigate the influence of wall friction. The variations of Young's modulus \( E_y \) and coefficient of earth pressure at rest \( K_0 \), with depth, are shown in Fig. 9.16. Actually, two analyses were carried out in case C which led to identical results, using both node numbering systems referred above (see Fig. 9.15). The movement of the wall is shown in Fig. 9.17.

Comparison of solutions B and C (Fig. 9.17) shows that wall friction (at least when Poisson's ratio of the heterogeneous isotropic soil is close to 0.5) reduces substantially not only the vertical movement of the wall but also the horizontal movement, resulting in greater curvature of the wall.

Comparison of solutions B and C also shows a limitation of the program due to the assumption of material model used, which, however, does not invalidate the conclusions that can be drawn from the results, in a qualitative sense. If there
is no wall friction, the wall must have a concave shape when viewed from the passive side. This is not shown in case B because, due to the deep excavation, tension is generated in the soil, near the top of the wall, and real soils may not be able to sustain this tension. However, this tension appears to be almost non existent in case A where the depth of excavation was taken smaller than the depth of penetration of the wall.

Comparison of the diagrams of bending moments, shear forces and axial forces on the wall, in cases B and C, shows that wall friction affects not only the bending moments and shear forces but also creates considerable compressive forces increasing with depth up to the level of excavation. This appears to be so important that it should not be disregarded in design. Obviously, the amount of wall friction that will be actually mobilized in the ground depends on the material properties but it also depends on the process of construction. As wall friction reduces the movement of the wall and the soil, as rough a surface as possible should be aimed at during construction.

9.2.6 Cross-Anisotropy

Table 9.2 and Fig. 9.16 describe the plane strain analyses carried out to investigate the influence of cross-anisotropy on the movement of the wall, using the triangular mesh $M_4$ shown in Fig. 9.19. This figure indicates a number of nodes that includes additional nodes.
reserved for anchors connecting the wall to any node in the soil mass. The author would have used this mesh for most of the parametric studies, if he had been able to go further on the subject of this chapter.

The vertical boundary CD on the active side was located at a distance equal to 12 times the depth of the wall and the horizontal boundary BC was placed at a depth equal to 4.5 times the depth of the wall. This appears to be a good compromise between economy of the solution and accuracy of the results, taking into account the experience gained in solving the previous problems of the same type mentioned in this thesis.

The node numbering system, having numbers available for nodes as yet non existent, leaves open the possibility of considering up to five anchors with free lengths connecting any pair of nodes in the mesh, without increasing the bandwidth. It also permits the simultaneous solutions corresponding to three different load systems (say, derived from three different laws of variation of $K_o$ with depth), using under 80 k in the computer ICL 1905F of the University of Surrey.

Incidentally, note that the program can solve problems in which zero Young's moduli (not just a small value, as usual in many programs) are assigned to some elements (e.g. removed by excavation) and this is the reason why some numbers can be left available for special purposes such as the inclusion of anchors whose geometry and position may not be known in advance.
Fig. 9.20 shows the distributed horizontal pressures on the wall corresponding to the values of $K_0$ represented by the three pairs of curves in Fig. 9.16. Fig. 9.21 shows the horizontal and vertical displacements of the wall corresponding to isotropic soil ($A_1$, $A_2$ and $A_3$) and cross-anisotropic soil ($B_1$, $B_2$ and $B_3$), keeping all other factors unchanged.

As it has been found before for other problems, even of different type, cross-anisotropy only reduces the magnitude of the displacements (when $E_H > E_V$) without changing the general pattern of deformations.

9.2.7 Coefficient of Earth Pressure at Rest $K_0$

Fig. 9.21 shows how the movement of the wall increases when the value of $K_0$ increases, whether the soil is isotropic or cross-anisotropic. However, the general pattern of deformation is very much the same, in spite of the different laws of variation assumed for $K_0$ near the ground surface.

The movement of the wall is caused by loads resulting from two sources:

1. Horizontal forces applied to the wall, corresponding to the in-situ horizontal stresses, which depend on $K_0$;

2. Vertical forces applied to the base of the excavation, equivalent to the overburden pressure, which are independent of $K_0$.

To investigate the influence of the vertical forces (and, therefore, of the horizontal forces depending on $K_0$), solution $A_0$ (Table 9.2) was obtained for comparison with $A_1$, $A_2$ and $A_3$. 
Fig. 9.21 shows the movement of the wall and the displacements of the soil mass are represented by scaled vectors in Fig. 9.22. The principal stresses corresponding to the same case $A_0$ are represented in Fig. 9.23 by scaled vectors which include the in-situ stresses. It is interesting to note that the movement is essentially a clockwise rotation about a point within the soil mass behind the wall.

The dashed line $(A_2 - A_0)$ in Fig. 9.21 represents the effect of the horizontal loads. These alone reduce the uplifting of the wall, reversing its tilt and producing a horizontal movement of the bottom of the wall which happens to be in this case equal to that due to vertical forces. The horizontal movement of the top of the wall is almost entirely due to the horizontal forces and this emphasizes the importance for the accuracy of the solution of the assumption made about the variation of $K_0$ with depth. Unfortunately, of all the relevant factors, this is perhaps the least known.

Figs. 9.24 to 9.26 show, respectively, the diagrams of bending moments, shear forces and axial forces on the wall for cases $A_1$, $A_2$ and $A_3$. The results produced by the program were plotted directly on those figures without any rounding off. They show that variations in $K_0$ of the type indicated in Fig. 9.16 only change the magnitude of the results without changing the general pattern.
The diagrams of bending moments were deliberately drawn as shown in Fig. 9.24 to emphasize two points:

1. Vertical forces due to wall friction cause bending moments which appear to be important and should not be disregarded in design, in addition to compressive forces;

2. If the results evaluated by the program, especially the bending moments, are used directly, the corresponding diagrams may be misleading, besides being incorrect, for example on the passive side.

The diagrams of bending moments (Fig. 9.27) drawn on the basis of the average of nodal values, as explained in chapter 8, have been shown to be the correct ones. Obviously, the curves could and should be smoothed out, although this was not done to emphasize the relation between Figs. 9.24 and 9.27.

Finally, to show the accuracy of the program in finding the consistent nodal forces acting on the wall after deformation, the resultants obtained in cases A₁, A₂ and A₃ are presented in Fig. 6.28. The system is obviously in equilibrium.

The pairs of nodal values evaluated by the program, in case A₂, for the bending moments, shear forces and axial forces, are presented in Table 9.3. The second pair of values for node 947 actually refers to a point outside the wall, immediately below its bottom. Therefore, all values must be zero if the system of consistent nodal forces
is in equilibrium. The computed values are so close to
zero that great confidence can be placed on the results
contained in Table 9.3.

9.3 **Conclusions**

A preliminary parametric study of a diaphragm wall
has been presented. In spite of the relative small number
of cases analysed, some conclusions can be drawn and are
summarized as follows:

(a) The vertical boundary on the active side,
    assumed in the FEM analysis, must be located
    at a distance of not less than ten times the
    depth of the wall and the lower horizontal
    boundary at a depth of about half that
distance;

(b) The magnitude and direction of the displacements
    depends greatly on Poisson's ratio, since it
    determines the shape of the deformed ground
    surface;

(c) Plane stress and plane strain analysis give
    similar patterns of deformation but, on the
    active side, greater displacements are
    computed when using plane strain analysis;

(d) In practical cases, a number of about 200
    elements in the FEM mesh appears to lead to
    an accuracy which would be only slightly
    improved by a much greater number of elements;
(e) Wall friction reduces the movement of the wall, generating bending moments and considerable compressive forces which should be considered in design;

(f) If $E_H > E_V$, cross-anisotropy reduces the magnitude of the displacements without changing the pattern of deformations;

(g) The contribution, to the deformation of the wall, of the vertical forces due to release of in-situ vertical stresses at the bottom of the excavation is to cause a clockwise rotation of the wall about a point within the soil mass behind the wall (assuming that the excavation takes place on the left hand side);

(h) The movement of the top of a wall in practice is almost entirely due to the horizontal forces applied to the wall which depend on the coefficient of earth pressure at rest $K_o$;

(i) The program gives very accurate values of bending moments, shear forces and axial forces along the wall, but the corresponding diagrams must be drawn using the average, for each node, of the pairs of computed values, as explained in chapter 8.
### TABLE 9.1 Assumed Properties and Location of Boundary Conditions

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<tr>
<th>CASE</th>
<th>GEOMETRY (see Fig. 9.2)</th>
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* Plane stress analysis
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* No wall friction was assumed in this case.

** Only vertical loads due to release of in-situ vertical stresses were considered.
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\( E(z) = E(0) + \lambda \cdot z \)

![Diagram showing parameters](image)

### Table: Material Parameters Used in FEM Analysis

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<th>Type</th>
<th>( v_{\text{RH}} )</th>
<th>( n )</th>
<th>( E_{\text{c}} )</th>
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**Legend:**
- \( D_R \): Base of Excavation
- \( D_L \): GWL
- \( H \): Depth
- \( N \): Width

**Formula:**
\( \sigma_{\text{f}} = k_s' (E(0) + \lambda_k \cdot z) \)
FIG. 9.5 Profiles of Horizontal Displacements (A to E)
FIG. 9.6 Profiles of Vertical Displacements (A to E)
FIG. 9.7 Movement of the Wall (A to E)
FIG. 9.11 Profiles of Vertical Displacements (F to I)
FIG. 9.12 Movement of the Wall (F to I)
FIG. 9.14 Computed Displacements of the Wall Corresponding to Different Triangular Meshes ($M_1$ and $M_2$)
FIG. 9.15 Triangular Mesh $M_3$ for FEM Analysis
FIG. 9.17 Movement of the Wall (A to C)
FIG. 9.18 Diagrams of Bending Moments, Shear Forces and Axial Forces (A to C)
FIG. 9.19 Triangular Mesh $M_4$ for FEM Analysis

SCALE: 0 10 20 30 40 50 m

NE = 262
NP = 610
G = 1166
FIG. 9.20 Initial Horizontal Pressures on the Wall ($A_1, A_2, A_3$)
FIG. 9.21 Movement of the Wall (A₀ to B₃)
FIG. 9.22 Scaled Vectors of Displacement ($A_o$)
FIG. 9.24 Computed Diagrams of Bending Moments on the Wall (A₁, A₂, A₃)

BENDING MOMENTS, kN · m

Key: 

A₁  A₂  A₃

DEPTH, m
FIG. 9.25 Computed Diagrams of Shear Forces on the Wall \((A_1, A_2, A_3)\)
FIG. 9.26 Computed Diagrams of Axial Forces on the Wall ($A_1$, $A_2$, $A_3$)
FIG. 9.27 Correct Diagrams of Bending Moments on the Wall (A₁, A₂, A₃)
<table>
<thead>
<tr>
<th>CASE</th>
<th>HORIZONTAL FORCES, kN</th>
<th>VERTICAL FORCES, kN</th>
<th>θ (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F'_x \times 10^3$</td>
<td>$F_x \times 10^3$</td>
<td>$\Sigma F_x \times 10^{-3}$</td>
</tr>
<tr>
<td>A₁</td>
<td>2.942127</td>
<td>-2.942125</td>
<td>2.797723</td>
</tr>
<tr>
<td>A₂</td>
<td>3.437401</td>
<td>-3.437397</td>
<td>3.838956</td>
</tr>
<tr>
<td>A₃</td>
<td>3.988694</td>
<td>-3.988690</td>
<td>3.926307</td>
</tr>
</tbody>
</table>

* $F'_x$ and $F'_y$ include the nodal components of the midside node of the triangular element at the bottom of the wall.

**FIG. 9.28** Resultants of the Consistent Forces Acting on the Wall, after Deformation ($A_1$, $A_2$, $A_3$)
Chapter 10

SUMMARY AND MAIN CONCLUSIONS

10.1 The Program

A program based on the finite element displacement method, entirely developed by the author and his supervisor, thoroughly tested and applied to the solution of a number of major geotechnical problems, has proved to be versatile, efficient and accurate. Therefore, it can be used with confidence to solve plane stress, plane strain and axisymmetric problems, and will be called RODSIM for identification purposes.

All element matrices were evaluated by an exact method of integration and a quadratic expansion was assumed for the displacements over the six node triangular element. These factors appear to be mainly responsible for the remarkable accuracy of the computed solutions.

The consideration of isotropic, cross-anisotropic and orthotropic material, with constant Poisson's ratios and Young's moduli varying linearly in any direction within each element, makes it possible to simulate a variety of material behaviour, especially after the facile introduction into the program of constitutive material models other than the linear elastic model. Incidentally, any improvements are considerably facilitated by the setting up of the program as a relatively small number of statements making use of a great number of routines.
Due to use of magnetic tape and disc backing store, large problems can be solved in a medium size computer, depending mainly on the bandwidth of the structure. This bandwidth can be reduced through the use of an algorithm to optimize the node numbering system which was also developed and a paper describing this work has been accepted for publication in a technical journal, Rodrigues (1975). This algorithm will be especially useful to an engineer unfamiliar with the finite element method but interested in using the program.

The program was shown to be particularly suitable to analyse diaphragm walls, with or without anchors, considering either wall friction or sliding along the surfaces of contact between the soil and the wall. The consistent system of loads to simulate sequential excavation is evaluated directly from the displacements corresponding to the previous excavations, without using the values of the stresses. This leads to great accuracy in the evaluation of the bending moments, shear forces, axial forces and pressures on the wall, after deformation.

10.2 Main Conclusions

The tests and applications of the program described in the present thesis proved that it satisfies the requirements mentioned in 1.2.1. From the analyses of four major geotechnical problems referred to in chapters 6 to 9, a number of conclusions can also be drawn, some of which may be summarized as follows:
(a) The use of the program has confirmed the validity of Gibson's Law which states that any surface loaded area on an incompressible, isotropic elastic half-space, in which the undrained Young's modulus increases from zero at the surface linearly with depth, will settle by an amount proportional to the pressure on that area, the surface settlement outside the loaded area being zero. This behaviour, similar to a Winkler spring model, also applies to a stratum of the same material (Gibson soil) resting on a smooth rigid base, the settlement of the loaded area then being independent of the depth of the stratum. The vertical settlement of the surface outside the loaded area is zero;

(b) The shape of the deformed surface of a semi-infinite half-space, due to surface loading, depends on the material properties of the medium, (and also on the roughness or smoothness and location of the boundaries assumed in the FEM analysis). Therefore, adequate information about the soil parameters in each practical case is indispensable for the correct design of a raft and the structure it supports;

(c) Since the boundaries have to be assumed at a finite distance, when solving any geotechnical problem by the finite element method, great care
must be exercised when choosing their location, which is always a compromise between the economy of the solution and the accuracy of the results. In the case of a diaphragm wall in London Clay, for instance, the vertical boundary on the active side of the excavation must be located at a distance of not less than ten times the depth of the excavation and the horizontal boundary at a depth equal to about half that distance;

(d) The influence of cross-anisotropy on the deformation of a medium in which \( E_H > E_V \) is to reduce the magnitude of the displacements without affecting the general pattern of deformation;

(e) The value of Poisson's ratio not only affects the magnitude of the displacements but also their direction;

(f) In the case of an excavation supported by a diaphragm wall, the effect of releasing the in-situ vertical stresses at the base of the excavation is to cause a rotation of the wall about a point within the soil mass, on the active side, the curvature of the wall and the horizontal movement of its top being caused, almost exclusively, by the release of in-situ horizontal stresses (and stressing of anchors,
if any). This emphasizes the necessity for the use of all possible information about the variation of the coefficient of earth pressure at rest $K_0$ and the pore water pressure $u$, with depth;

(g) The deformation and stresses of a diaphragm wall, especially if the soil is in a situation close to the condition of incompressibility (saturated), are very much affected by the existence of wall friction. If there is no sliding along the surfaces of contact between the soil and the wall, additional bending moments and considerable compressive forces will be induced in the wall by the downward movement of the soil behind it. This must be taken into account in design.

10.3 Further Work

In the development of the program a simple constitutive material model, i.e. a linear elastic material and static loading independent of time, was deliberately assumed. However, all possible care was taken to ensure the generality and efficiency of all routines, in such a way that any future improvements will be easily introduced into the basic program described herein, without major changes.

There is, obviously, room for improvement, especially in the area concerning the characterization of the material. The author will mention only a few of the improvements which could be considered in the near future:
1. To make minor alterations, not affecting the basic formulation nor even the efficiency and accuracy of the solution; for instance, in order to make the data preparation more familiar to an engineer used to other programs, some statements could be added to permit the use of a node numbering system including even and odd numbers (all natural numbers, not just odd numbers as the author has used to date). Thus, after reading the input data, the computer could modify, if necessary, the given values and converse modification would occur before outputting the results, the user being unaware of such operations;

2. To make it possible to solve dynamic problems. This was not considered in the description of the program presented in chapter 3 but the relevant expressions were included in the basic formulation of the finite element displacement method detailed in chapter 2. Two routines would have to be programmed: one to evaluate the element mass matrix, another to solve the system of equations (2.51) and produce the eigenvalues of the structure.

3. To program a routine for the evaluation of the element stiffness matrix of a one-dimensional friction element. The program can already simulate two extreme situations concerning the surface of contact between the soil and
the structure (e.g. diaphragm wall or raft), assuming no friction or no sliding. Due to the importance of the earth-structure interaction in a variety of geotechnical problems, it is of interest to simulate the existence of the interface in an intermediate and more realistic situation: for example, a combination of sliding with some friction;

4. To program a routine to evaluate the element stiffness matrix of a volume element for three-dimensional analysis. The author would choose the linear strain tetrahedron of ten nodes and would evaluate the element stiffness matrix by exact integration using tetrahedral coordinates and the procedure outlined in 3.2.3. Whether the tetrahedral, pentahedral, hexahedral or any other element is employed, the program is suitable to solve three-dimensional problems without any change other than the inclusion of the new routine which could be called ELSTIFMAT2. The assembly and solution of the system equations would be obtained by using NDF = 3 and ENP equal to the number of nodes of the volume element chosen.

5. To establish a nonlinear time-independent stress-strain relationship (nonlinear material model, of which the elastic-plastic model is a particular case) and then use an incremental procedure (increments in load), considering the nonlinear
model as a piecewise linear model. A hyperbolic stress-strain relationship, for instance, making use of a small number of material parameters derived from laboratory or field tests, appears to be particularly suitable from the point of view of computation;

6. To consider a constitutive model taking into account the time-dependency of displacements due to drainage of the water. Three-dimensional problems could be disregarded but the element stiffness matrix could be evaluated considering simultaneously the variables x, y, t (i.e. cartesian coordinates and time);

7. Even before introducing into the program any of the improvements mentioned above and especially after the easy evaluation of the stiffness matrix of a friction element (referred to in number 4), it would be of interest to carry further the parametric study of a diaphragm wall discussed in chapter 9.
REFERENCES


APPENDIX I

Auxiliary Calculations for Element Matrices in Plane Strain and Plane Stress Analysis

I.1 Expansions of matrices $C_{ij}$ in (3.115)

\[ C_{11} = D_{11} \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{bmatrix} + D_{13} \begin{bmatrix} 2a_1 b_1 & (a_2 b_1 + a_1 b_2) & (a_3 b_1 + a_1 b_3) \\ (a_1 b_2 + a_2 b_1) & 2a_2 b_2 & (a_3 b_2 + a_2 b_3) \\ (a_1 b_3 + a_3 b_1) & (a_2 b_3 + a_3 b_2) & 2a_3 b_3 \end{bmatrix} + D_{33} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} \]

\[ C_{12} = D_{12} \begin{bmatrix} a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} + D_{23} \begin{bmatrix} b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} \]

\[ C_{21} = D_{12} \begin{bmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_1 b_3 & a_2 b_3 & a_3 b_3 \end{bmatrix} + D_{13} \begin{bmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \\ a_1 b_3 & a_2 b_3 & a_3 b_3 \end{bmatrix} \]

\[ C_{22} = D_{12} \begin{bmatrix} a_1 a_2 & a_1 a_3 \\ a_2 a_3 & a_3 \end{bmatrix} + D_{13} \begin{bmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \end{bmatrix} + D_{23} \begin{bmatrix} b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} + D_{33} \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \]
\[ C_{22} = D_{22} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} + D_{23} \begin{bmatrix} 2a_1 b_1 & (a_1 b_2 + a_2 b_1) & (a_1 b_3 + a_3 b_1) \\ (a_1 b_2 + a_2 b_1) & 2a_2 b_2 & (a_2 b_3 + a_3 b_2) \\ (a_1 b_3 + a_3 b_1) & (a_2 b_3 + a_3 b_2) & 2a_3 b_3 \end{bmatrix} \]

\[ + D_{33} \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{bmatrix} \]
I.2 Expansion of matrix $\mathbf{R} = \hat{\mathbf{Q}}^T \mathbf{Q} \psi$ in (3.118)

\[
\mathbf{R} = \begin{bmatrix}
\psi_{11} & 0 & 0 \\
0 & \psi_{22} & 0 \\
0 & 0 & \psi_{33}
\end{bmatrix}
\begin{bmatrix}
Q_{1,1} & Q_{1,2} & Q_{1,3} \\
Q_{2,1} & Q_{2,2} & Q_{2,3} \\
Q_{3,1} & Q_{3,2} & Q_{3,3}
\end{bmatrix}
\begin{bmatrix}
\psi_{11} & 0 & 0 & \psi_{14} & 0 & \psi_{16} \\
0 & \psi_{22} & 0 & \psi_{24} & \psi_{16} & 0 \\
0 & 0 & \psi_{33} & 0 & \psi_{14} & \psi_{24}
\end{bmatrix}
\]

$R_{1,1} = Q_{1,1} \psi_{11}^2$

$R_{1,3} = Q_{1,3} \psi_{11} \psi_{33}$

$R_{1,5} = Q_{1,2} \psi_{11} \psi_{16} + Q_{1,3} \psi_{11} \psi_{14}$

$R_{2,1} = Q_{2,1} \psi_{11} \psi_{22}$

$R_{2,3} = Q_{2,3} \psi_{22} \psi_{33}$

$R_{2,5} = Q_{2,2} \psi_{16} \psi_{22} + Q_{2,3} \psi_{14} \psi_{22}$

$R_{3,1} = Q_{3,1} \psi_{11} \psi_{33}$

$R_{3,3} = Q_{3,3} \psi_{33}$

$R_{3,5} = Q_{3,2} \psi_{16} \psi_{33} + Q_{3,3} \psi_{14} \psi_{33}$

$R_{4,1} = Q_{1,1} \psi_{11} \psi_{14} + Q_{2,1} \psi_{11} \psi_{24}$

$R_{4,3} = Q_{1,3} \psi_{14} \psi_{33} + Q_{2,3} \psi_{24} \psi_{33}$

$R_{4,4} = Q_{1,1} \psi_{14}^2 + Q_{2,1} \psi_{14} \psi_{24} + Q_{1,3} \psi_{24}^2 + Q_{2,3} \psi_{24} \psi_{14}$

$R_{4,5} = Q_{1,2} \psi_{14} \psi_{16} + Q_{2,2} \psi_{14} \psi_{24} + Q_{1,3} \psi_{14} + Q_{2,3} \psi_{14} \psi_{24}$

$R_{4,6} = Q_{1,1} \psi_{14} \psi_{16} + Q_{2,1} \psi_{16} \psi_{24} + Q_{1,3} \psi_{14} + Q_{2,3} \psi_{24}$

$R_{5,1} = Q_{2,1} \psi_{11} \psi_{16} + Q_{3,1} \psi_{11} \psi_{14}$

$R_{5,2} = Q_{2,2} \psi_{16} \psi_{22} + Q_{3,2} \psi_{14} \psi_{22}$

$R_{5,3} = Q_{2,3} \psi_{16} \psi_{33} + Q_{3,3} \psi_{14} \psi_{33}$

$R_{5,4} = Q_{2,1} \psi_{14} \psi_{16} + Q_{3,1} \psi_{14} + Q_{2,2} \psi_{24} + Q_{3,2} \psi_{16} + Q_{3,3} \psi_{14}$

$R_{5,5} = Q_{2,2} \psi_{16} + Q_{3,2} \psi_{14} \psi_{16} + Q_{3,3} \psi_{14} + Q_{2,3} \psi_{14}$

$R_{5,6} = Q_{2,1} \psi_{16} + Q_{3,1} \psi_{14} \psi_{16} + Q_{2,3} \psi_{16} \psi_{24} + Q_{3,3} \psi_{14} \psi_{24}$
\[ R_{6,1} = Q_{1,1} \psi_{11} \psi_{16 + Q_{3,1}} \psi_{11} \psi_{24} \]
\[ R_{6,2} = Q_{1,2} \psi_{16} \psi_{22 + Q_{3,2}} \psi_{24} \psi_{22} \]
\[ R_{6,3} = Q_{1,3} \psi_{16} \psi_{33 + Q_{3,3}} \psi_{24} \psi_{33} \]
\[ R_{6,4} = Q_{1,1} \psi_{14} \psi_{16 + Q_{3,1}} \psi_{14} \psi_{24 + Q_{1,1}} \psi_{24} \psi_{16 + Q_{3,2}} \psi_{24}^{2} \]
\[ R_{6,5} = Q_{1,2} \psi_{16 + Q_{3,2}} \psi_{16} \psi_{24 + Q_{1,2}} \psi_{14} \psi_{16 + Q_{3,3}} \psi_{14} \psi_{24} \]
\[ R_{6,6} = Q_{1,1} \psi_{16 + Q_{3,1}} \psi_{16} \psi_{24 + Q_{3,3}} \psi_{24 + Q_{1,3}} \psi_{16} \psi_{24} \]
1.3 Auxiliary Integrals

\[ I_1 = \iint_{11} (E_i) dA = (9E_1 + 3E_2 + 3E_3) \ A/15 \]

\[ I_2 = \iint_{22} (E_i) dA = (3E_1 + 9E_2 + 3E_3) \ A/15 \]

\[ I_3 = \iint_{33} (E_i) dA = (3E_1 + 3E_2 + 9E_3) \ A/15 \]

\[ I_4 = \iint_{11} \psi_{22} (E_i) dA = (-2E_1 - 2E_2 - E_3) \ A/15 \]

\[ I_5 = \iint_{11} \psi_{33} (E_i) dA = (-2E_1 - E_2 - 2E_3) \ A/15 \]

\[ I_6 = \iint_{22} \psi_{33} (E_i) dA = (-E_1 - 2E_2 - 2E_3) \ A/15 \]

\[ I_7 = \iint_{11} \psi_{24} (E_i) dA = (14E_1 + 3E_2 + 3E_3) \ A/15 \]

\[ I_8 = \iint_{22} \psi_{14} (E_i) dA = (3E_1 + 14E_2 + 3E_3) \ A/15 \]

\[ I_9 = \iint_{33} \psi_{16} (E_i) dA = (3E_1 + 3E_2 + 14E_3) \ A/15 \]

\[ I_{10} = \iint_{11} \psi_{14} (E_i) dA = (3E_1 - 2E_2 - E_3) \ A/15 \]

\[ I_{11} = \iint_{11} \psi_{16} (E_i) dA = (3E_1 - E_2 - 2E_3) \ A/15 \]

\[ I_{12} = \iint_{22} \psi_{16} (E_i) dA = (-E_1 + 3E_2 - 2E_3) \ A/15 \]

\[ I_{13} = \iint_{22} \psi_{24} (E_i) dA = (-2E_1 + 3E_2 - E_3) \ A/15 \]

\[ I_{14} = \iint_{33} \psi_{14} (E_i) dA = (-E_1 - 2E_2 + 3E_3) \ A/15 \]

\[ I_{15} = \iint_{33} \psi_{24} (E_i) dA = (-2E_1 - E_2 + 3E_3) \ A/15 \]

\[ I_{16} = \iint_{14} \psi_{14} (E_i) dA = (8E_1 + 24E_2 + 8E_3) \ A/15 \]

\[ I_{17} = \iint_{16} \psi_{16} (E_i) dA = (8E_1 + 8E_2 + 24E_3) \ A/15 \]

\[ I_{18} = \iint_{24} \psi_{24} (E_i) dA = (24E_1 + 8E_2 + 8E_3) \ A/15 \]

\[ I_{19} = \iint_{14} \psi_{16} (E_i) dA = (8E_1 + 4E_2 + 8E_3) \ A/15 \]

\[ I_{20} = \iint_{14} \psi_{24} (E_i) dA = (8E_1 + 8E_2 + 4E_3) \ A/15 \]

\[ I_{21} = \iint_{16} \psi_{24} (E_i) dA = (8E_1 + 4E_2 + 8E_3) \ A/15 \]
When the Young's modulus is constant over the element, i.e. when \( E_1 = E_2 = E_3 = E \), then the previous integrals will simply be:

\[
\begin{align*}
I_1 & = I_2 = I_3 = A \\
I_4 & = I_5 = I_6 = -A/3 \\
I_7 & = I_8 = I_9 = -4A/3 \\
I_{10} & = I_{11} = I_{12} = I_{13} = I_{14} = I_{15} = 0 \\
I_{16} & = I_{17} = I_{18} = 8A/3 \\
I_{19} & = I_{20} = I_{21} = 4A/3 \\
\end{align*}
\]

I.4 Expansion of symmetric matrix \( \Omega = \Phi^T \Phi \) in (3.128)

\[
\begin{align*}
Q_{1,1} & = \xi_1^2 (2\xi_1-1)^2 \\
Q_{1,3} & = \xi_1 \xi_3 (2\xi_1-1) (2\xi_3-1) \\
Q_{1,5} & = 4\xi_1 \xi_2 \xi_3 (2\xi_1-1) \\
Q_{2,2} & = \xi_2^2 (2\xi_2-1) \\
Q_{2,4} & = 4\xi_1 \xi_2^2 (2\xi_2-1) \\
Q_{2,6} & = 4\xi_1 \xi_2 \xi_3 (2\xi_2-1) \\
Q_{3,4} & = 4\xi_1 \xi_2 \xi_3 (2\xi_3-1) \\
Q_{3,6} & = 4\xi_1 \xi_3^2 (2\xi_3-1) \\
Q_{4,5} & = 16\xi_1 \xi_2^2 \xi_3 \\
Q_{5,5} & = 16\xi_2^2 \xi_3 \\
Q_{6,6} & = 16\xi_2^2 \xi_3 \\
Q_{1,2} & = \xi_1 \xi_2 (2\xi_1-1) (2\xi_2-1) \\
Q_{1,4} & = 4\xi_1^2 \xi_2 (2\xi_1-1) \\
Q_{1,6} & = 4\xi_1 \xi_3^2 (2\xi_1-1) \\
Q_{2,3} & = \xi_2 \xi_3 (2\xi_2-1) (2\xi_3-1) \\
Q_{2,5} & = 4\xi_2^2 \xi_3 (2\xi_2-1) \\
Q_{3,3} & = \xi_3^2 (2\xi_3-1)^2 \\
Q_{3,5} & = 4\xi_2 \xi_3^2 (2\xi_3-1) \\
Q_{4,4} & = 16\xi_1^2 \xi_2^2 \\
Q_{4,6} & = 16\xi_1^2 \xi_2 \xi_3 \\
Q_{5,6} & = 16\xi_1^2 \xi_2 \xi_3 \\
\end{align*}
\]
I.5 Auxiliary integrals

(a) Taking $Q = \Phi^T \Phi$ in (3.128) and using formula

$$J_0 = \int_2 \xi_1^j \xi_2^j \, dl = \frac{1!1!1!}{(1+j+1)!} \ell$$

$J_1 = \int Q_{1,1} \, dl = \int Q_{2,2} \, dl = \int Q_{3,3} \, dl = 4 \ell/30$

$J_2 = \int Q_{1,2} \, dl = \int Q_{1,3} \, dl = \int Q_{2,3} \, dl = -\ell/30$

$J_3 = \int Q_{1,4} \, dl = \int Q_{1,6} \, dl = \int Q_{2,4} \, dl = \int Q_{2,5} \, dl = \int Q_{3,5} \, dl = \int Q_{3,6} \, dl = 2 \ell/30$

$J_4 = \int Q_{1,5} \, dl = \int Q_{2,6} \, dl = \int Q_{3,4} \, dl = -\ell/30$

$J_5 = \int Q_{4,4} \, dl = \int Q_{5,5} \, dl = \int Q_{6,6} \, dl = 16 \ell/30$

$J_6 = \int Q_{4,5} \, dl = \int Q_{4,6} \, dl = \int Q_{5,6} \, dl = 8 \ell/30$

(b) Taking $Q = \Phi^T \Phi$ in (3.135) and using formula

$$I_0 = \int \int \xi_1^i \xi_2^j \xi_3^k \, dA = \frac{1!1!1!}{(i+j+k+2)!} \frac{2A}{A}$$

$I_1 = \int Q_{1,1} \, dA = 6A/180$

$I_2 = \int Q_{1,2} \, dA = -A/180$

$I_3 = \int Q_{1,4} \, dA = 0$

$I_4 = \int Q_{1,5} \, dA = -4A/180$

$I_5 = \int Q_{4,4} \, dA = 32A/180$

$I_6 = \int Q_{4,5} \, dA = 16A/180$
I.6 Auxiliary Integrals and Expressions

(a) Expansion of $\psi^T C_1$ in (3.145):

$$\psi^T C_1 = \begin{bmatrix} \psi_{11} & 0 & 0 \\ 0 & \psi_{22} & 0 \\ 0 & 0 & \psi_{33} \\ \psi_{14} & \psi_{24} & 0 \\ 0 & \psi_{16} & \psi_{14} \\ \psi_{16} & 0 & \psi_{24} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \psi_{11} C_1 \\ \psi_{22} C_2 \\ \psi_{33} C_3 \\ \psi_{14} C_1 + \psi_{24} C_2 \\ \psi_{16} C_2 + \psi_{14} C_3 \\ \psi_{16} C_1 + \psi_{24} C_3 \end{bmatrix}$$

$$E \xi = E_1 \xi_1 + E_2 \xi_2 + E_3 \xi_3$$

$$\hat{T} = \xi_1 (2 \xi_1 - 1) T_1 + \xi_2 (2 \xi_2 - 1) T_2 + \xi_3 (2 \xi_3 - 1) T_3 + 4 \xi_1 \xi_2 T_4 + 4 \xi_2 \xi_3 T_5 + 4 \xi_3 \xi_1 T_6$$

(b) Auxiliary integrals to evaluate (3.146) and (3.147):

$$I_1 = \iint \psi_{11}(E \xi) (\hat{T}) dA = [(18 T_1 - 5T_2 - 5T_3 + 24T_4 + 4T_5 + 24T_6) E_1$$
$$+ (3T_1 - 6T_2 - 8T_3 + 8T_4 + 8T_5 + 8T_6) E_2$$
$$+ (3T_1 - T_2 - 6T_3 + 4T_4 + 4T_5 + 4T_6) E_3] A/180$$

$$I_2 = \iint \psi_{22}(E \xi) (\hat{T}) dA = [(-6T_1 + 3T_2 - 3T_3 + 24T_4 + 4T_5 - 8T_6) E_1$$
$$+ (-5T_1 + 18T_2 - 5T_3 + 24T_4 + 24T_5 + 4T_6) E_2$$
$$+ (-T_1 + 3T_2 - 6T_3 + 4T_4 + 8T_5 + 8T_6) E_3] A/180$$

$$I_3 = \iint \psi_{33}(E \xi) (\hat{T}) dA = [(-6T_1 - T_2 + 3T_3 - 8T_4 + 4T_5 + 8T_6) E_1$$
$$+ (-T_1 - 6T_2 + 3T_3 - 8T_4 + 8T_5 + 4T_6) E_2$$
$$+ (-5T_1 - 5T_2 + 18T_3 + 4T_4 + 24T_5 + 24T_6) E_3] A/180$$

$$I_4 = \iint \psi_{24}(E \xi) (\hat{T}) dA = [(6T_1 - 2T_2 - 2T_3 + 12T_4 + 4T_5 + 12T_6) E_1$$
$$+ (-T_3 + 8T_4 + 4T_5 + 4T_6) E_2$$
$$+ (-T_2 + 4T_4 + 4T_5 + 8T_6) E_3] A/45$$
\[ I_5 = \int \int \psi_{14}(E \xi) (\xi T) dA = \left[ (-T_3 + 8T_4 + 4T_5 + 4T_6)E_1 \\
+ (-2T_1 + 6T_2 - 2T_3 + 12T_4 + 12T_5 + 4T_6)E_2 \\
+ (-T_1 + 4T_4 + 8T_5 + 4T_6)E_3 \right] A/45 \]

\[ I_6 = \int \int \psi_{16}(E \xi) (\xi T) dA = \left[ (-T_2 + 4T_4 + 4T_5 + 8T_6)E_1 \\
+ (-T_1 + 4T_4 + 8T_5 + 4T_6)E_2 \\
+ (-2T_1 - 2T_2 + 6T_3 + 4T_4 + 12T_5 + 12T_6)E_3 \right] A/45 \]

If the Young's modulus is constant, i.e. \( E_1 = E_2 + E_3 = E \), and the temperature varies linearly, then

\[ I_1 = EAT_1/3 \]
\[ I_2 = EAT_2/3 \]
\[ I_3 = EAT_3/3 \]
\[ I_4 = (2T_1 + T_2 + T_3)EA/3 \]
\[ I_5 = (T_1 + 2T_2 + T_3)EA/3 \]
\[ I_6 = (T_1 + T_2 + 2T_3)EA/3 \]

If the Young's modulus is constant and the temperature varies quadratically:

\[ I_1 = (2T_1 - T_2 - T_3 + 3T_4 - T_5 + 3T_6)EA/15 \]
\[ I_2 = (-T_1 + 2T_2 - T_3 + 3T_4 + 3T_5 - T_6)EA/15 \]
\[ I_3 = (-T_1 - T_2 + 2T_3 - T_4 + 3T_5 + 3T_6)EA/15 \]
\[ I_4 = (2T_1 - T_2 - T_3 + 8T_4 + 4T_5 + 8T_6)EA/15 \]
\[ I_5 = (-T_1 + 2T_2 - T_3 + 8T_4 + 8T_5 + 4T_6)EA/15 \]
\[ I_6 = (-T_1 - T_2 + 2T_3 + 4T_4 + 8T_5 + 8T_6)EA/15 \]

If both the Young's Modulus E and the temperature variation T are constant:

\[ I_1 = I_2 = I_3 = EAT/3 \]
\[ I_4 = I_5 = I_6 = 4EAT/3 \]
APPENDIX II

Auxiliary Calculations for Element Matrices in Axi-Symmetric Analysis

II.1 Auxiliary integrals for the evaluation of $KA$ in (3.170).

(a)

\begin{align*}
J_1 &= \int \int \xi_1 dA = A/30 \\
J_2 &= \int \int \xi_2^2 dA = A/6 \\
J_3 &= \int \int \xi_1 \xi_2 dA = A/12 \\
J_4 &= \int \int \xi_3^3 dA = A/10 \\
J_5 &= \int \int \xi_1^2 \xi_3 dA = A/30 \\
J_6 &= \int \int \xi_1^4 dA = A/15 \\
J_7 &= \int \int \xi_2^3 \xi_3 dA = A/60 \\
J_8 &= \int \int \xi_1^2 \xi_2^2 dA = A/90 \\
J_9 &= \int \int \xi_3^2 \xi_2^3 dA = A/180 \\
J_{10} &= \int \int \xi_1^5 dA = A/21 \\
J_{11} &= \int \int \xi_1^4 \xi_2^2 dA = A/105 \\
J_{12} &= \int \int \xi_3^2 \xi_2^3 dA = A/210 \\
J_{13} &= \int \int \xi_1^3 \xi_2^3 \xi_3 dA = A/420 \\
J_{14} &= \int \int \xi_1^2 \xi_2^2 \xi_3^2 dA = A/630
\end{align*}

(b)

\begin{align*}
G_1 &= \int \int (E \xi) x dA = (E_1 x_1 + E_2 x_2 + E_3 x_3) J_2 + (E_1 x_2 + E_2 x_1 + E_1 x_3 + E_3 x_1 + E_2 x_3 + E_3 x_2) J_3 \\
G_2 &= \int \int \xi_1 (E \xi) x dA = E_1 x_1 J_4 + (E_2 x_2 + E_3 x_3 + E_1 x_2 + E_2 x_1 + E_1 x_3 + E_3 x_1) J_5 + (E_2 x_3 + E_3 x_2) J_7 \\
G_3 &= \int \int \xi_2 (E \xi) x dA = E_2 x_2 J_4 + (E_1 x_1 + E_3 x_3 + E_1 x_2 + E_2 x_1 + E_2 x_3 + E_3 x_2) J_5 + (E_1 x_3 + E_3 x_1) J_7 \\
G_4 &= \int \int \xi_3 (E \xi) x dA = E_3 x_3 J_4 + (E_2 x_2 + E_1 x_1 + E_3 x_2 + E_2 x_3 + E_1 x_3 + E_3 x_1) J_5 + (E_2 x_1 + E_1 x_2) J_7 \\
G_5 &= \int \int \xi_1^2 (E \xi) x dA = E_1 x_1 J_6 + (E_2 x_2 + E_3 x_3) J_8 + (E_1 x_2 + E_2 x_1 + E_1 x_3 + E_3 x_1) J_7 \\
&\quad + (E_2 x_3 + E_3 x_2) J_9 \\
G_6 &= \int \int \xi_2^2 (E \xi) x dA = (E_1 x_1 + E_3 x_3) J_6 + E_2 x_2 J_6 + (E_1 x_2 + E_2 x_1 + E_2 x_3 + E_3 x_2) J_7 \\
&\quad + (E_1 x_3 + E_3 x_1) J_9
\end{align*}
\[ G_7 = \int \psi_1^2 (E\xi) x dA = \left( E_1 x_1 + E_2 x_2 \right) J_8 + E_3 x_3 J_6 + \left( E_1 x_2 + E_2 x_1 \right) J_9 + \left( E_1 x_3 + E_3 x_1 + E_2 x_2 + E_3 x_2 \right) J_7 \]
\[ G_8 = \int \psi_2^2 (E\xi) x dA = \left( E_1 x_1 + E_2 x_2 \right) J_7 + \left( E_3 x_3 + E_1 x_3 + E_2 x_2 + E_3 x_2 \right) J_9 + \left( E_1 x_2 + E_2 x_1 \right) J_8 \]
\[ G_9 = \int \psi_3^2 (E\xi) x dA = \left( E_1 x_1 + E_3 x_3 \right) J_7 + \left( E_2 x_2 + E_1 x_2 + E_2 x_1 + E_3 x_2 \right) J_9 + \left( E_1 x_3 + E_3 x_1 \right) J_8 \]
\[ G_{10} = \int \psi_4^2 (E\xi) x dA = \left( E_2 x_2 + E_3 x_3 \right) J_7 + \left( E_1 x_1 + E_2 x_1 + E_2 x_1 + E_1 x_3 + E_3 x_1 \right) J_9 + \left( E_2 x_3 + E_3 x_2 \right) J_8 \]

\[ I_1 = \int \psi_1^2 (E\xi) x dA = 16G_5 - 8G_2 + G_1 \]
\[ I_2 = \int \psi_2^2 (E\xi) x dA = 16G_6 - 8G_3 + G_1 \]
\[ I_3 = \int \psi_3^2 (E\xi) x dA = 16G_7 - 8G_4 + G_1 \]
\[ I_4 = \int \psi_4^2 (E\xi) x dA = 16G_8 - 4G_3 \]
\[ I_5 = \int \psi_5^2 (E\xi) x dA = 16G_9 - 4G_4 \]
\[ I_6 = \int \psi_6^2 (E\xi) x dA = 16G_5 - 4G_2 \]
\[ I_7 = \int \psi_7^2 (E\xi) x dA = 16G_6 - 4G_2 - 4G_3 + G_1 \]
\[ I_8 = \int \psi_8^2 (E\xi) x dA = 16G_9 - 4G_2 - 4G_4 + G_1 \]
\[ I_9 = \int \psi_9^2 (E\xi) x dA = 16G_{10} - 4G_3 - 4G_4 + G_1 \]
\[ I_{10} = \int \psi_{10}^2 (E\xi) x dA = 16G_6 - 4G_3 \]
\[ I_{11} = \int \psi_{11}^2 (E\xi) x dA = 16G_{10} - 4G_4 \]
\[ I_{12} = \int \psi_{12}^2 (E\xi) x dA = 16G_6 - 4G_2 \]
\[ I_{13} = \int \psi_{13}^2 (E\xi) x dA = 16G_{10} - 4G_3 \]
\[ I_{14} = \int \psi_{14}^2 (E\xi) x dA = 16G_7 - 4G_4 \]
\[ I_{15} = \int \psi_{15}^2 (E\xi) x dA = 16G_9 - 4G_2 \]
\[ I_{16} = \int \psi_{16}^2 (E\xi) x dA = 16G_6 \]
\[ I_{17} = \int \psi_{17}^2 (E\xi) x dA = 16G_{10} \]
\[ I_{18} = \int \psi_{18}^2 (E\xi) x dA = 16G_8 \]
\[ I_{19} = \int \psi_{19}^2 (E\xi) x dA = 16G_7 \]
\[ I_{20} = \int \psi_{20}^2 (E\xi) x dA = 16G_9 \]
\[ I_{21} = \int \psi_{21}^2 (E\xi) x dA = 16G_5 \]
\[ \Gamma^T Q \psi = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \end{bmatrix} \begin{bmatrix} \psi_{11} & 0 & 0 & \psi_{14} & 0 & \psi_{16} \\ 0 & \psi_{22} & 0 & \psi_{24} & 0 & \psi_{16} \\ 0 & 0 & \psi_{33} & 0 & \psi_{14} & \psi_{24} \end{bmatrix} \]

\[ = \begin{bmatrix} Q_1 \gamma_{11} & Q_2 \gamma_{12} & Q_3 \gamma_{13} & Q_1 \gamma_{14} & Q_2 \gamma_{15} & Q_3 \gamma_{16} \\ Q_1 \gamma_{21} & Q_2 \gamma_{22} & Q_3 \gamma_{23} & Q_1 \gamma_{24} & Q_2 \gamma_{25} & Q_3 \gamma_{26} \\ Q_1 \gamma_{31} & Q_2 \gamma_{32} & Q_3 \gamma_{33} & Q_1 \gamma_{34} & Q_2 \gamma_{35} & Q_3 \gamma_{36} \end{bmatrix} \]

\[ \psi^T Q^T = (\Gamma^T Q \psi)^T \]

\[ \Gamma^T \Gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_2 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_3 & \gamma_3 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_4 & \gamma_4 & \gamma_4 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_5 & \gamma_5 & \gamma_5 & \gamma_5 & \gamma_5 & \gamma_6 \\ \gamma_6 & \gamma_6 & \gamma_6 & \gamma_6 & \gamma_6 & \gamma_6 \end{bmatrix} \]

\[ \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\ r_2 & r_3 & r_4 & r_5 & r_6 \\ r_3 & r_4 & r_5 & r_6 \\ r_4 & r_5 & r_6 \\ r_5 & r_6 \\ r_6 \end{bmatrix} \] symmetric
II.3 Auxiliary integrals to evaluate $K_B$ in (3.168) and (3.170).

\[ G_{11} = \int \int (E \xi) \, dA = (E_1 + E_2 + E_3) J_1 \]
\[ G_{12} = \int \int \xi_1 (E \xi) \, dA = E_1 J_{12} + (E_2 + E_3) J_3 \]
\[ G_{13} = \int \int \xi_2 (E \xi) \, dA = E_2 J_{12} + (E_1 + E_3) J_3 \]
\[ G_{14} = \int \int \xi_3 (E \xi) \, dA = E_3 J_{12} + (E_1 + E_2) J_3 \]
\[ G_{15} = \int \int \xi_1^2 (E \xi) \, dA = E_1 J_{45} + (E_2 + E_3) J_5 \]
\[ G_{16} = \int \int \xi_2^2 (E \xi) \, dA = E_2 J_{45} + (E_1 + E_3) J_5 \]
\[ G_{17} = \int \int \xi_3^2 (E \xi) \, dA = E_3 J_{45} + (E_1 + E_2) J_5 \]
\[ G_{18} = \int \int \xi_1 \xi_2 (E \xi) \, dA = E_3 J_{78} + (E_1 + E_2) J_5 \]
\[ G_{19} = \int \int \xi_1 \xi_3 (E \xi) \, dA = E_2 J_{78} + (E_1 + E_3) J_5 \]
\[ G_{20} = \int \int \xi_2 \xi_3 (E \xi) \, dA = E_1 J_{78} + (E_2 + E_3) J_5 \]
\[ G_{21} = \int \int \xi_1^3 (E \xi) \, dA = E_1 J_{67} + (E_2 + E_3) J_7 \]
\[ G_{22} = \int \int \xi_1^2 \xi_2 (E \xi) \, dA = E_1 J_{78} + E_2 J_8 + E_3 J_9 \]
\[ G_{23} = \int \int \xi_1 \xi_3^2 (E \xi) \, dA = E_1 J_{78} + E_3 J_8 + E_2 J_9 \]
\[ G_{24} = \int \int \xi_2^2 \xi_1 (E \xi) \, dA = E_2 J_{78} + E_1 J_8 + E_3 J_9 \]
\[ G_{25} = \int \int \xi_1 \xi_3 \xi_2 (E \xi) \, dA = E_3 J_{78} + E_1 J_8 + E_2 J_9 \]
\[ G_{26} = \int \int \xi_3 \xi_1^2 (E \xi) \, dA = E_3 J_{78} + E_2 J_8 + E_1 J_9 \]
\[ G_{27} = \int \int \xi_3 \xi_2^2 (E \xi) \, dA = E_3 J_{78} + E_2 J_8 + E_1 J_9 \]
\[ G_{28} = \int \int \xi_1 \xi_2 \xi_3 (E \xi) \, dA = (E_1 + E_2 + E_3) J_9 \]
\[ G_{29} = \int \int \xi_2 (E \xi) \, dA = E_2 J_{67} + (E_1 + E_3) J_7 \]
\[ G_{30} = \int \int \xi_3 (E \xi) \, dA = E_3 J_{67} + (E_1 + E_2) J_7 \]
\[ I_{22} = \int \int \tau_1 \psi_{11} (E \xi) \, dA = \int \int \tau_1 (2 \xi_1 - 1) \frac{2A}{x} (4 \xi_1 - 1) (E \xi) \, dA = 2A (8G_{21} - 6G_{15} + G_{12}) \]

\[ I_{23} = \int \int \tau_1 \psi_{22} (E \xi) \, dA = 2A (8G_{22} - 4G_{18} - 2G_{15} + G_{12}) \]

\[ I_{24} = \int \int \tau_1 \psi_{33} (E \xi) \, dA = 2A (8G_{23} - 4G_{19} - 2G_{15} + G_{12}) \]

\[ I_{25} = \int \int \tau_1 \psi_{14} (E \xi) \, dA = 2A (8G_{22} - 4G_{18}) \]

\[ I_{26} = \int \int \tau_1 \psi_{16} (E \xi) \, dA = 2A (8G_{23} - 4G_{19}) \]

\[ I_{27} = \int \int \tau_1 \psi_{24} (E \xi) \, dA = 2A (8G_{21} - 4G_{15}) \]

\[ I_{28} = \int \int \tau_2 \psi_{11} (E \xi) \, dA = 2A (8G_{24} - 4G_{18} - 2G_{16} + G_{13}) \]

\[ I_{29} = \int \int \tau_2 \psi_{22} (E \xi) \, dA = 2A (8G_{29} - 6G_{16} + G_{13}) \]

\[ I_{30} = \int \int \tau_2 \psi_{33} (E \xi) \, dA = 2A (8G_{25} - 4G_{20} - 2G_{16} + G_{13}) \]

\[ I_{31} = \int \int \tau_2 \psi_{14} (E \xi) \, dA = 2A (8G_{29} - 4G_{16}) \]

\[ I_{32} = \int \int \tau_2 \psi_{16} (E \xi) \, dA = 2A (8G_{25} - 4G_{20}) \]

\[ I_{33} = \int \int \tau_2 \psi_{24} (E \xi) \, dA = 2A (8G_{24} - 4G_{18}) \]

\[ I_{34} = \int \int \tau_3 \psi_{11} (E \xi) \, dA = 2A (8G_{26} - 4G_{19} - 2G_{17} + G_{14}) \]

\[ I_{35} = \int \int \tau_3 \psi_{22} (E \xi) \, dA = 2A (8G_{27} - 4G_{20} - 2G_{17} + G_{14}) \]

\[ I_{36} = \int \int \tau_3 \psi_{33} (E \xi) \, dA = 2A (8G_{30} - 6G_{17} + G_{14}) \]

\[ I_{37} = \int \int \tau_3 \psi_{14} (E \xi) \, dA = 2A (8G_{27} - 4G_{20}) \]

\[ I_{38} = \int \int \tau_3 \psi_{16} (E \xi) \, dA = 2A (8G_{30} - 4G_{17}) \]

\[ I_{39} = \int \int \tau_3 \psi_{24} (E \xi) \, dA = 2A (8G_{26} - 4G_{19}) \]

\[ I_{40} = \int \int \tau_4 \psi_{11} (E \xi) \, dA = 2A (16G_{22} - 4G_{18}) \]

\[ I_{41} = \int \int \tau_4 \psi_{22} (E \xi) \, dA = 2A (16G_{24} - 4G_{18}) \]

\[ I_{42} = \int \int \tau_4 \psi_{33} (E \xi) \, dA = 2A (16G_{28} - 4G_{18}) \]

\[ I_{43} = \int \int \tau_4 \psi_{14} (E \xi) \, dA = 2A \cdot 16G_{24} \]

\[ I_{44} = \int \int \tau_4 \psi_{16} (E \xi) \, dA = 2A \cdot 16G_{28} = \int \int \psi_{5 \psi_{24}} (E \xi) \, dA = \int \int \psi_{6 \psi_{14}} (E \xi) \, dA \]

\[ I_{45} = \int \int \tau_4 \psi_{24} (E \xi) \, dA = 2A \cdot 16G_{22} \]

\[ I_{46} = \int \int \tau_5 \psi_{11} (E \xi) \, dA = 2A (16G_{28} - 4G_{20}) \]

\[ I_{47} = \int \int \tau_5 \psi_{22} (E \xi) \, dA = 2A (16G_{25} - 4G_{20}) \]
\begin{align*}
\Gamma_{48} &= \int \Gamma_5 \psi_3 (E &x ) dA = 2A (16G_{27} - 4G_{20}) \\
\Gamma_{49} &= \int \Gamma_5 \psi_1 (E &x ) dA = 2A \cdot 16G_{25} \\
\Gamma_{50} &= \int \Gamma_5 \psi_6 (E &x ) dA = 2A \cdot 16G_{27} \\
\Gamma_{51} &= \int \Gamma_6 \psi_1 (E &x ) dA = 2A (16G_{23} - 4G_{19}) \\
\Gamma_{52} &= \int \Gamma_6 \psi_2 (E &x ) dA = 2A (16G_{28} - 4G_{19}) \\
\Gamma_{53} &= \int \Gamma_6 \psi_3 (E &x ) dA = 2A (16G_{26} - 4G_{19}) \\
\Gamma_{54} &= \int \Gamma_6 \psi_{16} (E &x ) dA = 2A \cdot 16G_{26} \\
\Gamma_{55} &= \int \Gamma_6 \psi_{24} (E &x ) dA = 2A \cdot 16G_{23} 
\end{align*}
II.4 Evaluation of the elements of $KC$ by approximate
integration:

$$KC_{1,1} = \frac{2\pi}{4A^2} D_{33} \int_1^2 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{1}{x_0} \right) \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(5E_1+E_2+E_3)}{210}$$

$$KC_{1,2} = \frac{2\pi}{4A^2} D_{33} \int_1^2 \Gamma_2 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(-4E_1-4E_2+E_3)}{1260}$$

$$KC_{1,3} = \frac{2\pi}{4A^2} D_{33} \int_1^2 \Gamma_3 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(-4E_1+E_2-4E_3)}{1260}$$

$$KC_{1,4} = \frac{2\pi}{4A^2} D_{33} \int_1^2 \Gamma_4 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(3E_1-2E_2-E_3)}{315}$$

$$KC_{1,5} = \frac{2\pi}{4A^2} D_{33} \int_1^2 \Gamma_5 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(-E_1-3E_2-3E_3)}{315}$$

$$KC_{1,6} = \frac{2\pi}{4A^2} D_{33} \int_1^2 \Gamma_6 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(3E_1-E_2-2E_3)}{315}$$

$$KC_{2,2} = \frac{2\pi}{4A^2} D_{33} \int_2^3 \Gamma_2 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(E_1+5E_2+E_3)}{210}$$

$$KC_{2,3} = \frac{2\pi}{4A^2} D_{33} \int_2^3 \Gamma_3 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(E_1-4E_2-4E_3)}{1260}$$

$$KC_{2,4} = \frac{2\pi}{4A^2} D_{33} \int_2^3 \Gamma_4 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(-2E_1+3E_2-E_3)}{315}$$

$$KC_{2,5} = \frac{2\pi}{4A^2} D_{33} \int_2^3 \Gamma_5 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(-E_1+3E_2-2E_3)}{315}$$

$$KC_{2,6} = \frac{2\pi}{4A^2} D_{33} \int_2^3 \Gamma_6 (E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \frac{2\pi}{4A^2} D_{33} \int_{\frac{1}{x}}^{\frac{2}{x}} \phi_1(E_x) \, dA = \left( 2\pi D_{33} \frac{A}{x_0} \right) \frac{(-3E_1-E_2-3E_3)}{315}$$
\[ K_{C,3} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (E_1 + E_2 + 5E_3) / 210 \]

\[ K_{C,4} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (-3E_1 - 3E_2 - E_3) / 315 \]

\[ K_{C,5} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (-E_1 - 2E_2 + 3E_3) / 315 \]

\[ K_{C,6} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (-2E_1 - E_2 + 3E_3) / 315 \]

\[ K_{C,4} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (E_1 + 3E_2 + E_3) \quad 8/315 \]

\[ K_{C,5} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (E_1 + 3E_2 + 2E_3) \quad 4/315 \]

\[ K_{C,6} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (2E_1 + 3E_2 + 2E_3) \quad 4/315 \]

\[ K_{C,5} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (E_1 + 3E_2 + 2E_3) \quad 8/315 \]

\[ K_{C,6} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (2E_1 + 2E_2 + 3E_3) \quad 4/315 \]

\[ K_{C,6} = \frac{2\pi}{4A^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E_{\xi}) \cdot dA = (2\pi D_{33} A/x_0) (3E_1 + E_2 + 3E_3) \quad 8/315 \]
II.5 Quadrature formulae for a triangular domain.

\[ I = \int_{\xi_2=0}^{1} \left[ \int_{\xi_1=0}^{1-\xi_2} f(\xi_1, \xi_2, \xi_3) d\xi_3 \right] d\xi_2 = \sum_{i=1}^{n} w_i f(\xi_1^i, \xi_2^i, \xi_3^i) \]

\[ \xi_3 = 1 - \xi_1 - \xi_2 \]

\[ dA = 2A \, d\xi_1 \, d\xi_2 \]

\[ \begin{array}{|c|c|c|c|c|c|} 
\hline
n & i & \xi_1^i & \xi_2^i & \xi_3^i & w_i \\
\hline
1 (linear) & 1 & 1/3 & 1/3 & 1/3 & 1/2 \\
2 (quadratic) & 1 & 1/2 & 1/2 & 0 & 1/6 \\
 & 2 & 0 & 1/2 & 1/2 & 1/6 \\
 & 3 & 1/2 & 0 & 1/2 & 1/6 \\
3 (cubic) & 1 & 1/3 & 1/3 & 1/3 & -9/32 \\
 & 2 & 3/5 & 1/5 & 1/5 & +25/96 \\
 & 3 & 1/5 & 3/5 & 1/5 & +25/96 \\
 & 4 & 1/5 & 1/5 & 3/5 & +25/96 \\
4 (quintic) & 1 & 0.333 & 333 & 33 & 0.333 & 333 & 33 & 0.112 & 500 \, CC \\
 & 2 & 0.797 & 426 & 99 & 0.101 & 286 & 51 & 0.101 & 286 & 51 & 0.062 & 969 & 59 \\
 & 3 & 0.101 & 286 & 51 & 0.797 & 426 & 99 & 0.101 & 286 & 51 & 0.062 & 969 & 59 \\
 & 4 & 0.101 & 286 & 51 & 0.101 & 286 & 51 & 0.797 & 426 & 99 & 0.062 & 969 & 59 \\
 & 5 & 0.059 & 715 & 87 & 0.470 & 142 & 06 & 0.470 & 142 & 06 & 0.066 & 197 & 06 \\
 & 6 & 0.470 & 142 & 06 & 0.059 & 715 & 87 & 0.470 & 142 & 06 & 0.066 & 197 & 06 \\
 & 7 & 0.470 & 142 & 06 & 0.470 & 142 & 06 & 0.059 & 715 & 87 & 0.066 & 197 & 06 \\
\hline
\end{array} \]